# Logic for Multiagent Systems

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## Propositional logic

## Language

*Definition 1.1*

*The language of propositional logic $PL$ consists of:*

- *a countable set $V = \{v_n \mid n \in \mathbb{N}\}$ of variables;*
- *the logic connectives $\neg$ (non), $\rightarrow$ (implies)*
- *parantheses: ( , ).*

- The set *Sym* of symbols of $PL$ is

$$Sym := V \cup \{\neg, \rightarrow, (, )\}.$$

- We denote variables by $u, v, x, y, z \ldots$

## Language

### Definition 1.2

*The set Expr of expressions of PL is the set of all finite sequences of symbols of PL.*

### Definition 1.3

*Let $\theta = \theta_0\theta_1 \dots \theta_{k-1}$ be an expression, where $\theta_i \in Sym$ for all $i = 0, \dots, k-1$.*

- *If $0 \leq i \leq j \leq k-1$, then the expression $\theta_i \dots \theta_j$ is called the $(i,j)$-subexpression of $\theta$.*
- *We say that an expression $\psi$ appears in $\theta$ if there exists $0 \leq i \leq j \leq k-1$ such that $\psi$ is the $(i,j)$-subexpression of $\theta$.*
- *We denote by $Var(\theta)$ the set of variables appearing in $\theta$.*

## Language

The definition of formulas is an example of an inductive definition.

### Definition 1.4

*The formulas of PL are the expressions of PL defined as follows:*

- *(F0) Any variable is a formula.*
- *(F1) If $\varphi$ is a formula, then $(\neg\varphi)$ is a formula.*
- *(F2) If $\varphi$ and $\psi$ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.*
- *(F3) Only the expressions obtained by applying rules (F0), (F1), (F2) are formulas.*

### Notations

The set of formulas is denoted by *Form*. Formulas are denoted by $\varphi, \psi, \chi, \ldots$.

### Proposition 1.5

*The set Form is countable.*

## *Language*

### *Unique readability*

If $\varphi$ is a formula, then exactly one of the following hold:

- $\varphi = v$, where $v \in V$.
- $\varphi = (\neg\psi)$, where $\psi$ is a formula.
- $\varphi = (\psi \to \chi)$, where $\psi, \chi$ are formulas.

Furthermore, $\varphi$ can be written in a unique way in one of these forms.

### *Definition 1.6*

*Let $\varphi$ be a formula. A subformula of $\varphi$ is any formula $\psi$ that appears in $\varphi$.*

## Language

*Proposition 1.7 (Induction principle on formulas)*

*Let* $\Gamma$ *be a set of formulas satisfying the following properties:*

- $V \subseteq \Gamma$.
- $\Gamma$ *is closed to* $\neg$, *that is:* $\varphi \in \Gamma$ *implies* $(\neg\varphi) \in \Gamma$.
- $\Gamma$ *is closed to* $\rightarrow$, *that is:* $\varphi, \psi \in \Gamma$ *implies* $(\varphi \rightarrow \psi) \in \Gamma$.

*Then* $\Gamma = \mathit{Form}$.

It is used to prove that all formulas have a property $\mathcal{P}$: we define $\Gamma$ as the set of all formulas satisfying $\mathcal{P}$ and apply induction on formulas to obtain that $\Gamma = \mathit{Form}$.

## Language

The derived connectives $\vee$ (or), $\wedge$ (and), $\leftrightarrow$ (if and only if) are introduced by the following abbreviations:

$$\begin{array}{lcl} \varphi \vee \psi & := & ((\neg\varphi) \to \psi) \\ \varphi \wedge \psi & := & \neg(\varphi \to (\neg\psi))) \\ \varphi \leftrightarrow \psi & := & ((\varphi \to \psi) \wedge (\psi \to \varphi)) \end{array}$$

### Conventions and notations

- The external parantheses are omitted, we put them only when necessary. We write $\neg\varphi$, $\varphi \to \psi$, but we write $(\varphi \to \psi) \to \chi$.
- To reduce the use of parentheses, we assume that
  - $\neg$ has higher precedence than $\to, \wedge, \vee, \leftrightarrow$;
  - $\wedge, \vee$ have higher precedence than $\to, \leftrightarrow$.
- Hence, the formula $(((\varphi \to (\psi \vee \chi)) \wedge ((\neg\psi) \leftrightarrow (\psi \vee \chi)))$ is written as $(\varphi \to \psi \vee \chi) \wedge (\neg\psi \leftrightarrow \psi \vee \chi)$.

## Semantics

### Truth values

We use the following notations for the truth values:

1 for true and 0 for false.

Hence, the set of truth values is $\{0, 1\}$.

Define the following operations on $\{0, 1\}$ using truth tables.

$$\neg : \{0, 1\} \to \{0, 1\},$$

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

$$\to : \{0, 1\} \times \{0, 1\} \to \{0, 1\},$$

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

## Semantics

$\vee : \{0,1\} \times \{0,1\} \to \{0,1\},$

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

$\wedge : \{0,1\} \times \{0,1\} \to \{0,1\},$

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

$\leftrightarrow : \{0,1\} \times \{0,1\} \to \{0,1\},$

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

## Semantics

*Definition 1.8*

*An evaluation (or interpretation) is a function $e : V \to \{0,1\}$.*

*Theorem 1.9*

*For any evaluation $e : V \to \{0,1\}$ there exists a unique function*

$$e^+ : Form \to \{0,1\}$$

*satisfying the following properties:*

- $e^+(v) = e(v)$ *for all* $v \in V$.
- $e^+(\neg\varphi) = \neg e^+(\varphi)$ *for any formula* $\varphi$.
- $e^+(\varphi \to \psi) = e^+(\varphi) \to e^+(\psi)$ *for any formulas* $\varphi$, $\psi$.

*Proposition 1.10*

*For any formula $\varphi$ and all evaluations $e_1, e_2 : V \to \{0,1\}$,*

*if $e_1(v) = e_2(v)$ for all $v \in Var(\varphi)$, then $e_1^+(\varphi) = e_2^+(\varphi)$.*

## Semantics

Let $\varphi$ be a formula.

### Definition 1.11

- *An evaluation $e : V \to \{0, 1\}$ is a model of $\varphi$ if $e^+(\varphi) = 1$.*
  *Notation: $e \vDash \varphi$.*
- *$\varphi$ is satisfiable if it has a model.*
- *If $\varphi$ is not satisfiable, we also say that $\varphi$ is unsatisfiable or contradictory.*
- *$\varphi$ is a tautology if every evaluation is a model of $\varphi$.*
  *Notation: $\vDash \varphi$.*

### Notation 1.12

*The set of models of $\varphi$ is denoted by $Mod(\varphi)$.*

## Semantics

*Remark 1.13*

- $\varphi$ *is a tautology iff* $\neg\varphi$ *is unsatisfiable.*
- $\varphi$ *is unsatisfiable iff* $\neg\varphi$ *is a tautology.*

*Proposition 1.14*

*Let* $e : V \to \{0, 1\}$ *be an evaluation. Then for all formulas* $\varphi$, $\psi$,

- $e \vDash \neg\varphi$ *iff* $e \nvDash \varphi$.
- $e \vDash \varphi \to \psi$ *iff* ($e \vDash \varphi$ *implies* $e \vDash \psi$) *iff* ($e \nvDash \varphi$ *or* $e \vDash \psi$).
- $e \vDash \varphi \vee \psi$ *iff* ($e \vDash \varphi$ *or* $e \vDash \psi$).
- $e \vDash \varphi \wedge \psi$ *iff* ($e \vDash \varphi$ *and* $e \vDash \psi$).
- $e \vDash \varphi \leftrightarrow \psi$ *iff* ($e \vDash \varphi$ *iff* $e \vDash \psi$).

## Semantics

### Definition 1.15

*Let $\varphi, \psi$ be formulas. We say that*

- *$\varphi$ is a semantic consequence of $\psi$ if $Mod(\psi) \subseteq Mod(\varphi)$.*
  *Notation: $\psi \vDash \varphi$.*
- *$\varphi$ and $\psi$ are (logically) equivalent if $Mod(\psi) = Mod(\varphi)$.*
  *Notation: $\varphi \sim \psi$.*

### Remark 1.16

*Let $\varphi, \psi$ be formulas.*

- *$\psi \vDash \varphi$ iff $\vDash \psi \rightarrow \varphi$.*
- *$\psi \sim \varphi$ iff ($\psi \vDash \varphi$ and $\varphi \vDash \psi$) iff $\vDash \psi \leftrightarrow \varphi$.*

## Semantics

For all formulas $\varphi, \psi, \chi$,

$$\begin{array}{ll}
\models & \varphi \vee \neg\varphi \\
\models & \neg(\varphi \wedge \neg\varphi) \\
\models & \varphi \wedge \psi \to \varphi \\
\models & \varphi \to \varphi \vee \psi \\
\models & \varphi \to (\psi \to \varphi) \\
\models & (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \\
\models & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\models & (\varphi \to \psi) \vee (\neg\varphi \to \psi) \\
\models & (\varphi \to \psi) \vee (\varphi \to \neg\psi) \\
\models & \neg\varphi \to (\neg\psi \leftrightarrow (\psi \to \varphi)) \\
\models & (\varphi \to \psi) \to (((\varphi \to \chi) \to \psi) \to \psi) \\
\models & \neg\psi \to (\psi \to \varphi)
\end{array}$$

## Semantics

$$\begin{aligned}
&\models \psi \to (\neg\psi \to \varphi)\\
&\models (\varphi \to \neg\varphi) \to \neg\varphi\\
&\models (\neg\varphi \to \varphi) \to \varphi\\
\psi &\models \varphi \to \psi\\
\neg\varphi &\models \varphi \to \psi\\
\neg\psi \wedge (\varphi \to \psi) &\models \neg\varphi\\
(\varphi \to \psi) \wedge (\psi \to \chi) &\models \varphi \to \chi\\
\varphi \wedge (\varphi \to \psi) &\models \psi
\end{aligned}$$

## Semantics

$$\begin{aligned}
\varphi \vee \psi &\sim \neg(\neg\varphi \wedge \neg\psi) \\
\varphi \wedge \psi &\sim \neg(\neg\varphi \vee \neg\psi) \\
\varphi \to (\psi \to \chi) &\sim \varphi \wedge \psi \to \chi \\
\varphi \sim \varphi \wedge \varphi &\sim \varphi \vee \varphi \\
\varphi \wedge \psi &\sim \psi \wedge \varphi \\
\varphi \vee \psi &\sim \psi \vee \varphi \\
\varphi \wedge (\psi \wedge \chi) &\sim (\varphi \wedge \psi) \wedge \chi \\
\varphi \vee (\psi \vee \chi) &\sim (\varphi \vee \psi) \vee \chi \\
\varphi \vee (\varphi \wedge \psi) &\sim \varphi \\
\varphi \wedge (\varphi \vee \psi) &\sim \varphi
\end{aligned}$$

## Semantics

$$\begin{array}{rcl}
\varphi \wedge (\psi \vee \chi) & \sim & (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \\
\varphi \vee (\psi \wedge \chi) & \sim & (\varphi \vee \psi) \wedge (\varphi \vee \chi) \\
\varphi \to \psi \wedge \chi & \sim & (\varphi \to \psi) \wedge (\varphi \to \chi) \\
\varphi \to \psi \vee \chi & \sim & (\varphi \to \psi) \vee (\varphi \to \chi) \\
\varphi \wedge \psi \to \chi & \sim & (\varphi \to \chi) \vee (\psi \to \chi) \\
\varphi \vee \psi \to \chi & \sim & (\varphi \to \chi) \wedge (\psi \to \chi) \\
\varphi \to (\psi \to \chi) & \sim & \psi \to (\varphi \to \chi) \\
 & \sim & (\varphi \to \psi) \to (\varphi \to \chi) \\
\neg\varphi \sim \varphi \to \neg\varphi & \sim & (\varphi \to \psi) \wedge (\varphi \to \neg\psi) \\
\varphi \to \psi \sim \neg\varphi \vee \psi & \sim & \neg(\varphi \wedge \neg\psi) \\
\varphi \vee \psi \sim \varphi \vee (\neg\varphi \wedge \psi) & \sim & (\varphi \to \psi) \to \psi \\
\varphi \leftrightarrow (\psi \leftrightarrow \chi) & \sim & (\varphi \leftrightarrow \psi) \leftrightarrow \chi
\end{array}$$

## Semantics

It is often useful to have a canonical tautology and a canonical unsatisfiable formula.

### Remark 1.17

*$v_0 \to v_0$ is a tautology and $\neg(v_0 \to v_0)$ is unsatisfiable.*

### Notation 1.18

*Denote $v_0 \to v_0$ by $\top$ and call it the truth.*
*Denote $\neg(v_0 \to v_0)$ by $\bot$ and call it the false.*

### Remark 1.19

- *$\varphi$ is a tautology iff $\varphi \sim \top$.*
- *$\varphi$ is unsatisfiable iff $\varphi \sim \bot$.*

## Semantics

Let $\Gamma$ be a set of formulas.

### Definition 1.20

*An evaluation $e : V \rightarrow \{0, 1\}$ is a model of $\Gamma$ if it is a model of every formula from $\Gamma$.*
*Notation: $e \vDash \Gamma$.*

### Notation 1.21

*The set of models of $\Gamma$ is denoted by $Mod(\Gamma)$.*

### Definition 1.22

*A formula $\varphi$ is a semantic consequence of $\Gamma$ if $Mod(\Gamma) \subseteq Mod(\varphi)$.*
*Notation: $\Gamma \vDash \varphi$.*

## Semantics

*Definition 1.23*

- $\Gamma$ *is satisfiable if it has a model.*
- $\Gamma$ *is finitely satisfiable if every finite subset of* $\Gamma$ *is satisfiable.*
- *If* $\Gamma$ *is not satisfiable, we say also that* $\Gamma$ *is unsatisfiable or contradictory.*

*Proposition 1.24*

*The following are equivalent:*

- $\Gamma$ *is unsatisfiable.*
- $\Gamma \models \bot$.

*Theorem 1.25 (Compactness Theorem)*

$\Gamma$ *is satisfiable iff* $\Gamma$ *is finitely satisfiable.*

## Syntax

We use a deductive system of Hilbert type for *LP*.

### Logical axioms

The set *Axm* of (logical) axioms of *LP* consists of:

(*A*1) $\varphi \to (\psi \to \varphi)$

(*A*2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(*A*3) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$,

where $\varphi$, $\psi$ and $\chi$ are formulas.

### The deduction rule

For any formulas $\varphi$, $\psi$, from $\varphi$ and $\varphi \to \psi$ infer $\psi$ (modus ponens or (MP)):

$$\frac{\varphi,\ \varphi \to \psi}{\psi}$$

## *Syntax*

Let $\Gamma$ be a set of formulas. The definition of $\Gamma$-theorems is another example of an inductive definition.

*Definition 1.26*

*The $\Gamma$-theorems of PL are the formulas defined as follows:*

(T0) *Every logical axiom is a $\Gamma$-theorem.*

(T1) *Every formula of $\Gamma$ is a $\Gamma$-theorem.*

(T2) *If $\varphi$ and $\varphi \to \psi$ are $\Gamma$-theorems, then $\psi$ is a $\Gamma$-theorem.*

(T3) *Only the formulas obtained by applying rules (T0), (T1), (T2) are $\Gamma$-theorems.*

If $\varphi$ is a $\Gamma$-theorem, then we also say that $\varphi$ is deduced from the hypotheses $\Gamma$.

## Syntax

### Notations

| | | |
|---|---|---|
| $\Gamma \vdash \varphi$ | $:\Leftrightarrow$ | $\varphi$ is a $\Gamma$-theorem |
| $\vdash \varphi$ | $:\Leftrightarrow$ | $\emptyset \vdash \varphi$. |

### *Definition 1.27*

*A formula $\varphi$ is called a theorem of LP if $\vdash \varphi$.*

By a reformulation of the conditions (T0), (T1), (T2) using the notation $\vdash$, we get

### *Remark 1.28*

- *If $\varphi$ is an axiom, then $\Gamma \vdash \varphi$.*
- *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*
- *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.*

## Syntax

*Definition 1.29*

*A $\Gamma$-proof (or proof from the hypotheses $\Gamma$) is a sequence of formulas $\theta_1$, ..., $\theta_n$ such that for all $i \in \{1, \dots, n\}$, one of the following holds:*

- *$\theta_i$ is an axiom.*
- *$\theta_i \in \Gamma$.*
- *there exist $k, j < i$ such that $\theta_k = \theta_j \to \theta_i$.*

*Definition 1.30*

*Let $\varphi$ be a formula. A $\Gamma$-proof of $\varphi$ or a proof of $\varphi$ from the hypotheses $\Gamma$ is a $\Gamma$-proof $\theta_1$, ..., $\theta_n$ such that $\theta_n = \varphi$.*

*Proposition 1.31*

*For any formula $\varphi$,*

$$\Gamma \vdash \varphi \quad \text{iff} \quad \text{there exists a } \Gamma\text{-proof of } \varphi.$$

## Syntax

*Theorem 1.32 (Deduction Theorem)*

*Let $\Gamma \cup \{\varphi, \psi\}$ be a set of formulas. Then*

$$\Gamma \cup \{\varphi\} \vdash \psi \quad \textit{iff} \quad \Gamma \vdash \varphi \to \psi.$$

*Proposition 1.33*

*For any formulas $\varphi, \psi, \chi$,*

$$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$$

*Proposition 1.34*

*Let $\Gamma \cup \{\varphi, \psi, \chi\}$ be a set of formulas.*

$$\begin{aligned} \Gamma \vdash \varphi \to \psi \textit{ and } \Gamma \vdash \psi \to \chi \quad &\Rightarrow \quad \Gamma \vdash \varphi \to \chi \\ \Gamma \cup \{\neg\psi\} \vdash \neg(\varphi \to \varphi) \quad &\Rightarrow \quad \Gamma \vdash \psi \\ \Gamma \cup \{\psi\} \vdash \varphi \textit{ and } \Gamma \cup \{\neg\psi\} \vdash \varphi \quad &\Rightarrow \quad \Gamma \vdash \varphi. \end{aligned}$$

## Consistent sets

Let Γ be a set of formulas.

*Definition 1.35*

*Γ is called consistent if there exists a formula $\varphi$ such that $\Gamma \nvdash \varphi$. Γ is said to be inconsistent if it is not consistent, that is $\Gamma \vdash \varphi$ for any formula $\varphi$.*

*Proposition 1.36*

- *$\emptyset$ is consistent.*
- *The set of theorems is consistent.*

*Proposition 1.37*

*The following are equivalent:*

- *Γ is inconsistent.*
- *$\Gamma \vdash \perp$.*

## Completeness Theorem

*Theorem 1.38 (Completeness Theorem (version 1))*

*Let $\Gamma$ be a set of formulas. Then*

$$\Gamma \textit{ is consistent} \quad \Longleftrightarrow \quad \Gamma \textit{ is satisfiable.}$$

*Theorem 1.39 (Completeness Theorem (version 2))*

*Let $\Gamma$ be a set of formulas. For any formula $\varphi$,*

$$\Gamma \vdash \varphi \quad \Longleftrightarrow \quad \Gamma \models \varphi.$$

## First-order logic

## First-order languages

### *Definition 2.1*

*A first-order language $\mathcal{L}$ consists of:*

- *a countable set $V = \{v_n \mid n \in \mathbb{N}\}$ of variables;*
- *the connectives $\neg$ and $\rightarrow$;*
- *parantheses ( , );*
- *the equality symbol $=$;*
- *the universal quantifier $\forall$;*
- *a set $\mathcal{R}$ of relation symbols;*
- *a set $\mathcal{F}$ of function symbols;*
- *a set $\mathcal{C}$ of constant symbols;*
- *an arity function $ari : \mathcal{F} \cup \mathcal{R} \rightarrow \mathbb{N}^*$.*

- $\mathcal{L}$ is uniquely determined by the quadruple $\tau := (\mathcal{R}, \mathcal{F}, \mathcal{C}, \text{ari})$.
- $\tau$ is called the signature of $\mathcal{L}$ or the similaritaty type of $\mathcal{L}$.

## First-order languages

Let $\mathcal{L}$ be a first-order language.

- The set $Sym_{\mathcal{L}}$ of symbols of $\mathcal{L}$ is

$$Sym_{\mathcal{L}} := V \cup \{\neg, \rightarrow, (, ), =, \forall\} \cup \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$$

- The elements of $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ are called non-logical symbols.
- The elements of $V \cup \{\neg, \rightarrow, (, ), =, \forall\}$ are called logical symbols.

- We denote variables by $x, y, z, v, \ldots$, relation symbols by $P, Q, R \ldots$, function symbols by $f, g, h, \ldots$ and constant symbols by $c, d, e, \ldots$.

- For every $m \in \mathbb{N}^*$ we denote:

$\mathcal{F}_m$ := the set of function symbols of arity $m$;

$\mathcal{R}_m$ := the set of relation symbols of arity $m$.

## *First-order languages*

*Definition 2.2*

*The set $Expr_{\mathcal{L}}$ of expressions of $\mathcal{L}$ is the set of all finite sequences of symbols of $\mathcal{L}$.*

*Definition 2.3*

*Let $\theta = \theta_0\theta_1 \ldots \theta_{k-1}$ be an expression of $\mathcal{L}$, where $\theta_i \in Sym_{\mathcal{L}}$ for all $i = 0, \ldots, k-1$.*

- *If $0 \leq i \leq j \leq k-1$, then the expression $\theta_i \ldots \theta_j$ is called the $(i, j)$-subexpression of $\theta$.*
- *We say that an expression $\psi$ appears in $\theta$ if there exists $0 \leq i \leq j \leq k-1$ such that $\psi$ is the $(i, j)$-subexpression of $\theta$.*
- *We denote by $Var(\theta)$ the set of variables appearing in $\theta$.*

## First-order languages

*Definition 2.4*

*The terms of $\mathcal{L}$ are the expressions defined as follows:*

(T0) *Every variable is a term.*

(T1) *Every constant symbol is a term.*

(T2) *If $m \geq 1$, $f \in \mathcal{F}_m$ and $t_1, \ldots, t_m$ are terms, then $ft_1 \ldots t_m$ is a term.*

(T3) *Only the expressions obtained by applying rules (T0), (T1), (T2) are terms.*

Notations:

- The set of terms is denoted by $Term_{\mathcal{L}}$.
- Terms are denoted by $t, s, t_1, t_2, s_1, s_2, \ldots$.
- $Var(t)$ is the set of variables that appear in the term $t$.

*Definition 2.5*

*A term $t$ is called closed if $Var(t) = \emptyset$.*

## First-order languages

*Proposition 2.6 (Induction on terms)*

*Let $\Gamma$ be a set of terms satisfying the following properties:*

- *$\Gamma$ contains the variables and the constant symbols.*
- *If $m \geq 1$, $f \in \mathcal{F}_m$ and $t_1, \ldots, t_m \in \Gamma$, then $ft_1 \ldots t_m \in \Gamma$.*

*Then $\Gamma = Term_{\mathcal{L}}$.*

It is used to prove that all terms have a property $\mathcal{P}$: we define $\Gamma$ as the set of all terms satisfying $\mathcal{P}$ and apply induction on terms to obtain that $\Gamma = Term_{\mathcal{L}}$.

## First-order languages

### Definition 2.7

*The atomic formulas of $\mathcal{L}$ are the expressions having one of the following forms:*

- $(s = t)$, *where* $s, t$ *are terms;*
- $(Rt_1 \dots t_m)$, *where* $R \in \mathcal{R}_m$ *and* $t_1, \dots, t_m$ *are terms.*

### Definition 2.8

*The formulas of $\mathcal{L}$ are the expressions defined as follows:*

(F0) *Every atomic formula is a formula.*

(F1) *If* $\varphi$ *is a formula, then* $(\neg\varphi)$ *is a formula.*

(F2) *If* $\varphi$ *and* $\psi$ *are formulas, then* $(\varphi \to \psi)$ *is a formula.*

(F3) *If* $\varphi$ *is a formula, then* $(\forall x \varphi)$ *is a formula for every variable* $x$.

(F4) *Only the expressions obtained by applying rules (F0), (F1), (F2), (F3) are formulas.*

## *First-order languages*

### *Notations*

- The set of formulas is denoted by *$Form_{\mathcal{L}}$*.
- Formulas are denoted by $\varphi, \psi, \chi, \ldots$.
- $Var(\varphi)$ is the set of variables that appear in the formula $\varphi$.

### *Unique readability*

If $\varphi$ is a formula, then exactly one of the following hold:

- $\varphi = (s = t)$, where $s, t$ are terms.
- $\varphi = (Rt_1 \ldots t_m)$, where $R \in \mathcal{R}_m$ and $t_1, \ldots, t_m$ are terms.
- $\varphi = (\neg\psi)$, where $\psi$ is a formula.
- $\varphi = (\psi \rightarrow \chi)$, where $\psi, \chi$ are formulas.
- $\varphi = (\forall x \psi)$, where $x$ is a variable and $\psi$ is a formula.

Furthermore, $\varphi$ can be written in a unique way in one of these forms.

## *First-order languages*

*Proposition 2.9 (Induction principle on formulas)*

*Let $\Gamma$ be a set of formulas satisfying the following properties:*

- *$\Gamma$ contains all atomic formulas.*
- *$\Gamma$ is closed to $\neg, \rightarrow$ and $\forall x$ (for any variable $x$), that is:*

$$\textit{if } \varphi, \psi \in \Gamma, \textit{ then } (\neg\varphi), (\varphi \rightarrow \psi), (\forall x \varphi) \in \Gamma.$$

*Then $\Gamma = Form_{\mathcal{L}}$.*

It is used to prove that all formulas have a property $\mathcal{P}$: we define $\Gamma$ as the set of all formulas satisfying $\mathcal{P}$ and apply induction on formulas to obtain that $\Gamma = Form_{\mathcal{L}}$.

## First-order languages

### Derived connectives

Connectives $\vee$, $\wedge$, $\leftrightarrow$ and the existential quantifier $\exists$ are introduced by the following abbreviations:

$$\begin{aligned}
\varphi \vee \psi &:= ((\neg\varphi) \rightarrow \psi)\\
\varphi \wedge \psi &:= \neg(\varphi \rightarrow (\neg\psi)))\\
\varphi \leftrightarrow \psi &:= ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))\\
\exists x\varphi &:= (\neg\forall x(\neg\varphi))
\end{aligned}$$

## First-order languages

Usually the external parantheses are omitted, we write them only when necessary. We write $s = t$, $Rt_1 \dots t_m$, $ft_1 \dots t_m$, $\neg\varphi$, $\varphi \to \psi$, $\forall x\varphi$. On the other hand, we write $(\varphi \to \psi) \to \chi$.

To reduce the use of parentheses, we assume that

- $\neg$ has higher precedence than $\to, \wedge, \vee, \leftrightarrow$;
- $\wedge, \vee$ have higher precedence than $\to, \leftrightarrow$;
- quantifiers $\forall$, $\exists$ have higher precedence than the other connectives. Thus, $\forall x\varphi \to \psi$ is $(\forall x\varphi) \to \psi$ and not $\forall x(\varphi \to \psi)$.

## First-order languages

- We write sometimes $f(t_1, \ldots, t_m)$ instead of $ft_1 \ldots t_m$ and $R(t_1, \ldots, t_m)$ instead of $Rt_1 \ldots t_m$.
- Function/relation symbols of arity 1 are called unary.
  Function/relation symbols of arity 2 are called binary.
- If $f$ is a binary function symbol, we write $t_1 f t_2$ instead of $f t_1 t_2$.
- If $R$ is a binary relation symbol, we write $t_1 R t_2$ instead of $R t_1 t_2$.

We identify often a language $\mathcal{L}$ with the set of its non-logical symbols and write $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$.

## First-order languages

### Definition 2.10

*Let $\varphi = \varphi_0\varphi_1\dots\varphi_{n-1}$ be a formula of $\mathcal{L}$ and $x$ be a variable.*

- *We say that $x$ occurs bound on position $k$ in $\varphi$ if $x = \varphi_k$ and there exists $0 \leq i \leq k \leq j \leq n-1$ such that the $(i,j)$-subexpression of $\varphi$ has the form $\forall x\psi$.*
- *We say that $x$ occurs free on position $k$ in $\varphi$ if $x = \varphi_k$, but $x$ does not occur bound on position $k$ in $\varphi$.*
- *$x$ is a bound variable of $\varphi$ if there exists $k$ such that $x$ occurs bound on position $k$ in $\varphi$.*
- *$x$ is a free variable of $\varphi$ if there exists $k$ such that $x$ occurs free on position $k$ in $\varphi$.*

### Example

Let $\varphi = \forall x(x = y) \to x = z$. Free variables: $x, y, z$. Bound variables: $x$.

## *First-order languages*

Notation: $FV(\varphi) :=$ the set of free variables of $\varphi$.

*Alternative definition*

The set $FV(\varphi)$ of free variables of a formula $\varphi$ can be also defined by induction on formulas:

$$\begin{aligned}
FV(\varphi) &= Var(\varphi), \quad \text{if } \varphi \text{ is an atomic formula}\\
FV(\neg\varphi) &= FV(\varphi)\\
FV(\varphi \to \psi) &= FV(\varphi) \cup FV(\psi)\\
FV(\forall x \varphi) &= FV(\varphi) \setminus \{x\}.
\end{aligned}$$

## $\mathcal{L}$-structures

### Definition 2.11

*An $\mathcal{L}$-structure is a quadruple*

$$\mathcal{A} = (A, \mathcal{F}^{\mathcal{A}}, \mathcal{R}^{\mathcal{A}}, \mathcal{C}^{\mathcal{A}}),$$

*where*

- *$A$ is a nonempty set.*
- *$\mathcal{F}^{\mathcal{A}} = \{f^{\mathcal{A}} \mid f \in \mathcal{F}\}$ is a set of functions on $A$; if $f$ has arity $m$, then $f^{\mathcal{A}} : A^m \to A$.*
- *$\mathcal{R}^{\mathcal{A}} = \{R^{\mathcal{A}} \mid R \in \mathcal{R}\}$ is a set of relations on $A$; if $R$ has arity $m$, then $R^{\mathcal{A}} \subseteq A^m$.*
- *$\mathcal{C}^{\mathcal{A}} = \{c^{\mathcal{A}} \in A \mid c \in \mathcal{C}\}$.*
- *$A$ is called the universe of the structure $\mathcal{A}$. Notation: $A = |\mathcal{A}|$*
- *$f^{\mathcal{A}}$ ( $R^{\mathcal{A}}$, $c^{\mathcal{A}}$, respectively) is called the interpretation of $f$ ($R$, $c$, respectively) in $\mathcal{A}$.*

## Examples - The language of equality $\mathcal{L}_=$

$\mathcal{L}_= = (\mathcal{R}, \mathcal{F}, \mathcal{C})$, where

- $\mathcal{R} = \mathcal{F} = \mathcal{C} = \emptyset$;
- this language is proper for expressing the properties of equality;
- $\mathcal{L}_=$-structures are the nonempty sets.

*Examples of formulas:*

- equality is symmetric:

$$\forall x \forall y (x = y \to y = x)$$

- the universe has at least three elements:

$$\exists x \exists y \exists z (\neg(x = y) \wedge \neg(y = z) \wedge \neg(z = x))$$

## Examples - The language of arithmetics $\mathcal{L}_{ar}$

$\mathcal{L}_{ar} = (\mathcal{R}, \mathcal{F}, \mathcal{C})$, where

- $\mathcal{R} = \{\dot{<}\}$; $\dot{<}$ is a binary relation symbol;
- $\mathcal{F} = \{\dot{+}, \dot{\times}, \dot{S}\}$; $\dot{+}, \dot{\times}$ are binary function symbols and $\dot{S}$ is a unary function symbol;
- $\mathcal{C} = \{\dot{0}\}$.

We write $\mathcal{L}_{ar} = (\dot{<}; \dot{+}, \dot{\times}, \dot{S}; \dot{0})$ or $\mathcal{L}_{ar} = (\dot{<}, \dot{+}, \dot{\times}, \dot{S}, \dot{0})$.

The natural example of $\mathcal{L}_{ar}$-structure:

$$\mathcal{N} := (\mathbb{N}, <, +, \cdot, S, 0),$$

where $S : \mathbb{N} \to \mathbb{N}, S(m) = m + 1$ is the successor function. Thus,

$$\dot{<}^{\mathcal{N}} =<,\ \dot{+}^{\mathcal{N}} = +,\ \dot{\times}^{\mathcal{N}} = \cdot,\ \dot{S}^{\mathcal{N}} = S,\ \dot{0}^{\mathcal{N}} = 0.$$

- Another example of $\mathcal{L}_{ar}$-structure: $\mathcal{A} = (\{0, 1\}, <, \vee, \wedge, \neg, 1)$.

## Examples - The language with a binary relation symbol

$\mathcal{L}_R = (\mathcal{R}, \mathcal{F}, \mathcal{C})$, where

- $\mathcal{R} = \{R\}$; $R$ is a binary relation symbol;
- $\mathcal{F} = \mathcal{C} = \emptyset$;
- $\mathcal{L}$-structures are nonempty sets together with a binary relation.

- If we are interested in partially ordered sets $(A, \leq)$, we use the symbol $\dot{\leq}$ instead of $R$ and we denote the language by $\mathcal{L}_{\leq}$.
- If we are interested in strictly ordered sets $(A, <)$, we use the symbol $\dot{<}$ instead of $R$ and we denote the language by $\mathcal{L}_{<}$.
- If we are interested in graphs $G = (V, E)$, we use the symbol $\dot{E}$ instead of $R$ and we denote the language by $\mathcal{L}_{Graf}$.
- If we are interested in structures $(A, \in)$, we use the symbol $\dot{\in}$ instead of $R$ and we denote the language by $\mathcal{L}_{\in}$.

## Semantics

Let $\mathcal{L}$ be a first-order language and $\mathcal{A}$ be an $\mathcal{L}$-structure.

*Definition 2.12*

*An $\mathcal{A}$-assignment or $\mathcal{A}$-evaluation is a function $e : V \to A$.*

When the $\mathcal{L}$-structure $\mathcal{A}$ is clear from the context, we also write simply $e$ is an assignment.

In the following, $e : V \to A$ is an $\mathcal{A}$-assignment.

*Definition 2.13 (Interpretation of terms)*

*The interpretation $t^{\mathcal{A}}(e) \in A$ of a* term $t$ under the $\mathcal{A}$-assignment *$e$ is defined by induction on terms :*

- *if $t = x \in V$, then $t^{\mathcal{A}}(e) := e(x)$;*
- *if $t = c \in \mathcal{C}$, then $t^{\mathcal{A}}(e) := c^{\mathcal{A}}$;*
- *if $t = ft_1 \ldots t_m$, then $t^{\mathcal{A}}(e) := f^{\mathcal{A}}(t_1^{\mathcal{A}}(e), \ldots, t_m^{\mathcal{A}}(e))$.*

## *Semantics*

The interpretation

$$\varphi^{\mathcal{A}}(e) \in \{0,1\}$$

of a *formula $\varphi$ under the $\mathcal{A}$-assignment $e$* is defined by induction on formulas.

$$\begin{aligned} (s = t)^{\mathcal{A}}(e) &= \begin{cases} 1 & \text{if } s^{\mathcal{A}}(e) = t^{\mathcal{A}}(e) \\ 0 & \text{otherwise.} \end{cases} \\ (Rt_1 \dots t_m)^{\mathcal{A}}(e) &= \begin{cases} 1 & \text{if } R^{\mathcal{A}}(t_1^{\mathcal{A}}(e), \dots, t_m^{\mathcal{A}}(e)) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

## Semantics

### *Negation and implication*

- $(\neg\varphi)^{\mathcal{A}}(e) = 1 - \varphi^{\mathcal{A}}(e)$;
- $(\varphi \to \psi)^{\mathcal{A}}(e) = \varphi^{\mathcal{A}}(e) \rightarrow \psi^{\mathcal{A}}(e)$, where,

$$\rightarrow : \{0,1\} \times \{0,1\} \to \{0,1\},$$

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Hence,

- $(\neg\varphi)^{\mathcal{A}}(e) = 1$ iff $\varphi^{\mathcal{A}}(e) = 0$.
- $(\varphi \to \psi)^{\mathcal{A}}(e) = 1$ iff $\big(\, \varphi^{\mathcal{A}}(e) = 0$ or $\psi^{\mathcal{A}}(e) = 1\big)$.

## Semantics

### Notation

For any variable $x \in V$ and any $a \in A$, we define a new $\mathcal{A}$-assignment $e_{x \mapsto a} : V \to A$ by

$$e_{x \mapsto a}(v) = \begin{cases} e(v) & \text{if } v \neq x \\ a & \text{if } v = x. \end{cases}$$

### Universal quantifier

$$(\forall x \varphi)^{\mathcal{A}}(e) \quad = \quad \begin{cases} 1 & \text{if } \varphi^{\mathcal{A}}(e_{x \mapsto a}) = 1 \text{ for all } a \in A \\ 0 & \text{otherwise.} \end{cases}$$

## Semantics

Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $e : V \to A$ be an $\mathcal{A}$-assignment.

### *Definition 2.14*

*Let $\varphi$ be a formula. We say that:*

- *$e$ satisfies $\varphi$ in $\mathcal{A}$ if $\varphi^{\mathcal{A}}(e) = 1$. Notation: $\mathcal{A} \vDash \varphi[e]$.*
- *$e$ does not satisfy $\varphi$ in $\mathcal{A}$ if $\varphi^{\mathcal{A}}(e) = 0$. Notation: $\mathcal{A} \nvDash \varphi[e]$.*

### *Proposition 2.15*

*For all formulas $\varphi, \psi$ and any variable $x$,*

*(i)* $\mathcal{A} \vDash \neg\varphi[e]$ *iff* $\mathcal{A} \nvDash \varphi[e]$.

*(ii)* $\mathcal{A} \vDash (\varphi \to \psi)[e]$ *iff* ($\mathcal{A} \vDash \varphi[e]$ *implies* $\mathcal{A} \vDash \psi[e]$)
*iff* ($\mathcal{A} \nvDash \varphi[e]$ *or* $\mathcal{A} \vDash \psi[e]$).

*(iii)* $\mathcal{A} \vDash (\forall x\varphi)[e]$ *iff for all* $a \in A$, $\mathcal{A} \vDash \varphi[e_{x \mapsto a}]$.

## Semantics

*Proposition 2.16*

*For all formulas $\varphi, \psi$ and any variable $x$,*

*(i)* $\mathcal{A} \vDash (\varphi \wedge \psi)[e]$ *iff* ($\mathcal{A} \vDash \varphi[e]$ *and* $\mathcal{A} \vDash \psi[e]$).

*(ii)* $\mathcal{A} \vDash (\varphi \vee \psi)[e]$ *iff* ($\mathcal{A} \vDash \varphi[e]$ *or* $\mathcal{A} \vDash \psi[e]$).

*(iii)* $\mathcal{A} \vDash (\varphi \leftrightarrow \psi)[e]$ *iff* ($\mathcal{A} \vDash \varphi[e]$ *iff* $\mathcal{A} \vDash \psi[e]$).

*(iv)* $\mathcal{A} \vDash (\exists x \varphi)[e]$ *iff there exists* $a \in A$ *s.t.* $\mathcal{A} \vDash \varphi[e_{x \mapsto a}]$.

## Semantics

Let $\varphi$ be a formula of $\mathcal{L}$.

*Definition 2.17*

*$\varphi$ is satisfiable if there exists an $\mathcal{L}$-structure $\mathcal{A}$ and an $\mathcal{A}$-assignment $e$ such that $\mathcal{A} \vDash \varphi[e]$.*
*We also say that $(\mathcal{A}, e)$ is a model of $\varphi$.*

*Definition 2.18*

*$\varphi$ is true in an $\mathcal{L}$-structure $\mathcal{A}$ if $\mathcal{A} \vDash \varphi[e]$ for all $\mathcal{A}$-assignments $e$.*
*We also say that $\mathcal{A}$ satisfies $\varphi$ or that $\mathcal{A}$ is a model of $\varphi$.*
*Notation:* $\mathcal{A} \vDash \varphi$

*Definition 2.19*

*$\varphi$ is universally true (or logically valid or, simply, valid) if $\mathcal{A} \vDash \varphi$ for all $\mathcal{L}$-structures $\mathcal{A}$.*
*Notation:* $\vDash \varphi$

## Semantics

Let $\varphi, \psi$ be formulas of $\mathcal{L}$.

*Definition 2.20*

*$\psi$ is a logical consequence of $\varphi$ if for all $\mathcal{L}$-structures $\mathcal{A}$ and all $\mathcal{A}$-assignments $e$,*

$$\mathcal{A} \vDash \varphi[e] \quad \textit{implies} \quad \mathcal{A} \vDash \psi[e].$$

*Notation:* $\varphi \vDash \psi$

*Definition 2.21*

*$\varphi$ and $\psi$ are logically equivalent or, simply, equivalent if for all $\mathcal{L}$-structures $\mathcal{A}$ and all $\mathcal{A}$-assignments $e$,*

$$\mathcal{A} \vDash \varphi[e] \textit{ iff } \mathcal{A} \vDash \psi[e].$$

*Notation:* $\varphi \mathrel{\text{⫤}\!\vDash} \psi$

*Remark*

- $\varphi \vDash \psi$ iff $\vDash \varphi \to \psi$.
- $\varphi \mathrel{\text{⫤}\!\vDash} \psi$ iff ($\psi \vDash \varphi$ and $\varphi \vDash \psi$) iff $\vDash \psi \leftrightarrow \varphi$.

## Semantics

For all formulas $\varphi$, $\psi$ and all variables $x, y$,

$$\begin{aligned}
\neg\exists x\varphi &\;\models\!\!\!\dashv\; \forall x\neg\varphi && (1)\\
\neg\forall x\varphi &\;\models\!\!\!\dashv\; \exists x\neg\varphi && (2)\\
\forall x(\varphi \wedge \psi) &\;\models\!\!\!\dashv\; \forall x\varphi \wedge \forall x\psi && (3)\\
\forall x\varphi \vee \forall x\psi &\;\models\; \forall x(\varphi \vee \psi) && (4)\\
\exists x(\varphi \wedge \psi) &\;\models\; \exists x\varphi \wedge \exists x\psi && (5)\\
\exists x(\varphi \vee \psi) &\;\models\!\!\!\dashv\; \exists x\varphi \vee \exists x\psi && (6)\\
\forall x(\varphi \rightarrow \psi) &\;\models\; \forall x\varphi \rightarrow \forall x\psi && (7)\\
\forall x(\varphi \rightarrow \psi) &\;\models\; \exists x\varphi \rightarrow \exists x\psi && (8)\\
\forall x\varphi &\;\models\; \exists x\varphi && (9)
\end{aligned}$$

## Semantics

$$\begin{aligned}
\varphi &\models \exists x\varphi && (10)\\
\forall x\varphi &\models \varphi && (11)\\
\forall x\forall y\varphi &\mathrel{\models\!\!\!\dashv} \forall y\forall x\varphi && (12)\\
\exists x\exists y\varphi &\mathrel{\models\!\!\!\dashv} \exists y\exists x\varphi && (13)\\
\exists y\forall x\varphi &\models \forall x\exists y\varphi. && (14)
\end{aligned}$$

## Semantics

*Proposition 2.22*

*For all terms $s, t, u$,*

*(i)* $\models t = t$;

*(ii)* $\models s = t \to t = s$;

*(iii)* $\models s = t \wedge t = u \to s = u$.

*Proposition 2.23*

*For all $m \geq 1$, $f \in \mathcal{F}_m, R \in \mathcal{R}_m$ and all terms $t_i, u_i, i = 1, \dots, m$,*

$$\models (t_1 = u_1) \wedge \dots \wedge (t_m = u_m) \to ft_1 \dots t_m = fu_1 \dots u_m$$

$$\models (t_1 = u_1) \wedge \dots \wedge (t_m = u_m) \to (Rt_1 \dots t_m \leftrightarrow Ru_1 \dots u_m)$$

## Semantics

*Proposition 2.24*

*For any $\mathcal{L}$-structure $\mathcal{A}$ and any $\mathcal{A}$-assignments $e_1, e_2$,*

*(i)* *for any term $t$,*

$$\text{if } e_1(v) = e_2(v) \text{ for all variables } v \in Var(t), \text{ then } t^{\mathcal{A}}(e_1) = t^{\mathcal{A}}(e_2).$$

*(ii)* *for any formula $\varphi$,*

$$\text{if } e_1(v) = e_2(v) \text{ for all variables } v \in FV(\varphi), \text{ then } \mathcal{A} \vDash \varphi[e_1] \text{ iff } \mathcal{A} \vDash \varphi[e_2].$$

## Semantics

### Definition 2.25

*A formula $\varphi$ is called a sentence if $FV(\varphi) = \emptyset$, that is $\varphi$ does not have free variables.*
*Notation: $Sent_{\mathcal{L}} :=$ the set of sentences of $\mathcal{L}$.*

### Proposition 2.26

*Let $\varphi$ be a sentence. For all $\mathcal{A}$-assignments $e_1, e_2$,*

$$\mathcal{A} \vDash \varphi[e_1] \iff \mathcal{A} \vDash \varphi[e_2]$$

### Definition 2.27

*Let $\varphi$ be a sentence. An $\mathcal{L}$-structure $\mathcal{A}$ is a model of $\varphi$ if $\mathcal{A} \vDash \varphi[e]$ for an (any) $\mathcal{A}$-assignment $e$. Notation: $\mathcal{A} \vDash \varphi$*

## Semantics

Let $\varphi$ be a sentence and $\Gamma$ be a set of sentences of $\mathcal{L}$.

*Definition 2.28*

*An $\mathcal{L}$-structure $\mathcal{A}$ is a model of $\Gamma$ if*

$$\mathcal{A} \vDash \gamma \text{ for all } \gamma \in \Gamma.$$

*Notation:* $\mathcal{A} \vDash \Gamma$

*Definition 2.29*

*$\Gamma$ is satisfiable if $\Gamma$ has a model.*

*Definition 2.30*

*$\varphi$ is a logical consequence of $\Gamma$ if for all $\mathcal{L}$-structures $\mathcal{A}$,*

$$\mathcal{A} \vDash \Gamma \implies \mathcal{A} \vDash \varphi.$$

*Notation:* $\Gamma \vDash \varphi$

## Examples

For all $n \geq 2$, we denote by $\exists^{\geq n}$ the following sentence:

$$\exists x_1 \dots \exists x_n (\neg(x_1 = x_2) \wedge \neg(x_1 = x_3) \wedge \dots \wedge \neg(x_{n-1} = x_n)),$$

written in a more compact way as follows:

$$\exists^{\geq n} = \exists x_1 \dots \exists x_n \left( \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j) \right).$$

*Proposition 2.31*

*For any $\mathcal{L}$-structure $\mathcal{A}$ and any $n \geq 2$,*

$$\mathcal{A} \models \exists^{\geq n} \iff A \textit{ has at least } n \textit{ elements.}$$

**Proof:** Easy exercise.

## Examples

### Notations

- For uniformity, let $\exists^{\geq 1} := \exists x(x = x)$.
- Denote $\exists^{\leq n} := \neg\exists^{\geq n+1}$ and $\exists^{=n} := \exists^{\leq n} \wedge \exists^{\geq n}$

### Proposition 2.32

*For any $\mathcal{L}$-structure $\mathcal{A}$ and any $n \geq 1$,*

$$\begin{aligned} \mathcal{A} \models \exists^{\leq n} &\iff A \text{ has at most } n \text{ elements} \\ \mathcal{A} \models \exists^{=n} &\iff A \text{ has exactly } n \text{ elements.} \end{aligned}$$

**Proof:** Easy exercise.

### Proposition 2.33

*Let $\Gamma := \{\exists^{\geq n} \mid n \geq 1\}$. Then for any $\mathcal{L}$-structure $\mathcal{A}$,*

$$\mathcal{A} \models \Gamma \iff A \text{ is an infinite set.}$$

**Proof:** Easy exercise.

## Substitution

Let $x$ be a variable of $\mathcal{L}$ and $u$ be a term of $\mathcal{L}$.

*Definition 2.34*

*For any term $t$ of $\mathcal{L}$, we define*

$t_x(u)$ := *the expression obtained from $t$ by replacing all occurences of $x$ with $u$.*

*Proposition 2.35*

*For any term $t$ of $\mathcal{L}$, $t_x(u)$ is a term of $\mathcal{L}$.*

## *Substitution*

- We would like to define, similarly, $\varphi_x(u)$ as the expression obtained from $\varphi$ by replacing all free occurences of $x$ in $\varphi$ with $u$.
- We expect that the following natural properties of substitution are true:

$$\vDash \forall x\varphi \to \varphi_x(u) \quad \text{and} \quad \vDash \varphi_x(u) \to \exists x\varphi.$$

As the following example shows, there are problems with this definition.

Let $\varphi := \exists y\neg(x = y)$ and $u := y$. Then $\varphi_x(u) = \exists y\neg(y = y)$. Avem

- For any $\mathcal{L}$-structure $\mathcal{A}$ with $|A| \geq 2$, $\mathcal{A} \vDash \forall x\varphi$.
- $\varphi_x(u)$ is not satisfiable.

## Substitution

Let $x$ be a variable, $u$ a term and $\varphi$ a formula.

### Definition 2.36

*We say that $x$ is free for $u$ in $\varphi$ or that $u$ is substitutable for $x$ in $\varphi$ if for any variable $y$ that occurs in $u$, no subformula of $\varphi$ of the form $\forall y \psi$ contains free occurences of $x$.*

### Remark

$x$ is free for $u$ in $\varphi$ in any of the following cases:

- $u$ does not contain variables;
- $\varphi$ does not contain variables that occur in $u$;
- no variable from $u$ occurs bound in $\varphi$;
- $x$ does not occur in $\varphi$;
- $\varphi$ does not contain free occurences of $x$.

## Substitution

Let $x$ be a variable, $u$ a term and $\varphi$ be a formula such that $x$ is free for $u$ in $\varphi$.

*Definition 2.37*

$\varphi_x(u) \quad := \quad$ *the expression obtained from $\varphi$ by replacing all free occurences of $x$ in $\varphi$ with $u$.*

*We say that $\varphi_x(u)$ is a free substitution.*

*Proposition 2.38*

*$\varphi_x(u)$ is a formula of $\mathcal{L}$.*

## Substitution

Free substitution rules out the problems mentioned above, it behaves as expected.

*Proposition 2.39*

*Let $\varphi$ be a formula and $x$ be a variable. For any term $u$ substitutable for $x$ in $\varphi$,*

$$\models \forall x\varphi \to \varphi_x(u) \quad \textit{and} \quad \models \varphi_x(u) \to \exists x\varphi.$$

## Prenex normal form

*Definition 2.40*

*A formula that does not contain quantifiers is called quantifier-free.*

*Definition 2.41*

*A formula $\varphi$ is in prenex normal form if*

$$\varphi = Q_1 x_1 Q_2 x_2 \dots Q_n x_n \, \psi,$$

*where $n \in \mathbb{N}$, $Q_1, \dots, Q_n \in \{\forall, \exists\}$, $x_1, \dots, x_n$ are variables and $\psi$ is a quantifier-free formula. $Q_1 x_1 Q_2 x_2 \dots Q_n x_n$ is the prefix of $\varphi$ and $\psi$ is called the matrix of $\varphi$.*

Any quantifier-free formula is in prenex normal form, as one can take $n = 0$ in the above definition.

## *Prenex normal form*

*Examples of formulas in prenex normal form:*

- universal formulas: $\varphi = \forall x_1 \forall x_2 \ldots \forall x_n \, \psi$, where $\psi$ is quantifier-free
- existential formulas: $\varphi = \exists x_1 \exists x_2 \ldots \exists x_n \, \psi$, where $\psi$ is quantifier-free
- $\forall\exists$-formulas: $\varphi = \forall x_1 \forall x_2 \ldots \forall x_n \exists y_1 \exists y_2 \ldots \exists y_k \psi$, where $\psi$ is quantifier-free
- $\forall\exists\forall$-formulas: $\varphi = \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_k \forall z_1 \ldots \forall z_p \psi$, where $\psi$ is quantifier-free

*Theorem 2.42 (Prenex normal form theorem)*

*For any formula $\varphi$ there exists a formula $\varphi^*$ in prenex normal form such that $\varphi \mathrel{\text{⧦}} \varphi^*$ and $FV(\varphi) = FV(\varphi^*)$.*
*$\varphi^*$ is called a prenex normal form of $\varphi$.*

## Prenex normal form

**Proposition 2.43**
*For all formulas $\varphi$, $\psi$ and any variable $x \notin FV(\varphi)$,*

$$
\begin{aligned}
\forall x(\varphi \wedge \psi) &\;\;⟚\;\; \varphi \wedge \forall x\psi && (15)\\
\forall x(\varphi \vee \psi) &\;\;⟚\;\; \varphi \vee \forall x\psi && (16)\\
\exists x(\varphi \wedge \psi) &\;\;⟚\;\; \varphi \wedge \exists x\psi && (17)\\
\exists x(\varphi \vee \psi) &\;\;⟚\;\; \varphi \vee \exists x\psi && (18)\\
\forall x(\varphi \to \psi) &\;\;⟚\;\; \varphi \to \forall x\psi && (19)\\
\exists x(\varphi \to \psi) &\;\;⟚\;\; \varphi \to \exists x\psi && (20)\\
\forall x(\psi \to \varphi) &\;\;⟚\;\; \exists x\psi \to \varphi && (21)\\
\exists x(\psi \to \varphi) &\;\;⟚\;\; \forall x\psi \to \varphi && (22)
\end{aligned}
$$

**Proposition 2.44**
*For any formula $\varphi$, distinct variables $x$ and $y$ such that $y$ does not occur in $\varphi$,*

$$\exists x\varphi ⟚ \exists y\varphi_x(y) \quad \textit{and} \quad \forall x\varphi ⟚ \forall y\varphi_x(y).$$

## Prenex normal form

Let $\mathcal{L}$ be a first-order language containing

- two unary relation symbols $R, S$ and two binary relation symbols $P, Q$;
- a unary function symbol $f$ and a binary function symbol $g$;
- two constant symbols $c, d$.

*Example*

Find a prenex normal form of the formula

$$\varphi := \exists y(g(y,z) = c) \wedge \neg\exists x(f(x) = d)$$

We have that

$$\begin{aligned} \varphi \quad &\vDash\!\!\!\dashv \quad \exists y\big(g(y,z) = c \wedge \neg\exists x(f(x) = d)\big) \\ &\vDash\!\!\!\dashv \quad \exists y\big(g(y,z) = c \wedge \forall x\neg(f(x) = d)\big) \\ &\vDash\!\!\!\dashv \quad \exists y\forall x\big(g(y,z) = c \wedge \neg(f(x) = d)\big) \end{aligned}$$

Thus, $\varphi^* = \exists y\forall x\big(g(y,z) = c \wedge \neg(f(x) = d)\big)$ is a prenex normal form of $\varphi$.

## Prenex normal form

### Example

Find a prenex normal form of the formula

$$\varphi := \neg\forall y(S(y) \to \exists z R(z)) \wedge \forall x(\forall y P(x,y) \to f(x) = d).$$

$$\begin{aligned}
\varphi \;&\mathrel{\rlap{\models}{\;\dashv}}\; \exists y \neg(S(y) \to \exists z R(z)) \wedge \forall x(\forall y P(x,y) \to f(x) = d) \\
&\mathrel{\rlap{\models}{\;\dashv}}\; \exists y \neg\exists z(S(y) \to R(z)) \wedge \forall x(\forall y P(x,y) \to f(x) = d) \\
&\mathrel{\rlap{\models}{\;\dashv}}\; \exists y \neg\exists z(S(y) \to R(z)) \wedge \forall x \exists y(P(x,y) \to f(x) = d) \\
&\mathrel{\rlap{\models}{\;\dashv}}\; \exists y \forall z \neg(S(y) \to R(z)) \wedge \forall x \exists y(P(x,y) \to f(x) = d) \\
&\mathrel{\rlap{\models}{\;\dashv}}\; \exists y \forall z \big(\neg(S(y) \to R(z)) \wedge \forall x \exists y(P(x,y) \to f(x) = d)\big) \\
&\mathrel{\rlap{\models}{\;\dashv}}\; \exists y \forall z \forall x \big(\neg(S(y) \to R(z)) \wedge \exists y(P(x,y) \to f(x) = d)\big) \\
&\mathrel{\rlap{\models}{\;\dashv}}\; \exists y \forall z \forall x \big(\neg(S(y) \to R(z)) \wedge \exists v(P(x,v) \to f(x) = d)\big) \\
&\mathrel{\rlap{\models}{\;\dashv}}\; \exists y \forall z \forall x \exists v \big(\neg(S(y) \to R(z)) \wedge (P(x,v) \to f(x) = d)\big)
\end{aligned}$$

$\varphi^* = \exists y \forall z \forall x \exists v \big(\neg(S(y) \to R(z)) \wedge (P(x,v) \to f(x) = d)\big)$ is a prenex normal form of $\varphi$.

## Syntax

### Definition 2.45

*The set $LogAx_{\mathcal{L}} \subseteq Form_{\mathcal{L}}$ of logical axioms of $\mathcal{L}$ consists of:*

(i) *the axioms of the propositional logic LP.*

(ii) *formulas of the form*

$$t = t, \quad s = t \to t = s, \quad s = t \wedge t = u \to s = u,$$

*for any terms $s, t, u$.*

(iii) *formulas of the form*

$$t_1 = u_1 \wedge \ldots \wedge t_m = u_m \to ft_1 \ldots t_m = fu_1 \ldots u_m,$$

$$t_1 = u_1 \wedge \ldots \wedge t_m = u_m \to (Rt_1 \ldots t_m \leftrightarrow Ru_1 \ldots u_m),$$

*for any $m \geq 1$, $f \in \mathcal{F}_m$, $R \in \mathcal{R}_m$ and any terms $s_i, t_i$ $(i = 1, \ldots, m)$.*

(iv) *formulas of the form*

$$\varphi_x(t) \to \exists x \varphi,$$

*where $\varphi_x(t)$ is a free substitution ($\exists$-axioms).*

## Syntax

*Definition 2.46*

*The deduction rules (or inference rules) are the following: for any formulas $\varphi$, $\psi$,*

*(i)* *from $\varphi$ and $\varphi \to \psi$ infer $\psi$ (modus ponens or (MP)):*

$$\frac{\varphi,\ \varphi \to \psi}{\psi}$$

*(ii)* *if $x \notin FV(\psi)$, then from $\varphi \to \psi$ infer $\exists x\varphi \to \psi$ ($\exists$-introduction):*

$$\frac{\varphi \to \psi}{\exists x\varphi \to \psi} \quad \text{if } x \notin FV(\psi).$$

## *Syntax*

Let $\Gamma$ be a set of formulas of $\mathcal{L}$.

*Definition 2.47*

*The* *$\Gamma$-theorems* *of $\mathcal{L}$ are the formulas defined as follows:*

- *($\Gamma$0)* *Every logical axiom is a $\Gamma$-theorem.*
- *($\Gamma$1)* *Every formula of $\Gamma$ is a $\Gamma$-theorem.*
- *($\Gamma$2)* *If $\varphi$ and $\varphi \to \psi$ are $\Gamma$-theorems, then $\psi$ is a $\Gamma$-theorem.*
- *($\Gamma$3)* *If $\varphi \to \psi$ is a $\Gamma$-theorem and $x \notin FV(\psi)$, then $\exists x\varphi \to \psi$ is a $\Gamma$-theorem.*
- *($\Gamma$4)* *Only the formulas obtained by applying rules ($\Gamma$0), ($\Gamma$1), ($\Gamma$2) and ($\Gamma$3) are $\Gamma$-theorems.*

If $\varphi$ is a $\Gamma$-theorem, then we also say that $\varphi$ is deduced from the hypotheses $\Gamma$.

## Syntax

### Notations

| | | |
|---|---|---|
| $\Gamma \vdash_{\mathcal{L}} \varphi$ | $:=$ | $\varphi$ is a $\Gamma$-theorem |
| $\vdash_{\mathcal{L}} \varphi$ | $:=$ | $\emptyset \vdash_{\mathcal{L}} \varphi$ |

### Definition 2.48

*A formula $\varphi$ is called a (logical) theorem of $\mathcal{L}$ if $\vdash_{\mathcal{L}} \varphi$.*

### Convention

When $\mathcal{L}$ is clear from the context, we write $\Gamma \vdash \varphi$, $\vdash \varphi$, etc..

## Syntax

*Definition 2.49*

*A $\Gamma$-proof (or proof from the hypotheses $\Gamma$) of $\mathcal{L}$ is a sequence of formulas $\theta_1$, ..., $\theta_n$ such that for all $i \in \{1, \dots, n\}$, one of the following holds:*

(i) *$\theta_i$ is an axiom;*

(ii) *$\theta_i \in \Gamma$;*

(iii) *there exist $k, j < i$ such that $\theta_k = \theta_j \to \theta_i$;*

(iv) *there exists $j < i$ such that*

$$\theta_j = \varphi \to \psi \text{ and } \theta_i = \exists x \varphi \to \psi,$$

*where $\varphi, \psi$ are formulas and $x \notin FV(\psi)$.*

*A $\emptyset$-proof is called simply a proof.*

## Syntax

*Definition 2.50*

*Let $\varphi$ be a formula. A $\Gamma$-proof of $\varphi$ or a proof of $\varphi$ from the hypotheses $\Gamma$ is a $\Gamma$-proof $\theta_1, \ldots, \theta_n$ such that $\theta_n = \varphi$.*

*Proposition 2.51*

*Let $\Gamma$ be a set of formulas. For any formula $\varphi$,*

$$\Gamma \vdash \varphi \quad \textit{iff} \quad \textit{there exists a } \Gamma\textit{-proof of } \varphi.$$

## Syntax

### Proposition 2.52

*For any formula $\varphi$ and variable $x$,*

$$\Gamma \vdash \varphi \quad \Longleftrightarrow \quad \Gamma \vdash \forall x \varphi.$$

### Definition 2.53

*Let $\varphi$ be a formula with $FV(\varphi) = \{x_1, \ldots, x_n\}$. The universal closure of $\varphi$ is the sentence*

$$\overline{\forall \varphi} := \forall x_1 \ldots \forall x_n \varphi.$$

### Notation 2.54

$\overline{\forall \Gamma} := \{\overline{\forall \psi} \mid \psi \in \Gamma\}$.

### Proposition 2.55

*For any formula $\varphi$,*

$$\Gamma \vdash \varphi \quad \Longleftrightarrow \quad \Gamma \vdash \overline{\forall \varphi} \quad \Longleftrightarrow \quad \overline{\forall \Gamma} \vdash \varphi \quad \Longleftrightarrow \quad \overline{\forall \Gamma} \vdash \overline{\forall \varphi}.$$

## Completeness Theorem

*Theorem 2.56 (Completeness Theorem)*

*For any set of sentences* $\Gamma$ *and any sentence* $\varphi$,

$$\Gamma \vdash \varphi \quad \Longleftrightarrow \quad \Gamma \models \varphi.$$

- The Completeness Theorem was proved by Gödel in 1929 in his PhD thesis.
- Henkin gave in 1949 a simplified proof.

## Intelligent Agents

Textbook:

Michael Wooldridge, An Introduction to MultiAgent Systems, Second Edition, John Wiley & Sons, 2009

We also use

Lecture slides/handouts, made available by Michael Wooldridge here

## Agents

- The question What is an agent? does not have a definitive answer.
- Many competing, mutually inconsistent answers have been offered in the past.

### Definition

An agent is a (computer) system that is situated in some environment, and that is capable of autonomous action in the environment in order to meet its delegated objectives.

The agent figures out what needs to be done to satisfy design objectives, rather than constantly being told by its user or owner.

## Agents

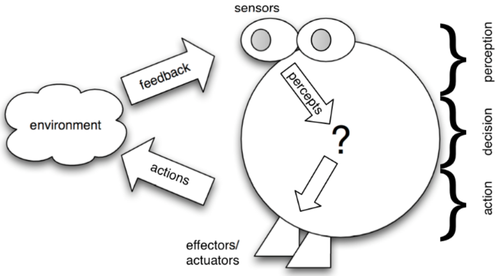


*Figure 1:* An agent in its environment

- Figure 1 gives an abstract view of an agent in its environment
- The agent takes sensory input from the environment, and produces, as output, actions that affect it. The interaction is usually an ongoing, non-terminating one.

- Usually, an agent will not have complete control over its environment.
- It will have at best partial control, in that it can influence it.
- From the point of view of the agent, this means that the same action performed twice in apparently identical circumstances might appear to have entirely different effects, and in particular, it may fail to have the desired effect.
- Thus agents in all but the most trivial of environments must be prepared for the possibility of failure.

## *Intelligent agents*

What is an intelligent agent? is another difficult question. There are some capabilities that we expect an intelligent agent to have.

### *Reactivity*

- Maintain an ongoing interaction with the environment.
- Respond to changes that occur in it (in time for the response to be useful).
- Most environments are dynamic.

### *Proactiveness*

- Goal directed behaviour.
- Generate and attempt to achieve goals.
- Take the initiative.
- Recognise opportunities.

## *Intelligent agents*

### *Social ability*

- The ability to interact with other agents (and possibly humans) via cooperation, coordination, and negotiation.
- Cooperation is working together as a team to achieve a shared goal. It gives a better result.
- Coordination is managing the interdependencies between activities. For example, if there is a non-sharable resource that you want to use and I want to use, then we need to coordinate.
- Negotiation is the ability to reach agreements on matters of common interest. Typically involves offer and counter-offer, with compromises made by participants.

## *Abstract architectures for intelligent agents*

Make formal the abstract view of agents.

- Assume the environment may be in any of a finite set $E$ of discrete, instantaneous states:

$$E = \{e', e", \ldots\}$$

- An agent is assumed to have a repertoire of possible actions available:

$$Ac = \{\alpha', \alpha", \ldots\}$$

- Actions transform the state of the environment.
- We assume that the set $Ac$ of actions contains a special action *null*, with the meaning that nothing will be done.
- States are denoted also by $e_0$, $e_1$, ....
- Actions are denotes also by $\alpha_0$, $\alpha_1$, ....

## *Abstract architectures for intelligent agents*

The basic model of agents interacting with their environments is as follows:

- The environment starts in some state.
- The agent begins by choosing an action to perform on that state.
- As a result of this action, the environment can respond with a number of possible states. However, only one state will actually result — though, of course, the agent does not know in advance which it will be.
- On the basis of this second state, the agent again chooses an action to perform.
- The environment responds with one of a set of possible states.
- The agent then chooses another action; and so on.

## Abstract architectures for intelligent agents

A run over $E$ and $Ac$ is a finite sequence of interleaved environment states and actions.

*Definition 3.1*

*A run $r$ over $E$ and $Ac$ is a finite sequence*

$$r = (x_0, x_1, x_2, \ldots, x_n),$$

*where $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, $x_{2k} \in E$ and $x_{2k+1} \in Ac$.*

Runs are denoted by $r, r', \ldots$. We write a run $r$ as follows:

$$r : \quad e_0 \xrightarrow{\alpha_0} e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{u-2}} e_{u-1} \xrightarrow{\alpha_{u-1}} e_u$$

or

$$r : \quad e_0 \xrightarrow{\alpha_0} e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{u-2}} e_{u-1} \xrightarrow{\alpha_{u-1}}$$

## Abstract architectures for intelligent agents

### Notation 3.2

- $\mathcal{R}$ denotes the set of all runs (over $E$ and $Ac$).
- $\mathcal{R}^{Ac}$ is the subset of these that end with an action.
- $\mathcal{R}^{E}$ is the subset of these that end with an environment state.

## *Abstract architectures for intelligent agents*

*Definition 3.3*

*A function $\tau : \mathcal{R}^{Ac} \to 2^E$ is said to be a state transformer function.*

- A state transformer function maps a run $r \in \mathcal{R}^{Ac}$ to a set $\tau(r)$ of possible environment states that could result from performing the action.
- State transformer functions represent the effect that an agent's actions have on an environment.

If $\tau(r) = \emptyset$, then there are no possible successor states to $r$. In this case, we say that the run $r$ has ended or that $r$ is a terminated run.

Recall: For any set $A$, $2^A$ is the set of all subsets of $A$:

$$2^A = \{B \mid B \subseteq A\}.$$

## *Abstract architectures for intelligent agents*

*Definition 3.4*

*An environment is a triple $Env = (E, e_0, \tau)$, where $E$ is the set of environment states, $e_0 \in E$ is an initial state, and $\tau$ is a state transformer function.*

Environments are:

- history dependent. The next state of an environment is not solely determined by the action performed by the agent and the current state of the environment. The actions made earlier by the agent also play a part in determining the current state.
- non-deterministic. There is uncertainty about the result of performing an action in some state.

## Abstract architectures for intelligent agents

We introduce a model of the agents that inhabit systems.

*Definition 3.5*

*An agent is a function $Ag : \mathcal{R}^E \rightarrow Ac$ mapping runs (assumed to end with an environment state) to actions.*

- An agent makes a decision about what action to perform based on the history of the system.
- Agents are deterministic.

*Definition 3.6*

*A system is a pair $(Ag, Env)$ containing an agent $Ag$ and an environment $Env = (E, e_0, \tau)$.*

## Abstract architectures for intelligent agents

### Definition 3.7

*A run in the system $(Ag, Env)$ is a run $r = (x_0, x_1, x_2, \ldots, x_n)$ over $E$ and $Ac$ satisfying the following:*

- $x_0 = e_0$.
- *for all* $k \geq 0$:

$$x_{2k+1} = Ag(x_0, \ldots, x_{2k}) \text{ and } x_{2k+2} \in \tau(x_0, \ldots, x_{2k+1}).$$

- *$r$ is terminated in the following sense:*
  - *if* $x_n \in E$, *then* $Ag(r) = null$;
  - *if* $x_n \in Ac$, *then* $\tau(r) = \emptyset$.

*We also say that $r$ is a run of the agent $Ag$ in the environment $Env$.*

### Notation 3.8

*We denote by $\mathcal{R}(Ag, Env)$ the set of all runs in the system $(Ag, Env)$.*

## *Abstract architectures for intelligent agents*

Let $r \in \mathcal{R}^{Ac}$,

$$r = e_0 \xrightarrow{\alpha_0} e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{u-2}} e_{u-1} \xrightarrow{\alpha_{u-1}}$$

Then $r \in \mathcal{R}(Ag, Env)$ iff the following are satisfied:

- $\alpha_0 = Ag(e_0)$.
- For all $j = 1, \dots, u-1$,

$$e_j \in \tau(e_0 \xrightarrow{\alpha_0} e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{j-2}} e_{j-1} \xrightarrow{\alpha_{j-1}})$$
$$\alpha_j = Ag(e_0 \xrightarrow{\alpha_0} e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{j-1}} e_j).$$

- $\tau(r) = \emptyset$.

## *Abstract architectures for intelligent agents*

Let $r \in \mathcal{R}^E$,

$$r = e_0 \xrightarrow{\alpha_0} e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{u-2}} e_{u-1} \xrightarrow{\alpha_{u-1}} e_u$$

Then $r \in \mathcal{R}(Ag, Env)$ iff the following are satisfied:

- $\alpha_0 = Ag(e_0)$.
- For all $j = 1, \dots, u$,

$$e_j \in \tau(e_0 \xrightarrow{\alpha_0} e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{j-2}} e_{j-1} \xrightarrow{\alpha_{j-1}})$$
$$\alpha_{j-1} = Ag(e_0 \xrightarrow{\alpha_0} e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{j-2}} e_{j-1}).$$

- $Ag(r) = null$.

## *Abstract architectures for intelligent agents*

*Definition 3.9*

*Two agents $Ag_1$ and $Ag_2$ are said to be*

- *behaviourally equivalent with respect to environment Env if and only if $\mathcal{R}(Ag_1, Env) = \mathcal{R}(Ag_2, Env)$.*
- *behaviourally equivalent if they are behaviourally equivalent with respect to all environments.*

## *Purely reactive agents*

- Certain types of agents decide what to do without reference to their history. They base their decision-making entirely on the present, with no reference at all to the past.
- We call such agents purely reactive, since they simply respond directly to their environment.

*Definition 3.10*

*A purely reactive agent is a mapping* $Ag_{pure} : E \to Ac$.

## Example: Thermostat

A thermostat is a very simple example of a purely reactive agent.

- A thermostat has a sensor for detecting room temperature, and it produces as output one of two signals:
  - one that indicates that the temperature is too low;
  - another one which indicates that the temperature is OK.
- Its delegated goal is to maintain room temperature, the available actions being 'heating on' and 'heating off'.
- The decision-making component of the thermostat implements the following rules:

$$\begin{array}{lcl} \text{temperature is too low} & \rightarrow & \text{heating on,} \\ \text{temperature is OK} & \rightarrow & \text{heating off.} \end{array}$$

## *Example: Thermostat*

- Let us denote

  $e_1 :=$ temperature too low, $\quad e_2 :=$ temperature OK,

  $\alpha_1 :=$ heating on, $\quad \alpha_2 :=$ heating off.

- Then $E := \{e_1, e_2\}$ and $Ac := \{\alpha_1, \alpha_2\}$.

- The thermostat is the purely reactive agent *Therm* defined as follows:

$$Therm : E \to Ac, \quad Therm(e) = \begin{cases} \alpha_1 & \text{if } e = e_1, \\ \alpha_2 & \text{if } e = e_2. \end{cases}$$

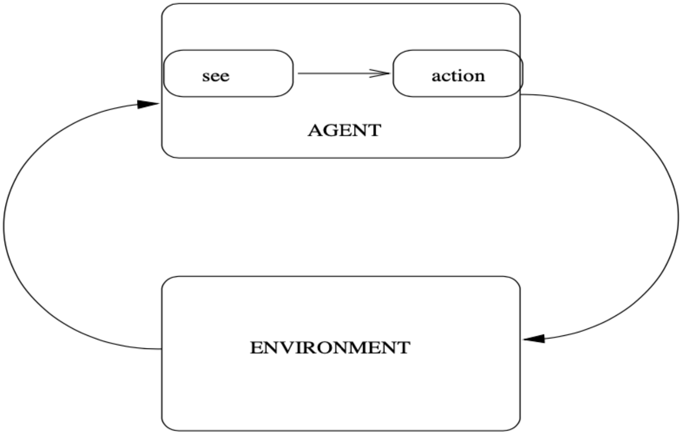
## *Intelligent agents*

This view of agents is too abstract. It does not help us to construct them, since it gives us no clues about how to design the decision function action.

- We refine our abstract model of agents, by breaking it down into subsystems.
- We make design choices on these subsystems — what data and control structures will be present.
- An agent architecture is essentially a map of the internals of an agent — its data structures, the operations that may be performed on these data structures, and the control flow between these data structures.
- There are different types of agent architectures, with very different views on the data structures and algorithms that will be present within an agent.

## Perception

One high-level design decision is the separation of an agent's decision function into perception and action subsystems.



## *Perception*

- The perception function see captures the agent's ability to observe its environment. Example: a video camera or an infra-red sensor on a mobile robot.
- The output of the see function is a percept — a perceptual input.
- The action function represents the agent's decision making process.

Let Per be a nonempty set of percepts.

### *Definition 3.11*

*The see and action functions are defined as follows:*

$$see : E \to Per \qquad and \qquad action : Per^* \to Ac.$$

Recall: For any set $A$, $A^*$ is the set of all finite sequences of elements of $A$:

$$A^* = \{a_1 a_2 \ldots a_n \mid n \in \mathbb{N} \text{ and } a_i \in A \text{ for all } i = 1, \ldots, n\}.$$

## Perception

These simple definitions allow us to explore some interesting properties of agents and perception.

Suppose that we have two environment states $e_1, e_2 \in E$ such that $e_1 \neq e_2$, but $see(e_1) = see(e_2)$. Then $e_1$ and $e_2$ are mapped to the same percept, and the agent receives the same perceptual information from each of them. As far as the agent is concerned, $e_1$ and $e_2$ are indistinguishable.

*Definition 3.12*

*The relation $\equiv$ on $E$ is defined as follows: for every $e_1, e_2 \in E$,*

$$e_1 \equiv e_2 \quad \textit{iff} \quad see(e_1) = see(e_2).$$

*Remark 3.13*

*$\equiv$ is an equivalence relation on $E$.*

## Perception

- $\equiv$ partitions $E$ into mutually indistinguishable sets of states, namely the different equivalence classes $[e]$, were $e \in E$.

- If $[e] = \{e\}$ for every $e \in E$, then $see(e_1) \neq see(e_2)$ for every states $e_1 \neq e_2$. The agent can distinguish every state — the agent has perfect perception in the environment.
- If $[e] = E$ for every $e \in E$, then $see(e_1) = see(e_2)$ for every states $e_1, e_2$. The agent's perceptual ability is non-existent, it cannot distinguish between any different states. As far as the agent is concerned, all environment states are identical.

## Example

- Let us consider the statements

$$x := \text{“the room temperature is OK”}$$
$$y := \text{“Angela Merkel is the German Chancellor”}$$

- Assume that these are the only two facts about our environment that we are concerned with. Then

$$E = \{e_1 := \{x, y\}, e_2 := \{x, \neg y\}, e_3 := \{\neg x, y\}, e_4 := \{\neg x, \neg y\}\}$$

- In state $e_1$ the room temperature is OK and Angela Merkel is the German Chancellor; in state $e_2$ the room temperature is OK and Angela Merkel is not the German Chancellor; and so on.
- The thermostat is sensitive only to temperatures in the room.
- The room temperature is not casually related to whether or not Angela Merkel is the German Chancellor.
- The states where Angela Merkel is and is not the German Chancellor are indistinguishable to the thermostat.

## *Perception - an example*

The set of percepts is $Per = \{p_1, p_2\}$, where

$$p_1 \mathrel{:=} \text{temperature OK}, \qquad p_2 \mathrel{:=} \text{temperature too low}.$$

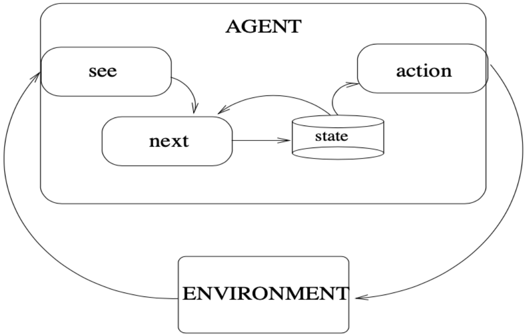
The perception function *see* is defined as follows:

$$see : E \to Per, \quad see(e) = \begin{cases} p_1 & \text{if } e = e_1 \text{ or } e = e_2, \\ p_2 & \text{if } e = e_3 \text{ or } e = e_4. \end{cases}$$

- $[e_1] = [e_2] = \{e_1, e_2\}$
- $[e_3] = [e_4] = \{e_3, e_4\}$

## Agents with state

We now consider agents that maintain state.



These agents have some internal data structure, which is typically used to record information about the environment state and history.

## Agents with state

Let $I$ be the set of all internal states of the agent.

*Definition 3.14*

*The see and action functions are defined as follows:*

$$see : E \to Per \qquad and \qquad action : I \to Ac.$$

The perception function *see* is unchanged. The action-selection function *action* takes as inputs internal states.

*Definition 3.15*

*The function next is defined as follows:*

$$next : I \times Per \to I.$$

## *Agents with state*

*The behaviour of a state-based agent:*

- The agent starts in some initial internal state $i_0$.
- It then observes its environment state $e$, and generates a percept $see(e)$.
- The internal state of the agent is then updated to $i_1 := next(i_0, see(e))$.
- The action selected by the agent is $\alpha := action(i_1)$.
- The agents performs action $\alpha$.
- The agent enters another cycle: perceives the world via *see*, updates its state via *next*, and chooses an action to perform via *action*.

## *Tasks for agents*

- We build agents in order to carry out tasks for us.
- The tasks to be carried out must be specified by us in some way
- How to specify these tasks? How to tell the agent what to do?

One way to to do this: write a program for the agent to execute.

- Advantage: no uncertainty about what the agent will do; it will do exactly what we told it to, and no more.
- Disadvantage: we have to think about exactly how the task will be carried out ourselves; if unforeseen circumstances arise, the agent executing the task will be unable to respond accordingly.

## *Tasks for agents*

- We want to tell our agent what to do without telling it how to do it.
- One way of doing this is to define tasks indirectly, via some kind of performance measure.
- One possibility: associate utilities with states of the environment.

- A utility is a numeric value representing how ‘good’ a state is: the higher the utility, the better.
- The task of the agent is then to bring about states that maximize utility.
- We do not specify to the agent how this is to be done.

## *Utility functions*

*Definition 3.16*

*A utility function (or task specification) is a function $u : E \to \mathbb{R}$.*

What is the overall utility of a run?

- minimum utility of a state on run?
- maximum utility of a state on run?
- sum of utilities of all states on run?
- average utility of all states on run?

Main disadvantage:

- assigns utilities to local states.
- difficult to specify a long-term view when assigning utilities to individual states.

## Utility functions

Idea: assign a utility not to individual states, but to runs.

*Definition 3.17*

*A utility function (or task specification) is a function $u : \mathcal{R} \to \mathbb{R}$.*

- If we are concerned with agents that must operate independently over long periods of time, then this approach is appropriate.

- The utility-based approach works well in certain scenarios.
- Problems:
  - Sometimes it is difficult to define a utility function.
  - People don't think in terms of utilities. It is hard for people to specify tasks in these terms.

## Tileworld

Tileworld was proposed as an experimental environment for evaluating agent architectures in
*Martha E. Pollock, Marc Ringuette, Introducing the Tileworld: Experimentally Evaluating Agent Architectures, AAAI-90 Proceedings, 1990*

- Simulated two dimensional grid environment on which there are agents, tiles, obstacles, and holes.
- An agent can move in four directions, up, down, left, or right, and if it is located next to a tile, it can push it.
- An obstacle is a group of immovable grid cells.
- Holes have to be filled up with tiles by the agent.
- An agent scores points by filling holes with tiles, the aim being to fill as many holes as possible.

## Tileworld

- Holes appear randomly and exist for as long as their life expectancy, unless they disappear because of the agent's actions. The interval between the appearance of successive holes is called the hole gestation time.
- Tileworld is an example of a dynamic environment: starting in some randomly generated world state, based on parameters set by the experimenter, it changes over time in discrete steps, with the random appearance and disappearance of holes.
- The performance of an agent in the Tileworld is measured by running the Tileworld testbed for a predetermined number of time steps, and measuring the number of holes that the agent succeeds in filling.
- Experimental error is eliminated by running the agent in the environment a number of times, and computing the average of the performance.

## Tileworld

*Definition 3.18*

*The utility function is defined as follows:*

$$u : \mathcal{R} \to \mathbb{R}, \qquad u(r) = \frac{\textit{number of holes filled in } r}{\textit{number of holes that appeared in } r}$$

- $u$ is normalized: $u(r) \in [0, 1]$ for every run $r$
- $u(r) = 1$: agent filled every hole that appeared in $r$
- $u(r) = 0$: agent did not fill any hole that appeared in $r$

## *Tileworld*

- Despite its simplicity, Tileworld allows us to examine several important capabilities of agents.
- Examples of abilities of agents:
  - to react to changes in the environment
  - to exploit opportunities when they arise.

## Tileworld

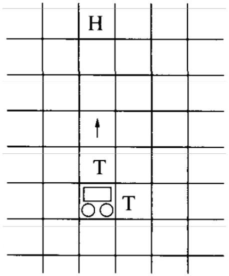


*Figure 2:* Tileworld example

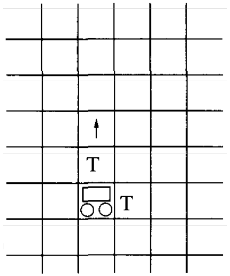


*Figure 3:* Tileworld example

### *Example 3.19*

Suppose an agent is pushing a tile to a hole (Figure 2), when this hole disappears (Figure 3).
The agent should recognize this change in the environment, and modify its behaviour appropriately.

## Tileworld

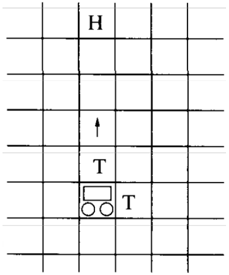


*Figure 4:* Tileworld example

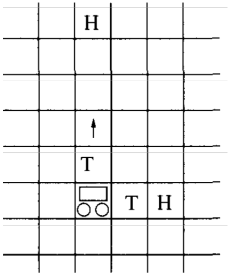


*Figure 5:* Tileworld example

### *Example 3.20*

Suppose an agent is pushing a tile to a hole (Figure 4), when a hole appears to the right of the agent (Figure 5).
It would do better to push the tile to the right, than to continue to head north, for the simple reason that it only has to push the tile one step, rather than three.

## *Expected utility*

Let us denote $P(r \mid Ag, Env)$ the probability that run $r$ occurs when agent $Ag$ is placed in environment $Env$.
Obviously, $\sum_{r \in \mathcal{R}(Ag,Env)} P(r \mid Ag, Env) = 1$.

*Definition 3.22*

*The expected utility of agent Ag in environment Env (given P, u) is defined as follows:*

$$EU(Ag, Env) = \sum_{r \in \mathcal{R}(Ag,Env)} u(r)P(r \mid Ag, Env).$$

## *Expected utility - an example*

Consider the environment $(E, e_0, \tau)$ defined as follows:

- $E = \{e_0, e_1, e_2, e_3, e_4, e_5\}$ and $Ac = \{\alpha_0, \alpha_1, \mathit{null}\}$.
- Define

$$\tau : \mathcal{R}^{Ac} \to 2^E, \qquad \tau(r) = \begin{cases} \{e_1, e_2\} & \text{if } r = e_0 \xrightarrow{\alpha_0}, \\ \{e_3, e_4, e_5\} & \text{if } r = e_0 \xrightarrow{\alpha_1}, \\ \emptyset & \text{otherwise.} \end{cases}$$

We consider the following two agents in this environment:

$$Ag_1 : \mathcal{R}^E \to Ac, \quad Ag_1(r) = \begin{cases} \alpha_0 & \text{if } r = e_0 \\ \mathit{null} & \text{otherwise} \end{cases}$$

$$Ag_2 : \mathcal{R}^E \to Ac, \quad Ag_2(r) = \begin{cases} \alpha_1 & \text{if } r = e_0 \\ \mathit{null} & \text{otherwise} \end{cases}$$

## *Expected utility - an example*

*Runs of $Ag_1$ and $Ag_2$ in the environment $Env$*

$$\mathcal{R}(Ag_1, Env) = \{r_1 := e_0 \xrightarrow{\alpha_0} e_1, r_2 := e_0 \xrightarrow{\alpha_0} e_2\},$$
$$\mathcal{R}(Ag_2, Env) = \{r_3 := e_0 \xrightarrow{\alpha_1} e_3, r_4 := e_0 \xrightarrow{\alpha_1} e_4, r_5 := e_0 \xrightarrow{\alpha_1} e_5\}.$$

The probabilities of the various runs are defined as follows:

- for the agent $Ag_1$:

$$P(r_1 \mid Ag_1, Env) = 0.4, \quad P(r_2 \mid Ag_1, Env) = 0.6.$$

- for the agent $Ag_2$:

$$P(r_3 \mid Ag_2, Env) = 0.1, \quad P(r_4 \mid Ag_2, Env) = 0.2,$$
$$P(r_5 \mid Ag_2, Env) = 0.7.$$

## Expected utility - an example

Assume the utility function $u$ is defined as follows:

$$u(r_1) = 8, \quad u(r_2) = 11,$$
$$u(r_3) = 70, \quad u(r_4) = 9, \quad u(r_5) = 10.$$

What are the expected utilities of the agents for this utility function?

$$\begin{aligned}
EU(Ag_1, Env) &= u(r_1)P(r_1 \mid Ag_1, Env) + u(r_2)P(r_2 \mid Ag_1, Env) \\
&= 8 \times 0.4 + 11 \times 0.6 = 9.6 \\
EU(Ag_2, Env) &= u(r_3)P(r_3 \mid Ag_2, Env) + u(r_4)P(r_4 \mid Ag_2, Env) \\
&\quad + u(r_5)P(r_5 \mid Ag_2, Env) \\
&= 70 \times 0.1 + 9 \times 0.2 + 10 \times 0.7 = 15.7
\end{aligned}$$

## Optimal agents

### Notation 3.23

*Let AG denote the finite set of all agents acting in some environment.*

### Definition 3.24

*An optimal agent in an environment Env is an agent $Ag_{opt}$ that maximizes the expected utility:*

$$Ag_{opt} = arg \max_{Ag \in AG} EU(Ag, Env).$$

- The fact that an agent is optimal does not mean that it will be best; only that on average, we can expect it to do best.
- The definition does not not give us any clues about how to implement this agent.
- There are agents that cannot be implemented on a real computer.

## *Bounded optimal agents*

Suppose $m$ is a particular computer.

*Notation 3.25*

*$AG_m$ denotes the set of agents that can be implemented on m:*

$$AG_m = \{Ag \mid Ag \in AG \text{ and } Ag \text{ can be implemented on } m\}.$$

*Definition 3.26*

*A bounded optimal agent in an environment Env, with respect to m, is an agent $Ag_{bopt} \in AG_m$ that maximizes the expected utility:*

$$Ag_{bopt} = arg \max_{Ag \in AG_m} EU(Ag, Env).$$

- We consider only the agents that can actually be implemented on the machine that we have for the task.

## Predicate task specifications

- A predicate task specification is one where the utility function acts as a predicate over runs.
- A utility function $u : \mathcal{R} \to \mathbb{R}$ is a predicate if the range of $u$ is the set $\{0, 1\}$, that is if $u$ assigns a run either 1 (true) or 0 (false).
- If $u(r) = 1$, we say that the run $r$ satisfies the specification; the agent succeeds on the run $r$.
- If $u(r) = 0$, we say that the run $r$ fails to satisfy the specification; the agent fails on the run $r$.

*Definition 3.27*

*A predicate task specification is a mapping* $\Psi : \mathcal{R} \to \{0, 1\}$.

## *Task environment*

*Definition 3.28*

*A task environment is a pair $(Env, \Psi)$, where Env is an environment, and $\Psi$ is a predicate task specification.*

*Notation 3.29*

*TE denotes the set of all task environments.*

A task environment specifies:

- the properties of the system the agent will inhabit (i.e. the environment $Env$);
- the criteria by which an agent will be judged to have either failed or succeeded (i.e. the specification $\Psi$).

## Task environment

*Notation 3.30*

$\mathcal{R}_\Psi(Ag, Env)$ *denotes the set of all runs of agent Ag that satisfy* $\Psi$*:*

$$\mathcal{R}_\Psi(Ag, Env) = \{r \mid r \in \mathcal{R}(Ag, Env) \text{ and } \Psi(r) = 1\}.$$

There are more possibilities to define the success of an agent in a task environment.

*The pessimistic definition:*

We say that an agent $Ag$ succeeds in task environment $(Env, \Psi)$ if $\mathcal{R}_\Psi(Ag, Env) = \mathcal{R}(Ag, Env)$.

Thus, the agent succeeds iff every possible run of the agent in the environment satisfies the predicate task specification.

## Task environment

*The optimistic definition:*

We say that an agent $Ag$ succeeds in task environment $(Env, \Psi)$ if $\mathcal{R}_\Psi(Ag, Env) \neq \emptyset$.

Thus, the agent succeeds iff at least one run of the agent in the environment satisfies the predicate task specification.

*The probabilistic definition:*

The success of an agent $Ag$ in task environment $(Env, \Psi)$ is defined as the probability $P(\Psi | Ag, Env)$ that the predicate task specification $\Psi$ is satisfied by the agent in the environment $Env$.

*Remark 3.31*

$P(\Psi | Ag, Env) = \sum_{r \in \mathcal{R}_\Psi(Ag, Env)} P(r | Ag, Env)$.

## *Achievement and maintenance tasks*

- The notion of a predicate task specification may seem rather abstract.
- It is a generalization of certain very common forms of tasks.

Two most common types of tasks are achievement tasks and maintenance tasks:

- Achievement tasks are those of the form achieve state of affairs $\varphi$.
- Maintenance tasks are those of the form maintain state of affairs $\varphi$.

## Achievement tasks

*Definition 3.32*

*The task environment* $(Env, \Psi)$ *specifies an achievement task if there exists some set of states* $G \subseteq E$ *such that for all* $r \in \mathcal{R}(Ag, Env)$,

$\Psi(r) = 1$ *iff there exists some state* $e \in G$ *such that* $e$ *occurs in* $r$.

*We also say that* $(Env, \Psi)$ *is an achievement task environment.*

The elements of $G$ are the goal states of the task.

*Notation 3.33*

*We use* $(Env, G)$ *to denote an achievement task environment with goal states* $G$ *and environment* $Env$.

An agent is successful if is guaranteed to bring about one of the goal states (we do not care which one — all are considered equally good).

## *Achievement tasks*

A useful way to think about achievement tasks is as the agent playing a game against the environment:

- The environment and agent both begin in some state.
- The agent executes an action, and the environment responds with some state.
- The agent then executes another action, and so on.
- The agent wins if it can force the environment into one of the goal states.

## Maintenance tasks

- Many other tasks can be classified as problems where the agent is required to avoid some state of affairs.
- We refer to such tasks as maintenance tasks.

### Definition 3.34

*The task environment $(Env, \Psi)$ specifies a maintenance task if there exists some set of states $B \subseteq E$ such that for all $r \in \mathcal{R}(Ag, Env)$,*

$$\Psi(r) = 1 \textit{ iff for any state } e \in B, \; e \textit{ does not occur in } r.$$

*We also say that $(Env, \Psi)$ is a maintenance task environment.*

The elements of $B$ are the bad states of the task.

### Notation 3.35

*We use $(Env, B)$ to denote a maintenance task environment with bad states $B$ and environment $Env$.*

## *Maintenance tasks*

It is again useful to think of maintenance tasks as games:

- The agent wins if it manages to avoid all the bad states.
- The environment, in the role of opponent, is attempting to force the agent into $B$.
- The agent is successful if it has a winning strategy for avoiding $B$.

## *Achievement and maintenance tasks*

- More complex tasks might be specified by combinations of achievement and maintenance tasks.
- A simple combination:

  achieve any one of states $G$ while avoiding all states $B$.

## Agent synthesis

- Knowing that there exists an agent which will succeed in a given task environment is helpful.
- However, it would be more helpful if, knowing this, we obtain such an agent, we implement it.
- How do we do this?
- An obvious answer is: 'manually' implement the agent from the specification.

There is at least one other possibility:

develop an algorithm that will automatically synthesize such agents for us from task environment specifications.

## Agent synthesis

Agent synthesis is automatic programming: the goal is to have a program that will take as input a task environment, and from this task environment automatically generate an agent that succeeds in this environment.

*Definition 3.36*

*An agent synthesis algorithm is a function*

$$syn : TE \to AG \cup \{\perp\}.$$

A synthesis algorithm is

- sound if, whenever it returns an agent, this agent succeeds in the task environment that is passed as input, and
- complete if it is guaranteed to return an agent whenever there exists an agent that will succeed in the task environment given as input.

## *Agent synthesis*

*Definition 3.37*

*An agent synthesis algorithm syn is*

- *sound if for any task environment* $(Env, \Psi) \in TE$,
  $syn((Env, \Psi)) = Ag$ *implies* $\mathcal{R}(Ag, Env) = \mathcal{R}_\Psi(Ag, Env)$.
- *complete if for any* $(Env, \Psi) \in TE$,
  $\big($*there exists an agent* $Ag$ *s.t.* $\mathcal{R}(Ag, Env) = \mathcal{R}_\Psi(Ag, Env)\big)$
  *implies* $syn((Env, \Psi)) \neq \bot$.

- Soundness ensures that a synthesis algorithm always delivers agents that do their job correctly, but may not always deliver agents, even where such agents are in principle possible.
- Completeness ensures that an agent will always be delivered where such an agent is possible, but does not guarantee that these agents will do their job correctly.

## *Agent synthesis*

- Soundness and completeness ensure that a synthesis algorithm will output $\perp$ iff there is no agent that does the job correctly.
- Ideally, we seek synthesis algorithms that are both sound and complete.
- Of the two conditions, soundness is probably the more important.
- There is not much point in complete synthesis algorithms that deliver ‘buggy’ agents.

## Agent architectures

Textbook:

Michael Wooldridge, An Introduction to MultiAgent Systems, Second Edition, John Wiley & Sons, 2009

## *Agent architectures*

- An agent architecture is a software design for an agent.
- We have already seen a top-level decomposition into:

  perception – state – decision – action
- An agent architecture defines:
  - key data structures;
  - operations on data structures;
  - control flow between operations.

## *Agent architectures*

Some types of agents:

- logic-based or deductive reasoning or symbolic reasoning agents - decision making is realised through logical deduction;
- practical reasoning agents - reasoning directed towards actions, the process of figuring out what to do; action selection through deliberation and means-end reasoning;
- reactive agents - reacting to an environment, without reasoning about it; no representation, direct link from perceptual input to action;
- hybrid agents - combine reactive and deliberative reasoning, usually in layered architecture.

## Logic-based agents

The logic-based approach is the classical approach to building agents.

*Key ideas:*

- give a symbolic representation of the environment - logical formulas.
- manipulate syntactically this representation - logical deduction or theorem proving.

*Problems to be solved:*

- Transduction problem: the problem of translating the real world into an accurate, adequate symbolic description of the world, in time for that description to be useful.
- Representation/reasoning problem: the problem of representing information symbolically, and getting agents to manipulate/reason with it, in time for the results to be useful.

## *Agents as theorem provers*

Deliberate agents are a simple model of logic-based agents.

- An internal state of such an agent is a database of formulas of first-order logic.
- The agent's database might contain formulas such as

  *Open(valve1)*, *Temperature(reactor6, 32)*, *Pressure(tank6, 28)*.
- The database is the information that the agent has about its environment.
- An agent's database plays a somewhat analogous role to that of belief in humans.
- Some facts from the database could be wrong - agent's sensors may be faulty, its reasoning may be faulty, the information may be out of date.
- Thus, the fact that *Open(valve1)* is in the database does not mean that *valve1* is open; it could be closed.

## *Agents as theorem provers*

We use the model of agents with state.
Let $\mathcal{L}$ be a first-order language and $Form_{\mathcal{L}}$ be the set of its formulas.

We assume that $\mathcal{L}$ contains:

- a unary relation symbol $Do$;
- a constant symbol $c_\alpha$ for every action $\alpha \in Ac$. For simplicity, we write $\alpha$ instead of $c_\alpha$.

- A database is a set of formulas of $\mathcal{L}$.
- Let $\mathcal{D}$ be the set of all databases. Thus, $\mathcal{D} = 2^{Form_{\mathcal{L}}}$.
- We write $DB, DB_1, \ldots$ for members of $\mathcal{D}$.
- An internal state of the agent is a database. Thus, $I = \mathcal{D}$.

## Agents as theorem provers

- We fix a set $\Sigma \subseteq \mathit{Form}_{\mathcal{L}}$ of formulas of $\mathcal{L}$, whose elements are called deduction formulas.
- We use the notation $DB \vdash_{\Sigma} \varphi$ for $DB \cup \Sigma \vdash \varphi$.
- The idea is that

  if $DB \vdash_{\Sigma} Do(\alpha)$, then $\alpha$ is the action to be performed by the agent.
- The agent's behaviour is determined by its deduction formulas (its program) and its current database.

An agent's action selection function

$$\mathit{action} : \mathcal{D} \to Ac$$

is defined in terms of its deduction formulas. The pseudo-code definition of this function is given in Figure 6.

## *Agents as theorem provers*

```
1.   function $action(DB : \mathcal{D})$ returns an action $Ac$
2.   begin
3.      for each $\alpha \in Ac$ do
4.         if $DB \vdash_{\Sigma} Do(\alpha)$ then
5.            return $\alpha$
6.         end-if
7.      end-for
8.      for each $\alpha \in Ac$ do
9.         if $DB \nvdash_{\Sigma} \neg Do(\alpha)$ then
10.           return $\alpha$
11.        end-if
12.     end-for
13.     return $null$
14.  end function $action$
```

*Figure 6:* Agent selection as theorem proving.

## *Agents as theorem provers*

- In lines 3-7, the agent takes each of its possible actions $\alpha$ in turn, and attempts to prove the formula *Do*$(\alpha)$ from its database *DB* (passed as a parameter to the function) using its set $\Sigma$ of deduction formulas. If the agent succeeds in proving *Do*$(\alpha)$, then $\alpha$ is returned as the action to be performed.
- If the agent fails to prove *Do*$(\alpha)$, for all actions $\alpha \in$ *Ac*, then it tries to find an action that is consistent with the deduction formulas and the database, that is not explicitly forbidden.
- In lines 8-12, the agent attempts to find an action $\alpha \in$ *Ac* such that $\neg$*Do*$(\alpha)$ cannot be derived from its database using its deduction formulas. If it can find such an action, then this is returned as the action to be performed.
- If, however, the agent fails to find an action that is at least consistent, then it returns the special action *null*, indicating that no action has been selected.

## *Agents as theorem provers*

The perception function *see* remains unchanged:

$$see : E \rightarrow Per,$$

where *Per* is the set of percepts.

The *next* function has the form:

$$next : \mathcal{D} \times Per \rightarrow \mathcal{D}.$$

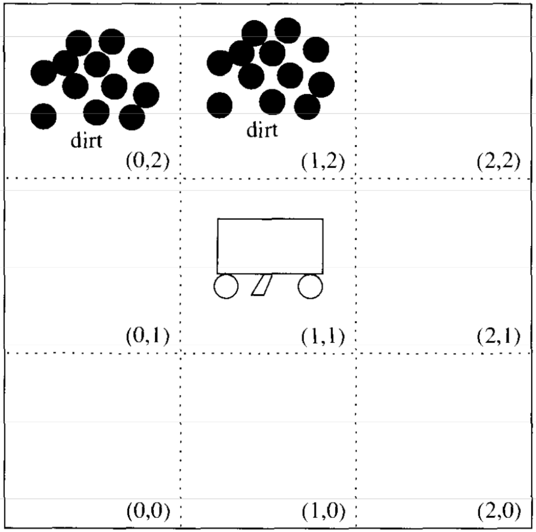
It maps a database and a percept to a new database.

## *Agents as theorem provers - example*

We consider an example: vacuum cleaning world

- We have a small robotic agent that will clean up a house.
- The robot is equipped with a sensor that will tell it whether it is over any dirt, and a vacuum cleaner that can be used to suck up dirt.
- The robot always has a definite orientation (one of north, south, east, or west).
- In addition to being able to suck up dirt, the agent can move forward one 'step' or turn right 90 degrees.
- The agent moves around a room, which is divided grid-like into a number of equally sized squares.
- Our agent does nothing but clean — it never leaves the room.
- Assume, for simplicity, that the room is a $3 \times 3$ grid, and the agent always starts in grid square $(0, 0)$ facing north.

## Agents as theorem provers - example



## Agents as theorem provers - example

- The set *Per* of percepts is defined as
$$Per = \{dirt, nothing\},$$
where *dirt* signifies that there is dirt beneath it, and *nothing* indicates no special information.
- The set *Ac* of actions is defined as
$$Ac = \{forward, suck, turn\},$$
where *forward* means ‘go forward’, *suck* means ‘suck up dirt’, and *turn* means ‘turn right 90 degrees’.
- The goal is to traverse the room continually searching for and removing dirt.

## *Agents as theorem provers - example*

We use three simple domain predicates:

| | |
|---|---|
| $In(i,j)$ | agent is at $(i,j)$, |
| $Dirt(i,j)$ | there is dirt at $(i,j)$, |
| $Facing(d)$ | the agent is facing direction $d$, |

where $i,j \in \{0,1,2\}$ and $d \in \{north, south, east, west\}$.

This means that the first-order language $\mathcal{L}$ contains:

- two binary relation symbols *In* and *Dirt*;
- a unary relation symbol *Facing*;
- constant symbols *north*, *south*, *east*, *west*;
- constant symbols $(i,j)$ for every $i,j \in \{0,1,2\}$.

## Agents as theorem provers - example

- The *next* function looks at the perceptual information obtained from the environment and at the actual database, and generates a new database which includes this information.
- It removes old or irrelevant information, and also, it tries to figure out the new location and orientation of the agent.
- We specify the *next* function in several parts.

Let old(DB) denote the set of 'old' information in a database, which we want the update function *next* to remove.

$$old(DB) = DB \cap \Delta,$$

where

$$\Delta = \{In(i,j) \mid i,j \in \{0,1,2\}\} \cup \{Dirt(i,j) \mid i,j \in \{0,1,2\}\} \cup \{Facing(d) \mid d \in \{north, south, east, west\}\}.$$

## Agents as theorem provers - example

- We require a function new, which gives the set of new formulas to add to the database:

$$new : \mathcal{D} \times Per \to \mathcal{D}.$$

- It must generate formulas
    - $In(\ldots)$, describing the new position of the agent;
    - $Facing(\ldots)$ describing the orientation of the agent;
    - $Dirt(\ldots)$ if dirt has been detected at the new position.

The next function is defined as follows:

$$next : \mathcal{D} \times Per \to \mathcal{D}, \quad next(DB, p) = (DB - old(DB)) \cup new(DB, p)$$

## *Agents as theorem provers - example*

The deduction formulas have the general form

$$\varphi \to \psi, \qquad \text{where } \varphi,\ \psi \text{ are formulas of } \mathcal{L}$$

*Cleaning*

$$\mathit{In}(x, y) \wedge \mathit{Dirt}(x, y) \to \mathit{Do}(\mathit{suck}) \qquad x, y \text{ are variables}$$

- If the agent is at location $(x, y)$ and it perceives dirt, then the prescribed action will be to suck up dirt.
- It takes priority over all other possible behaviours of the agent (such as navigation).

## *Agents as theorem provers - example*

### *Traversal*

- The basic action of the agent is to traverse the world.
- For simplicity, we assume that the robot will always move from $(0,0)$ to $(0,1)$ to $(0,2)$ and then to $(1,2)$, $(1,1)$ and so on. Once the agent reaches $(2,2)$, it must head back to $(0,0)$.
- The deduction formulas dealing with the traversal up to $(0,2)$:

$$\begin{aligned}
In(0,0) \wedge Facing(north) \wedge \neg Dirt(0,0) &\rightarrow Do(forward)\\
In(0,1) \wedge Facing(north) \wedge \neg Dirt(0,1) &\rightarrow Do(forward)\\
In(0,2) \wedge Facing(north) \wedge \neg Dirt(0,2) &\rightarrow Do(turn)\\
In(0,2) \wedge Facing(east) &\rightarrow Do(forward)
\end{aligned}$$

- Similar formulas can be easily generated that will get the agent to $(2,2)$ and back to $(0,0)$.

## Logic-based agents

Decision making is viewed as deduction, an agent's program is encoded as a logical theory, and actions selection reduces to a problem of proof.

*Advantages:*

- elegance and a clean (logical) semantics.

*Disadvantages:*

- inherent computational complexity of theorem proving;
- based on the assumption of calculative rationality:
  - world will not change in any significant way while the agent is deciding what to do;
  - an action which is rational when decision-making begins will be rational when it concludes.
- transduction and representation/reasoning problems essentially unsolved.

# Modal Logics

Textbook:

P. Blackburn, M. de Rijke, Y. Venema, Modal logic, Cambridge Tracts in Theoretical Computer Science 53, Cambridge University Press, 2001

## *Basic modal language*

*Definition 4.1*

*The basic modal language $ML_0$ consists of:*

- *a set PROP of atomic propositions (denoted $p, q, r, v, \ldots$);*
- *the propositional connectives: $\neg$, $\rightarrow$;*
- *parentheses: ( , );*
- *the modal operator $\Box$ (box).*

The set $Sym(ML_0)$ of symbols of $ML_0$ is

$$Sym(ML_0) := PROP \cup \{\neg, \rightarrow, (, ), \Box\}.$$

The expressions of $ML_0$ are the finite sequences of symbols of $ML_0$.

## Basic modal language

*Definition 4.2*

*The formulas of the basic modal language $ML_0$ are the expressions inductively defined as follows:*

(F0) *Every atomic proposition is a formula.*

(F2) *If $\varphi$ is a formula, then $(\neg\varphi)$ is a formula.*

(F3) *If $\varphi$ and $\psi$ are formulas, then $(\varphi \to \psi)$ is a formula.*

(F4) *If $\varphi$ is a formula, then $(\Box\varphi)$ is a formula.*

(F5) *Only the expressions obtained by applying rules (F0), (F1), (F2), (F3), (F4) are formulas.*

Notation: The set of formulas is denoted by $Form(ML_0)$.

## *Basic modal language*

Formulas of $ML_0$ are defined, using the Backus-Naur notation, as follows:

$$\varphi ::= p \mid (\neg\varphi) \mid (\varphi \to \psi) \mid (\Box\varphi), \quad \text{where } p \in PROP.$$

### *Derived connectives*

Connectives $\vee$, $\wedge$, $\leftrightarrow$ and the constants $\top$ (true), $\bot$ (false) are introduced as in classical propositional logic:

$$\varphi \vee \psi := ((\neg\varphi) \to \psi) \qquad \varphi \wedge \psi := \neg(\varphi \to (\neg\psi))$$

$$\varphi \leftrightarrow \psi := ((\varphi \to \psi) \wedge (\psi \to \varphi))$$

$$\top := p \to p, \text{ where } p \in PROP, \qquad \bot := \neg\top$$

### *Dual modal operator*

The dual of $\Box$ is denoted by $\Diamond$ (diamond) and is defined as:

$$\Diamond\varphi := (\neg(\Box(\neg\varphi)))$$

for every formula $\varphi$.

## Basic modal language

Usually the external parantheses are omitted, we write them only when necessary. We write $\neg\varphi, \varphi \to \psi$, $\Box\varphi$.

To reduce the use of parentheses, we assume that

- modal operators $\Diamond$ and $\Box$ have higher precedence than the other connectives.
- $\neg$ has higher precedence than $\to, \wedge, \vee, \leftrightarrow$;
- $\wedge, \vee$ have higher precedence than $\to, \leftrightarrow$.

## Basic modal language

### Classical modal logic

In classical modal logic,

- $\Box\varphi$ is read as is necessarily $\varphi$.
- $\Diamond\varphi$ means it is not necessary that not $\varphi$, that is it is possible the case that $\varphi$.

Examples of formulas we would probably regard as correct principles

- $\Box\varphi \to \Diamond\varphi$ (whatever is necessary is possible)
- $\varphi \to \Diamond\varphi$ (whatever is, is possible).

The status of other formulas is harder to decide. What can we say about $\varphi \to \Box\Diamond\varphi$ (whatever is, is necessarily possible) or $\Diamond\varphi \to \Box\Diamond\varphi$ (whatever is possible, is necessarily possible)? Can we consider them as general truths? In order to give an answer to such questions, one has to define a semantics for the classical modal logic.

## *Relational structures*

*Definition 4.3*

*A relational structure is a tuple $\mathcal{F}$ consisting of:*

- *a nonempty set $W$, called the universe (or domain) of $\mathcal{F}$, and*
- *a set of relations on $W$.*

We assume that every relational structure contains at least one relation. The elements of $W$ are called points, nodes, states, worlds, times, instances or situations.

*Example 4.4*

A partially ordered set $\mathcal{F} = (W, R)$, where $R$ is a partial order relation on $W$.

## *Relational structures*

Labeled Transition Systems (LTSs), or more simply, transition systems, are very simple relational structures widely used in computer science.

*Definition 4.5*

*An LTS is a pair $(W, \{R_a \mid a \in A\})$, where $W$ is a nonempty set of states, $A$ is a nonempty set of labels and, for every $a \in A$,*

$$R_a \subseteq W \times W$$

*is a binary relation on $W$.*

LTSs can be viewed as an abstract model of computation: the states are the possible states of a computer, the labels stand for programs, and $(u, v) \in R_a$ means that there is an execution of the program $a$ starting in state $u$ and terminating in state $v$.

## Relational structures

Let $W$ be a nonempty set and $R \subseteq W \times W$ be a binary relation.

We write usually $Rwv$ instead of $(w, v) \in R$. If $Rwv$, then we say that $v$ is $R$-accessible from $w$.

The inverse of $R$, denoted by $R^{-1}$, is defined as follows:

$$R^{-1}vw \quad \text{iff} \quad Rwv.$$

We define $R^n (n \geq 0)$ inductively:

$$R^0 = \{(w, w) \mid w \in R\}, \quad R^1 = R, \quad R^{n+1} = R \circ R^n.$$

Thus, for any $n \geq 2$, we have that $R^n wv$ iff there exists $u_1, \ldots, u_{n-1}$ such that $Rwu_1, Ru_1u_2, \ldots, Ru_{n-1}v$.

## *Frames and models*

In the sequel we give the semantics of the basic modal language $ML_0$.

We will do this in two distinct ways:

- at the level of models, where the fundamental notion of satisfaction (or truth) is defined.
- at the level of frames, where the key notion of validity is defined.

## Frames and models

*Definition 4.6*

*A frame for $ML_0$ is a pair $\mathcal{F} = (W, R)$ such that*

- *$W$ is a nonempty set;*
- *$R$ is a binary relation on $W$.*

That is, a frame for the basic modal language is simply a relational structure with a single binary relation.

*Interpretation using agents*

$Rwv$ holds iff the agent considers the world $v$ possible according to the informations available in the world $w$. We think of $R$ as a possibility relation, as $R$ defines worlds that are considered possible by the agent.

## Frames and models

*Definition 4.7*

*A model for $ML_0$ is a pair $\mathcal{M} = (\mathcal{F}, V)$, where*

- *$\mathcal{F} = (W, R)$ is a frame for $ML_0$;*
- *$V : PROP \to 2^W$ is a function called valuation.*

Thus, $V$ assigns to each atomic proposition $p \in PROP$ a subset $V(p)$ of $W$. Informally, we think of $V(p)$ as the set of points in the model $\mathcal{M}$ where $p$ is true.
Note that models for $ML_0$ can also be viewed as relational structures in a natural way:

$$\mathcal{M} = (W, R, \{V(p) \mid p \in PROP\}).$$

Thus, a model is a relational structure consisting of a domain, a single binary relation $R$ and the unary relations $V(p), p \in PROP$.
A frame $\mathcal{F}$ and a model $\mathcal{M}$ are two relational structures based on the same universe. However, as we shall see, frames and models are used very differently.

## Frames and models

Let $\mathcal{F} = (W, R)$ be a frame and $\mathcal{M} = (\mathcal{F}, V)$ be a model. We also write $\mathcal{M} = (W, R, V)$.

We say that the model $\mathcal{M} = (\mathcal{F}, V)$ is based on the frame $\mathcal{F} = (W, R)$ or that $\mathcal{F}$ is the frame underlying $\mathcal{M}$. Elements of $W$ are called states in $\mathcal{F}$ or in $\mathcal{M}$. We often write $w \in \mathcal{F}$ or $w \in \mathcal{M}$.

### *Remark*

Elements of $W$ are also called worlds or possible worlds, having as inspiration Leibniz's philosophy and the reading of basic modal language in which

$\Box\varphi$ means necessarily $\varphi$ and $\Diamond\varphi$ means possibly $\varphi$.

In Leibniz's view, necessity means truth in all possible worlds and possibility means truth in some possible world.

## *Frames and models*

We define now the notion of satisfaction.

*Definition 4.8*

*Let $\mathcal{M} = (W, R, V)$ be a model and $w$ a state in $\mathcal{M}$. We define inductively the notion*

*formula $\varphi$ is satisfied (or true) in $\mathcal{M}$ at state $w$,*
*Notation $\mathcal{M}, w \Vdash \varphi$*

$$\begin{array}{rll} \mathcal{M}, w \Vdash p & \textit{iff} & w \in V(p), \quad \textit{where } p \in \textit{PROP} \\ \mathcal{M}, w \Vdash \neg\varphi & \textit{iff} & \textit{it is not true that } \mathcal{M}, w \Vdash \varphi \\ \mathcal{M}, w \Vdash \varphi \to \psi & \textit{iff} & \mathcal{M}, w \Vdash \varphi \textit{ implies } \mathcal{M}, w \Vdash \psi \\ \mathcal{M}, w \Vdash \Box\varphi & \textit{iff} & \textit{for every } v \in W, Rwv \textit{ implies } \mathcal{M}, v \Vdash \varphi. \end{array}$$

## *Frames and models*

Let $\mathcal{M} = (W, R, V)$ be a model.

*Notation*

If $\mathcal{M}$ does not satisfy $\varphi$ at $w$, we write $\mathcal{M}, w \not\Vdash \varphi$ and we say that $\varphi$ is false in $\mathcal{M}$ at state $w$.

It follows from Definition 4.8 that for every state $w \in W$,

- $\mathcal{M}, w \Vdash \neg\varphi$ iff $\mathcal{M}, w \not\Vdash \varphi$.

*Notation*

We can extend the valuation $V$ from atomic propositions to arbitrary formulas $\varphi$ so that $V(\varphi)$ is the set of all states in $\mathcal{M}$ at which $\varphi$ is true:

$$V(\varphi) = \{w \mid \mathcal{M}, w \Vdash \varphi\}.$$

## Frames and models

Let $\mathcal{M} = (W, R, V)$ be a model and $w$ a state in $\mathcal{M}$.

*Proposition 4.9*

*For every formulas $\varphi$, $\psi$,*

$$\begin{array}{lcl} \mathcal{M}, w \Vdash \varphi \vee \psi & \textit{iff} & \mathcal{M}, w \Vdash \varphi \textit{ or } \mathcal{M}, w \Vdash \psi \\ \mathcal{M}, w \Vdash \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w \Vdash \varphi \textit{ and } \mathcal{M}, w \Vdash \psi \end{array}$$

*Proposition 4.10*

*For every formula $\varphi$,*

$\mathcal{M}, w \Vdash \Diamond\varphi$ *iff there exists* $v \in W$ *such that* $Rwv$ *and* $\mathcal{M}, v \Vdash \varphi$.

## Frames and models

Let $\mathcal{M} = (W, R, V)$ be a model and $w$ a state in $\mathcal{M}$.

*Proposition 4.11*

*For every $n \geq 1$ and every formula $\varphi$, define*

$$\Diamond^n \varphi := \underbrace{\Diamond\Diamond\ldots\Diamond}_{n \text{ times}} \varphi, \qquad \Box^n \varphi := \underbrace{\Box\Box\ldots\Box}_{n \text{ times}} \varphi.$$

*Then*

$$\begin{array}{lll} \mathcal{M}, w \Vdash \Diamond^n \varphi & \textit{iff} & \textit{there exists } v \in W \textit{ s.t. } R^n wv \textit{ and } \mathcal{M}, v \Vdash \varphi \\ \mathcal{M}, w \Vdash \Box^n \varphi & \textit{iff} & \textit{for every } v \in W, R^n wv \textit{ implies } \mathcal{M}, v \Vdash \varphi. \end{array}$$

## Frames and models

Let $\mathcal{M} = (W, R, V)$ be a model.

*Definition 4.12*

- *A formula $\varphi$ is globally true or simply true in $\mathcal{M}$ if $\mathcal{M}, w \Vdash \varphi$ for every $w \in W$. Notation: $\mathcal{M} \Vdash \varphi$*
- *A formula $\varphi$ is satisfiable in $\mathcal{M}$ if there exists a state $w \in W$ such that $\mathcal{M}, w \Vdash \varphi$.*

*Definition 4.13*

*Let $\Sigma$ be a set of formulas.*

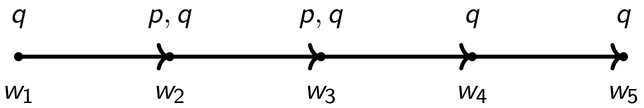
- *$\Sigma$ is true at state $w$ in $\mathcal{M}$ if $\mathcal{M}, w \Vdash \varphi$ for every $\varphi \in \Sigma$. Notation: $\mathcal{M}, w \Vdash \Sigma$*
- *$\Sigma$ is globally true or simply true in $\mathcal{M}$ if $\mathcal{M}, w \Vdash \Sigma$ for every state $w$ in $\mathcal{M}$. Notation: $\mathcal{M} \Vdash \Sigma$*
- *$\Sigma$ is satisfiable in $\mathcal{M}$ if there exists a state $w \in W$ such that $\mathcal{M}, w \Vdash \Sigma$.*

## Frames and models

A model $M = (W, R, V)$ is represented as a labeled directed graph:

- the nodes of the graph are the states of the model;
- the label of each node $w \in W$ describes which atomic propositions are true at state $w$;
- there exists an edge from node $w$ to node $v$ iff $Rwv$ holds.

*Example*



We know that $PROP = \{p, q, r\}$. Then $\mathcal{M} = (W, R, V)$, where $W = \{w_1, w_2, w_3, w_4, w_5\}$; $Rw_iw_j$ iff $j = i + 1$; $V(p) = \{w_2, w_3\}$, $V(q) = \{w_1, w_2, w_3, w_4, w_5\}$ and $V(r) = \emptyset$.

## Frames and models

### *Example*

Let $\mathcal{M} = (W, R, V)$ be the model represented by:

$$\begin{tikzcd} q & p, q & p, q & q & q \\ w_1 \arrow[r] & w_2 \arrow[r] & w_3 \arrow[r] & w_4 \arrow[r] & w_5 \end{tikzcd}$$

*(i)* $\mathcal{M}, w_1 \Vdash \Diamond\Box p$.
*(ii)* $\mathcal{M}, w_1 \nVdash \Diamond\Box p \to p$.
*(iii)* $\mathcal{M}, w_2 \Vdash \Diamond(p \wedge \neg r)$.
*(iv)* $\mathcal{M}, w_1 \Vdash q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q)))$.
*(v)* $\mathcal{M} \Vdash \Box q$.

**Proof:** (i) $\mathcal{M}, w_1 \Vdash \Diamond\Box p$ iff there exists $v \in W$ such that $Rw_1v$ and $\mathcal{M}, v \Vdash \Box p$. Take $v := w_2$. As $Rw_1w_2$, it remains to prove that $\mathcal{M}, w_2 \Vdash \Box p$. We have that $\mathcal{M}, w_2 \Vdash \Box p \iff$ for all $u \in W$, $Rw_2u$ implies $\mathcal{M}, u \Vdash p \iff \mathcal{M}, w_3 \Vdash p$ (as $w_3$ is the unique $u \in W$ s.t. $Rw_2u$) $\iff w_3 \in V(p)$, which is true.

## *Frames and models*

### *Example*

Let $\mathcal{M} = (W, R, V)$ be the model represented by:

$$\begin{tikzcd}
q & p, q & p, q & q & q \\
w_1 \arrow[r] & w_2 \arrow[r] & w_3 \arrow[r] & w_4 \arrow[r] & w_5
\end{tikzcd}$$

**Proof:** (ii) We have that $\mathcal{M}, w_1 \nVdash \Diamond\Box p \to p \iff \mathcal{M}, w_1 \Vdash \Diamond\Box p$ and $\mathcal{M}, w_1 \nVdash p$. Apply (i) and the fact that $w_1 \notin V(p)$.
(iii), (iv) Exercise.
(v) Let $w \in W$ be arbitray. Then $\mathcal{M}, w \Vdash \Box q \iff$ for all $v \in W$, $Rwv$ implies $\mathcal{M}, v \Vdash q \iff$ for all $v \in W$, $Rwv$ implies $v \in V(q)$, which is true, as $V(q) = W$.

## *Frames and models*

The notion of satisfaction is internal and local. We evaluate formulas inside models, at some particular state $w$ (the current state). Modal operators $\Diamond, \Box$ work locally: we verify the truth of $\varphi$ only in the states that are $R$-accesibile from the current one.

At first sight this may seem a weakness of the satisfaction definition. In fact, it is its greatest source of strength, as it gives us great flexibility.
For example, if we take $R = W \times W$, then all states are accessible from the current state; this corresponds to the Leibnizian idea in its purest form.
Going to the other extreme, if we take $R = \{(v, v) \mid v \in W\}$, then no state has access to any other.
Between these extremes there is a wide range of options to explore.

## *Frames and models*

We can ask ourselves the following natural questions:

- What happens if we impose some conditions on $R$ (for example, reflexivity, symmetry, transitivity, etc.)?
- What is the impact of these conditions on the notions of necessity and possibility?
- What principles or rules are justified by these conditions?

## Frames and validity

Validity in a frame is one of the key concepts in modal logic.

*Definition 4.14*

*Let $\mathcal{F}$ be a frame and $\varphi$ be a formula.*

- *$\varphi$ is valid at a state $w$ in $\mathcal{F}$ if $\varphi$ is true at $w$ in every model $\mathcal{M} = (\mathcal{F}, V)$ based on $\mathcal{F}$.*
- *$\varphi$ is valid in $\mathcal{F}$ if it is valid at every state $w$ in $\mathcal{F}$.*
  *Notation: $\mathcal{F} \Vdash \varphi$*

Hence, a formula is valid in a frame if it is true at every state in every model based on the frame.

## Frames and validity

Validity in a frame differs in an essential way from the truth in a model. Let us give a simple example.

*Example 4.15*

If $\varphi \vee \psi$ is true in a model $\mathcal{M}$ at $w$, then $\varphi$ is true in $\mathcal{M}$ at $w$ or $\psi$ is true in $\mathcal{M}$ at $w$ (by Proposition 4.9).

On the other hand, if $\varphi \vee \psi$ is valid in a frame $\mathcal{F}$ at $w$, it does not follow that $\varphi$ is valid in $\mathcal{F}$ at $w$ or $\psi$ is valid in $\mathcal{F}$ at $w$ ($p \vee \neg p$ is a counterexample).

## Frames and validity

*Definition 4.16*

*Let **M** be a class of models, **F** be a class of frames and $\varphi$ be a formula. We say that*

- *$\varphi$ is true in **M** if it is true in every model in **M**.*
  *Notation:* $\mathbf{M} \Vdash \varphi$
- *$\varphi$ is valid in **F** if it is valid in every frame in **F**.*
  *Notation:* $\mathbf{F} \Vdash \varphi$

*Definition 4.17*

*The set of all formulas of $ML_0$ that are valid in a class of frames **F** is called the logic of **F** and is denoted by $\Lambda_{\mathbf{F}}$.*

## *Frames and validity*

### *Example 4.18*

Formulas $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ and $\Box(p \to q) \to (\Box p \to \Box q)$ are valid in the class of all frames.

**Proof:** Let $\mathcal{F} = (W, R)$ be an arbitrary frame, $w$ a state in $\mathcal{F}$ and $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$. We have to show that

$$\mathcal{M}, w \Vdash \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q).$$

Suppose that $\mathcal{M}, w \Vdash \Diamond(p \vee q)$. Then there exists $v \in W$ such that $Rwv$ and $\mathcal{M}, v \Vdash p \vee q$. We have two cases:

- $\mathcal{M}, v \Vdash p$. Then $\mathcal{M}, w \Vdash \Diamond p$, so $\mathcal{M}, w \Vdash \Diamond p \vee \Diamond q$.
- $\mathcal{M}, v \Vdash q$. Then $\mathcal{M}, w \Vdash \Diamond q$, so $\mathcal{M}, w \Vdash \Diamond p \vee \Diamond q$.

We let as an exercise to prove that $\Box(p \to q) \to (\Box p \to \Box q)$ is valid in the class of all frames. $\Box$

## *Frames and validity*

### *Example 4.19*

Formula $\Box p \to \Box\Box p$ is not valid in the class of all frames.

**Proof:** We have to find a frame $\mathcal{F} = (W, R)$, a state $w$ in $\mathcal{F}$ and a model $\mathcal{M} = (\mathcal{F}, V)$ such that

$$\mathcal{M}, w \not\Vdash \Box p \to \Box\Box p.$$

Consider the following frame: $\mathcal{F} = (W, R)$, where

$$W = \{0, 1, 2\}, \quad R = \{(0, 1), (1, 2)\}$$

and take a valuation $V$ such that $V(p) = \{1\}$. Then $\mathcal{M}, 0 \Vdash \Box p$, since 1 is the only state $R$-accesible from 0 and $\mathcal{M}, 1 \Vdash p$, as $1 \in V(p)$.
On the other hand, $\mathcal{M}, 0 \not\Vdash \Box\Box p$, since $R^2 02$ and $\mathcal{M}, 2 \not\Vdash p$, as $2 \notin V(p)$. $\Box$

## *Frames and validity*

*Definition 4.20*

*We say that a frame $\mathcal{F} = (W, R)$ is transitive if $R$ is transitive.*

*Example 4.21*

Formula $\Box p \to \Box\Box p$ is valid in the class of all transitive frames.

**Proof:** Let $\mathcal{F} = (W, R)$ be a transitive frame, $w$ a state in $\mathcal{F}$ and $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$. Assume that $\mathcal{M}, w \Vdash \Box p$. Then for all $v \in W$,

$$(*) \qquad Rwv \text{ implies } \mathcal{M}, v \Vdash p.$$

Let us prove that $\mathcal{M}, w \Vdash \Box\Box p$. Let $u, u' \in W$ be such that $Rwu'$ and $Ru'u$. We have to prove that $\mathcal{M}, u \Vdash p$. Since $R$ is transitive, it follows that $Rwu$. Applying (*) with $v := u$ we get that $\mathcal{M}, u \Vdash p$. $\Box$

## *Modal consequence*

We introduce the consequence relation.

The basic ideas are the following;

- A relation of semantic consequence holds when the truth of the premises guarantees the truth of the conclusion.
- The inferences depend on the class of structures we are working with. (For example, inferences for transitive frames must be different than the ones for intransitive frames.)

Thus, the definition of the consequence relation must make reference to a class of structures $\boldsymbol{S}$.

## *Modal consequence*

Let $\boldsymbol{S}$ be a class of structures (frames or models) for $ML_0$.
If $\boldsymbol{S}$ is a class of models, then a model from $\boldsymbol{S}$ is simply an element $\mathcal{M}$ of $\boldsymbol{S}$. If $\boldsymbol{S}$ is a class of frames, then a model from $\boldsymbol{S}$ is a model based on a frame in $\boldsymbol{S}$.

*Definition 4.22*

*Let $\Sigma$ be a set of formulas and $\varphi$ be a formula. We say that $\varphi$ is a semantic consequence of $\Sigma$ over $\boldsymbol{S}$ if for all models $\mathcal{M}$ from $\boldsymbol{S}$ and all states $w$ in $\mathcal{M}$,*

$$\mathcal{M}, w \Vdash \Sigma \quad \textit{implies} \quad \mathcal{M}, w \Vdash \varphi.$$

*Notation:* $\Sigma \Vdash_{\boldsymbol{S}} \varphi$

Thus, if $\Sigma$ is true at a state of the model, then $\varphi$ must be true at the same state.

## Modal consequence

*Remark 4.23*

$$\{\psi\} \Vdash_{\boldsymbol{S}} \varphi \textit{ iff } \boldsymbol{S} \Vdash \psi \to \varphi.$$

*Example 4.24*

Let *Tran* be the class of transitive frames. Then

$$\{\Box\varphi\} \Vdash_{\mathit{Tran}} \Box\Box\varphi.$$

But $\Box\Box\varphi$ is NOT a semantic consequence of $\Box\varphi$ over the class of all frames.

## Normal modal logics

### Definition 4.25

A normal modal logic is a set $\Lambda$ of formulas of $ML_0$ satisfying the following properties:

- $\Lambda$ contains the following axioms:

$$\begin{array}{rl} (Taut) & \text{all propositional tautologies}, \\ (K) & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi), \end{array}$$

  where $\varphi, \psi$ are formulas of $ML_0$.
- $\Lambda$ is closed under the following deduction rules:
  - modus ponens (MP): $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$.
    Hence, if $\varphi \in \Lambda$ and $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$.
  - generalization or necessitation (GEN): $\dfrac{\varphi}{\Box\varphi}$.
    Hence, if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$.

## *Normal modal logics - tautologies*

We add all propositional tautologies as axioms for simplicity, it is not necessary. We could add a small number of tautologies, which generates all of them. For example,

$$\begin{array}{ll} (A1) & \varphi \to (\psi \to \varphi) \\ (A2) & (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \\ (A3) & (\neg\psi \to \neg\varphi) \to (\varphi \to \psi). \end{array}$$

### *Proposition 4.26*

*Any propositional tautology is valid in the class of all frames for $ML_0$.*

### *Remark 4.27*

*Tautologies may contain modalities, too. For example, $\Diamond\psi \vee \neg\Diamond\psi$ is a tautology, since it has the same form as $\varphi \vee \neg\varphi$.*

## Normal modal logics - axiom $(K)$

Axiom $(K)$ is sometimes called the distribution axiom and it is important because it allows us to transform $\Box(\varphi \to \psi)$ (a boxed formula) in an implication $\Box\varphi \to \Box\psi$, enabling further pure propositional reasoning to take place.
For example, assume that we want to prove $\Box\psi$ and we already have a proof that contains both $\Box(\varphi \to \psi)$ and $\Box\varphi$. Applying $(K)$ and modus ponens, we get $\Box\varphi \to \Box\psi$. Applying again modus ponens, we obtain $\Box\psi$.

By Example 4.18,

*Proposition 4.28*

$(K)$ *is valid in the class of all frames for* $ML_0$.

## Normal modal logics

*Theorem 4.29*

*For any class $\boldsymbol{F}$ of frames, $\Lambda_{\boldsymbol{F}}$, the logic of $\boldsymbol{F}$, is a normal modal logic.*

*Lemma 4.30*

- *The collection of all formulas is a normal modal logic, called the inconsistent logic.*
- *If $\{\Lambda_i \mid i \in I\}$ is a collection of normal modal logics, then $\bigcap_{i \in I} \Lambda_i$ is a normal modal logic.*

*Definition 4.31*

*$\boldsymbol{K}$ is the intersection of all normal modal logics.*

Hence, $\boldsymbol{K}$ is the smallest normal modal logic.

## $K$

*Definition 4.32*

*A **K**-proof is a sequence of formulas $\theta_1, \ldots, \theta_n$ such that for any $i \in \{1, \ldots, n\}$, one of the following conditions is satisfied:*

- *$\theta_i$ is an axiom (that is, a tautology or $(K)$);*
- *$\theta_i$ is obtained from previous formulas by applying modus ponens or generalization.*

*Definition 4.33*

*Let $\varphi$ be a formula. A **K**-proof of $\varphi$ is a **K**-proof $\theta_1, \ldots, \theta_n$ such that $\theta_n = \varphi$.*
*If $\varphi$ has a **K**-proof, we say that $\varphi$ is **K**-provable.*
*Notation: $\vdash_{\boldsymbol{K}} \varphi$.*

*Theorem 4.34*

$$\boldsymbol{K} = \{\varphi \mid \vdash_{\boldsymbol{K}} \varphi\}.$$

## K

*Definition 4.35*

*Let $\varphi, \psi_1, \ldots, \psi_n \,(n \geq 1)$ be formulas. We say that $\varphi$ is deducible in propositional logic from $\psi_1, \ldots, \psi_n$ if*

$$\psi_1 \wedge \ldots \wedge \psi_n \rightarrow \varphi \text{ is a tautology.}$$

*Lemma 4.36*

*Let $\varphi, \psi_1, \ldots, \psi_n \,(n \geq 1)$ be formulas. The following are equivalent:*

- *$\varphi$ is deducible in propositional logic from $\psi_1, \ldots, \psi_n$.*
- *$\psi_1 \rightarrow (\psi_2 \rightarrow \ldots \rightarrow (\psi_n \rightarrow \varphi))$ is a tautology.*

**Proof:** Use the fact that

$$\big(\psi_1 \wedge \ldots \wedge \psi_n \rightarrow \varphi\big) \leftrightarrow \big(\psi_1 \rightarrow (\psi_2 \rightarrow \ldots \rightarrow (\psi_n \rightarrow \varphi))\big)$$

is a tautology.

## $\mathbf{K}$

### *Proposition 4.37*

***K** is closed under propositional deduction: if $\varphi$ is deducible in propositional logic from assumptions $\psi_1, \ldots, \psi_n$, then*

$$\vdash_{\mathbf{K}} \psi_1, \ldots, \vdash_{\mathbf{K}} \psi_n \quad \textit{implies} \quad \vdash_{\mathbf{K}} \varphi.$$

**Proof:**

| | | |
|---|---|---|
| (1) | $\vdash_{\mathbf{K}} \psi_1$ | hypothesis |
| | $\vdots$ | |
| (n) | $\vdash_{\mathbf{K}} \psi_n$ | hypothesis |
| (n+1) | $\vdash_{\mathbf{K}} \psi_1 \to (\psi_2 \to \ldots \to (\psi_n \to \varphi))$ | (Taut) |
| (n+2) | $\vdash_{\mathbf{K}} \psi_2 \to \ldots \to (\psi_{n-1} \to (\psi_n \to \varphi))$ | (MP): (1), (n+1) |
| | $\vdots$ | |
| (2n-1) | $\vdash_{\mathbf{K}} \psi_{n-1} \to (\psi_n \to \varphi)$ | (MP): (n-2), (2n-2) |
| (2n) | $\vdash_{\mathbf{K}} \psi_n \to \varphi$ | (MP): (n-1), (2n-1) |
| (2n+1) | $\vdash_{\mathbf{K}} \varphi$ | (MP): (n), (2n) $\square$ |

## $\boldsymbol{K}$

*Example 4.38*

$$\vdash_{\boldsymbol{K}} \varphi \to \psi \text{ implies } \vdash_{\boldsymbol{K}} \Box\varphi \to \Box\psi.$$

**Proof:** We give the following $\boldsymbol{K}$-proof:

|  |  |  |
|---|---|---|
| (1) | $\vdash_{\boldsymbol{K}} \varphi \to \psi$ | hypothesis |
| (2) | $\vdash_{\boldsymbol{K}} \Box(\varphi \to \psi)$ | (GEN): (1) |
| (3) | $\vdash_{\boldsymbol{K}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ | (K) |
| (4) | $\vdash_{\boldsymbol{K}} \Box\varphi \to \Box\psi$ | (MP): (2), (3). |

## *K*

*Example 4.39*

$$\vdash_{\boldsymbol{K}} \varphi \to \psi \text{ implies } \vdash_{\boldsymbol{K}} \Diamond\varphi \to \Diamond\psi.$$

**Proof:** We give the following ***K***-proof:

| | | |
|---|---|---|
| (1) | $\vdash_{\boldsymbol{K}} \varphi \to \psi$ | hypothesis |
| (2) | $\vdash_{\boldsymbol{K}} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ | (Taut) |
| (3) | $\vdash_{\boldsymbol{K}} \neg\psi \to \neg\varphi$ | (MP): (1), (2) |
| (4) | $\vdash_{\boldsymbol{K}} \Box\neg\psi \to \Box\neg\varphi$ | Example 4.38: (3) |
| (5) | $\vdash_{\boldsymbol{K}} (\Box\neg\psi \to \Box\neg\varphi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$ | (Taut) |
| (6) | $\vdash_{\boldsymbol{K}} \neg\Box\neg\varphi \to \neg\Box\neg\psi$ | (MP): (4), (5) |
| (7) | $\vdash_{\boldsymbol{K}} \Diamond\varphi \to \Diamond\psi$ | definition of $\Diamond$ |

## $\boldsymbol{K}$

*Example 4.40*

$$\vdash_{\boldsymbol{K}} \Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi.$$

**Proof:** We give the following ***K***-proof:

| | | |
|---|---|---|
| (1) | $\vdash_{\boldsymbol{K}} \varphi \wedge \psi \to \varphi$ | (Taut) |
| (2) | $\vdash_{\boldsymbol{K}} \Box(\varphi \wedge \psi) \to \Box\varphi$ | Example 4.38: (1) |
| (3) | $\vdash_{\boldsymbol{K}} \varphi \wedge \psi \to \psi$ | (Taut) |
| (4) | $\vdash_{\boldsymbol{K}} \Box(\varphi \wedge \psi) \to \Box\psi$ | Example 4.38: (3) |
| (5) | $\vdash_{\boldsymbol{K}} \Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$ | Proposition 4.37, (2) and (4) |

(5) is obtained as an application of Proposition 4.37 with

$$\psi_1 = \sigma_1 \to \sigma_2, \quad \psi_2 = \sigma_1 \to \sigma_3, \quad \varphi = \sigma_1 \to \sigma_2 \wedge \sigma_3,$$

where

$$\sigma_1 = \Box(\varphi \wedge \psi), \quad \sigma_2 = \Box\varphi, \quad \sigma_3 = \Box\psi.$$

□

## $\boldsymbol{K}$

*Example 4.41*

$$\vdash_{\boldsymbol{K}} \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi).$$

**Proof:** We give the following $\boldsymbol{K}$-proof:

| | | |
|---|---|---|
| (1) | $\vdash_{\boldsymbol{K}} \varphi \to (\psi \to (\varphi \wedge \psi))$ | (Taut) |
| (2) | $\vdash_{\boldsymbol{K}} \Box\varphi \to \Box(\psi \to (\varphi \wedge \psi))$ | Ex. 4.38: (1) |
| (3) | $\vdash_{\boldsymbol{K}} \Box(\psi \to (\varphi \wedge \psi)) \to (\Box\psi \to \Box(\varphi \wedge \psi))$ | (K) |
| (4) | $\vdash_{\boldsymbol{K}} \Box\varphi \to (\Box\psi \to \Box(\varphi \wedge \psi))$ | Prop. 4.37, (2), (3) |
| (5) | $\vdash_{\boldsymbol{K}} \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ | Prop. 4.37, (4) |

(4) is obtained as an application of Proposition 4.37 with

$$\psi_1 = \sigma_1 \to \sigma_2, \quad \psi_2 = \sigma_2 \to \sigma_3, \quad \varphi = \sigma_1 \to \sigma_3,$$

where

$$\sigma_1 = \Box\varphi, \quad \sigma_2 = \Box(\psi \to (\varphi \wedge \psi)), \quad \sigma_3 = \Box\psi \to \Box(\varphi \wedge \psi).$$

(5) is obtained as an application of Proposition 4.37 with

$$\psi_1 = \sigma_1 \to (\sigma_2 \to \sigma_3), \quad \varphi = (\sigma_1 \wedge \sigma_2) \to \sigma_3,$$

where $\sigma_1 = \Box\varphi, \quad \sigma_2 = \Box\psi, \quad \sigma_3 = \Box(\varphi \wedge \psi).$ $\Box$

## $\boldsymbol{K}$

*Example 4.42*

$$\vdash_{\boldsymbol{K}} \Box\varphi \wedge \Box\psi \leftrightarrow \Box(\varphi \wedge \psi).$$

**Proof:** We give the following $\boldsymbol{K}$-proof:

|  |  |  |
|---|---|---|
| (1) | $\vdash_{\boldsymbol{K}} \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ | Example 4.41 |
| (2) | $\vdash_{\boldsymbol{K}} \Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$ | Example 4.40 |
| (3) | $\vdash_{\boldsymbol{K}} \Box\varphi \wedge \Box\psi \leftrightarrow \Box(\varphi \wedge \psi)$ |  |

(3) is obtained as an application of Proposition 4.37 with

$$\psi_1 = \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi), \quad \psi_2 = \Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$$

and

$$\varphi = \Box\varphi \wedge \Box\psi \leftrightarrow \Box(\varphi \wedge \psi).$$

$\Box$

## ***K***

The logic ***K*** is very weak. If we are interested in transitive frames, we would like a proof system which reflects this. For example, we know that $\Box\varphi \to \Box\Box\varphi$ is valid in the class of all transitive frames, so we would want a proof system that generates this formula.
***K*** does not do this, since $\Box\varphi \to \Box\Box\varphi$ is not valid in the class of all frames.

The idea is to extend ***K*** with additional axioms.

## $\boldsymbol{K}\Gamma$

By Lemma 4.30, for any set $\Gamma$ of formulas, there exists the smallest normal modal logic that contains $\Gamma$.

*Definition 4.43*

***K**Γ is the smallest normal modal logic that contains Γ. We say that **K**Γ is generated by Γ or axiomatized by Γ.*

*Definition 4.44*

*A **K**Γ-proof is a sequence of formulas $\theta_1, \ldots, \theta_n$ such that for any $i \in \{1, \ldots, n\}$, one of the following conditions is satisfied:*

- *$\theta_i$ is an axiom (that is, a tautology or $(K)$);*
- *$\theta_i \in \Gamma$;*
- *$\theta_i$ is obtained from previous formulas by applying modus ponens or generalization.*

## $\boldsymbol{K}\Gamma$

*Definition 4.45*

*Let $\varphi$ be a formula. A $\boldsymbol{K}\Gamma$-proof of $\varphi$ is a $\boldsymbol{K}\Gamma$-proof $\theta_1, \ldots, \theta_n$ such that $\theta_n = \varphi$.*
*If $\varphi$ has a $\boldsymbol{K}\Gamma$-proof, we say that $\varphi$ is $\boldsymbol{K}\Gamma$-provable.*
*Notation: $\vdash_{\boldsymbol{K}\Gamma} \varphi$.*

*Theorem 4.46*

$$\boldsymbol{K}\Gamma = \{\varphi \mid \vdash_{\boldsymbol{K}\Gamma} \varphi\}.$$

## *Normal modal logics*

Let $\Lambda$ be a normal modal logic.

*Definition 4.47*

*If $\varphi \in \Lambda$, we also say that $\varphi$ is a $\Lambda$-theorem or a theorem of $\Lambda$ and write $\vdash_\Lambda \varphi$. If $\varphi \notin \Lambda$, we write $\nvdash_\Lambda \varphi$.*

With these notations, the conditions from the definition of a normal modal logic are written as follows:

For any formulas $\varphi$, $\psi$, the following hold:

*(i)* If $\varphi$ is a tautology, then $\vdash_\Lambda \varphi$.

*(ii)* $\vdash_\Lambda (K)$.

*(iii)* If $\vdash_\Lambda \varphi$ and $\vdash_\Lambda \varphi \to \psi$, then $\vdash_\Lambda \psi$.

*(iv)* If $\vdash_\Lambda \varphi$, then $\vdash_\Lambda \Box\varphi$.

## Normal modal logics

*Remark 4.48*

- $\vdash_{\boldsymbol{K}} \varphi$ *implies* $\vdash_{\Lambda} \varphi$.
- *If* $\Gamma \subseteq \Lambda$, *then* $\vdash_{\boldsymbol{K}\Gamma} \varphi$ *implies* $\vdash_{\Lambda} \varphi$.

*Proposition 4.49*

*$\Lambda$ is closed under propositional deduction: if $\varphi$ is deducible in propositional logic from assumptions $\psi_1, \ldots, \psi_n$, then*

$$\vdash_{\Lambda} \psi_1, \ldots, \vdash_{\Lambda} \psi_n \quad \textit{implies} \quad \vdash_{\Lambda} \varphi.$$

**Proof:** Exercise.

## Normal modal logics

*Definition 4.50*

*Let $\Gamma \cup \{\varphi\}$ be a set of formulas. We say that $\varphi$ is deducible in $\Lambda$ from $\Gamma$ or that $\varphi$ is $\Lambda$-deducible from $\Gamma$ if there exist formulas $\psi_1, \ldots, \psi_n \in \Gamma$ $(n \geq 0)$ such that*

$$\vdash_\Lambda (\psi_1 \wedge \ldots \wedge \psi_n) \to \varphi.$$

*(When $n = 0$, this means that $\vdash_\Lambda \varphi$).*
*Notation: $\Gamma \vdash_\Lambda \varphi$ We write $\Gamma \not\vdash_\Lambda \varphi$ if $\varphi$ is not $\Lambda$-deducible from $\Gamma$.*

*Remark 4.51*

*The following are equivalent:*

*(i)* $\Gamma \vdash_\Lambda \varphi$.

*(ii) There exist formulas $\psi_1, \ldots, \psi_n \in \Gamma$ $(n \geq 0)$ such that*

$$\vdash_\Lambda \psi_1 \to (\psi_2 \to \ldots \to (\psi_n \to \varphi)).$$

## Normal modal logics

*Proposition 4.52 (Basic properties)*

*Let $\varphi$ be a formula and $\Gamma, \Delta$ be sets of formulas.*

*(i)* $\emptyset \vdash_\Lambda \varphi$ *iff* $\vdash_\Lambda \varphi$.

*(ii)* $\vdash_\Lambda \varphi$ *implies* $\Gamma \vdash_\Lambda \varphi$.

*(iii)* $\varphi \in \Gamma$ *implies* $\Gamma \vdash_\Lambda \varphi$.

*(iv)* *If* $\Gamma \vdash_\Lambda \varphi$ *and* $\Gamma \subseteq \Delta$, *then* $\Delta \vdash_\Lambda \varphi$.

**Proof:** Exercise.

## Normal modal logics

Let $\varphi, \psi$ be formulas and $\Gamma$ be a set of formulas,

*Proposition 4.53*

*$\Gamma \vdash_\Lambda \varphi$ iff there exists a finite subset $\Sigma$ of $\Gamma$ such that $\Sigma \vdash_\Lambda \varphi$.*

*Proposition 4.54*

(i) *If $\Gamma \vdash_\Lambda \varphi$ and $\psi$ is deducible in propositional logic from $\varphi$, then $\Gamma \vdash_\Lambda \psi$.*

(ii) *If $\Gamma \vdash_\Lambda \varphi$ and $\Gamma \vdash_\Lambda \varphi \to \psi$, then $\Gamma \vdash_\Lambda \psi$.*

(iii) *If $\Gamma \vdash_\Lambda \varphi$ and $\{\varphi\} \vdash_\Lambda \psi$, then $\Gamma \vdash_\Lambda \psi$.*

*Proposition 4.55 (Deduction Theorem)*

*For any set of formulas $\Gamma$ and any formulas $\varphi, \psi$,*

$$\Gamma \vdash_\Lambda \varphi \to \psi \quad \textit{iff} \quad \Gamma \cup \{\varphi\} \vdash_\Lambda \psi.$$

## Consistent sets

### Definition 4.56

*A set $\Gamma$ of formulas is Λ-consistent if $\Gamma \vdash_\Lambda \varphi$ for any formula $\varphi$.*
*If $\Gamma$ is not Λ-consistent, we say that $\Gamma$ is Λ-inconsistent.*
*A formula $\varphi$ is Λ-consistent if $\{\varphi\}$ is; otherwise, it is called Λ-inconsistent.*

### Proposition 4.57

*Let $\Gamma$ be a set of formulas. The following are equivalent:*

*(i)* $\Gamma$ *is Λ-inconsistent.*

*(ii)* *There exists a formula $\psi$ such that $\Gamma \vdash_\Lambda \psi$ and $\Gamma \vdash_\Lambda \neg\psi$.*

*(iii)* $\Gamma \vdash_\Lambda \bot$.

### Proposition 4.58

*$\Gamma$ is Λ-consistent iff any finite subset of $\Gamma$ is Λ-consistent.*

## *Normal logics - soundness*

In the following, we say "normal logic " instead of "normal modal logic".

Let $\boldsymbol{S}$ be a class of structures (frames or models) for $ML_0$.

*Notation:*

$$\Lambda_{\boldsymbol{S}} := \{\varphi \mid \mathcal{S} \Vdash \varphi \text{ for any structure } \mathcal{S} \text{ from } \boldsymbol{S}\}.$$

*Definition 4.59*

*A normal logic $\Lambda$ is sound with respect to $\boldsymbol{S}$ if $\Lambda \subseteq \Lambda_{\boldsymbol{S}}$.*

Thus, $\Lambda$ is sound with respect to $\boldsymbol{S}$ iff for any formula $\varphi$ and for any structure $\mathcal{S}$ in $\boldsymbol{S}$,

$$\vdash_{\Lambda} \varphi \quad \text{implies} \quad \mathcal{S} \Vdash \varphi.$$

If $\Lambda$ is sound with respect to $\boldsymbol{S}$, we say also that $\boldsymbol{S}$ is a class of frames (or models) for $\Lambda$.

## *Normal logics - soundness*

*Theorem 4.60 (Soundness theorem for **K**)*

***K** is sound with respect to the class of all frames.*

**Proof:** We apply Theorem 4.29 and the fact that ***K*** is the least normal logic. $\square$

## *Normal logics - completeness*

*Definition 4.61*

*A normal logic $\Lambda$ is*

*(i) strongly complete with respect to* $\boldsymbol{S}$ *if for any set of formulas* $\Gamma \cup \{\varphi\}$,

$$\Gamma \Vdash_{\boldsymbol{S}} \varphi \quad \textit{implies} \quad \Gamma \vdash_{\Lambda} \varphi.$$

*(ii) weakly complete with respect to* $\boldsymbol{S}$ *if for any formula* $\varphi$,

$$\boldsymbol{S} \Vdash \varphi \quad \textit{implies} \quad \vdash_{\Lambda} \varphi.$$

Obviously, weak completeness is a particular case of strong completeness; just take $\Gamma = \emptyset$ in Definition 4.61.(i).

## *Normal logics - completeness*

*Remark 4.62*

*$\Lambda$ is weakly complete with respect to $\boldsymbol{S}$ iff $\Lambda_{\boldsymbol{S}} \subseteq \Lambda$.*

If a normal logic $\Lambda$ is both sound and weakly complete with respect to a class of structures $\boldsymbol{S}$, then there is a perfect match between the syntactic and semantic perspectives: $\Lambda = \Lambda_{\boldsymbol{S}}$.

Given a semantically specified normal logic $\Lambda_{\boldsymbol{S}}$ (that is, the logic of some class of structures of interest), a very important problem is to find a simple set of formulas $\Gamma$ such that $\Lambda_{\boldsymbol{S}}$ is the logic generated by $\Gamma$; we say that $\Gamma$ axiomatizes $\boldsymbol{S}$.

## Completeness theorem for $\boldsymbol{K}$

*Theorem 4.63*

*$\boldsymbol{K}$ is sound and strongly complete with respect to the class of all frames for $ML_0$.*

## *Logic **K**4*

Let

$$(4) \quad \Box\varphi \to \Box\Box\varphi$$

We use the notation ***K*4** for the normal logic generated by (4). Thus, ***K*4** is the smallest normal logic that contains (4).

*Theorem 4.64*

***K*4** *is sound and strongly complete with respect to the class of transitive frames.*

## *Logic* ***T***

Let

$$(T) \quad \Box\varphi \to \varphi$$

We use the notation ***T*** for the normal logic generated by $(T)$.

*Definition 4.65*

*We say that a frame $\mathcal{F} = (W, R)$ is reflexive if $R$ is reflexive.*

*Theorem 4.66*

***T*** *is sound and strongly complete with respect to the class of reflexive frames.*

## *Logic B*

Let

$$(B) \quad \varphi \to \Box\Diamond\varphi$$

We use the notation ***B*** for the normal logic ***KB*** generated by $(B)$.

*Definition 4.67*

*We say that a frame $\mathcal{F} = (W, R)$ is symmetric if $R$ is symmetric.*

*Theorem 4.68*

***B*** *is sound and strongly complete with respect to the class of symmetric frames.*

## *Logic **KD***

Let

$$\begin{array}{rc} (D) & \Box\varphi \to \Diamond\varphi \\ (D') & \neg\Box(\varphi \wedge \neg\varphi) \end{array}$$

One can prove that $\vdash_{\boldsymbol{K}} (D) \leftrightarrow (D')$ (exercise).

Let ***KD*** be the normal logic generated by $(D)$ (or, equivalently, by $(D')$).

*Definition 4.69*

*We say that a frame $\mathcal{F} = (W, R)$ is serial if for all $w \in W$ there exists $v \in W$ such that $Rwv$.*

*Theorem 4.70*

***KD** is sound and strongly complete with respect to the class of serial frames.*

## *Logic* $\boldsymbol{K}5$

Let

$$\begin{array}{ll} (5) & \Diamond\varphi \to \Box\Diamond\varphi \\ (5') & \neg\Box\varphi \to \Box\neg\Box\varphi \end{array}$$

One can prove that $\vdash_{\boldsymbol{K}} (5) \leftrightarrow (5')$ (exercise).

Let $\boldsymbol{K}5$ be the normal logic generated by (5) (or, equivalently, by (5′)).

*Definition 4.71*

*We say that a frame $\mathcal{F} = (W, R)$ is Euclidean if for all $w, v, u \in W$,*

*if $Rwv$ and $Rwu$, then $Rvu$.*

*Theorem 4.72*

*$\boldsymbol{K}5$ is sound and strongly complete with respect to the class of Euclidean frames.*

## *Logic* $\boldsymbol{S}4$

We use the notation $\boldsymbol{S}4$ for the normal logic $\boldsymbol{KT}4$ generated by $(T)$ and $(4)$.

*Theorem 4.73*

$\boldsymbol{S}4$ *is sound and strongly complete with respect to the class of reflexive and transitive frames.*

## *Logic $\boldsymbol{S}5$*

We use the notation $\boldsymbol{S}5$ for the normal logic $\boldsymbol{KT}4\boldsymbol{B}$ generated by $(T)$, (4) and $(B)$.

*Proposition 4.74*

$\boldsymbol{S}5 = \boldsymbol{KDB}4 = \boldsymbol{KDB}5 = \boldsymbol{KT}5.$

*Theorem 4.75*

$\boldsymbol{S}5$ *is sound and strongly complete with respect to the class of frames whose relation is an equivalence relation.*

## Multimodal logics

The whole theory presented so far adapts easily to languages with more modal operators.

Let $I$ be a nonempty set.

- The multimodal language $ML_I$ consists of: a set $PROP$ of atomic propositions, $\neg$, $\rightarrow$, the parentheses ( , ) and a set of modal operators $\{\Box_i \mid i \in I\}$.
- Formulas of $ML_I$ are defined, using the Backus-Naur notation, as follows:
$$\varphi ::= p \mid (\neg\varphi) \mid (\varphi \rightarrow \varphi) \mid (\Box_i\varphi),$$
where $p \in PROP$ and $i \in I$.
- The dual of $\Box_i$ is denoted by $\Diamond_i$ and is defined as:
$$\Diamond_i\varphi := \neg\Box_i\neg\varphi$$

## *Multimodal logics*

- A frame for $ML_I$ is a relational structure $\mathcal{F} = (W, \{R_i \mid i \in I\})$, where $R_i$ is a binary relation on $W$ for every $i \in I$.
- A model for $ML_I$ is, as previously, a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a frame and $V : PROP \to 2^W$ is a valuation.
- The last clause from the definition of the satisfaction relation $\mathcal{M}, w \Vdash \varphi$ is changed to: for all $i \in I$,

  $$\mathcal{M}, w \Vdash \Box_i \varphi \text{ iff for every } v \in W, R_i wv \text{ implies } \mathcal{M}, v \Vdash \varphi.$$
- It follows that

  $$\mathcal{M}, w \Vdash \Diamond_i \varphi \text{ iff there exists } v \in W \text{ s.t. } R_i wv \text{ and } \mathcal{M}, v \Vdash \varphi.$$
- The definitions of truth in a model ($\mathcal{M} \Vdash \varphi$), of validity in a frame ($\mathcal{F} \Vdash \varphi$) and of the consequence relation are unchanged.

## Multimodal logics

Definition 4.76

*A normal multimodal logic is a set $\Lambda$ of formulas of $ML_I$ satisfying the following properties:*

- *$\Lambda$ contains all propositional tautologies and is closed under modus ponens.*
- *$\Lambda$ contains all formulas*

  $$(K_i) \quad \Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi),$$

  *where $\varphi, \psi$ are formulas and $i \in I$.*
- *$\Lambda$ is closed under generalization: for any formula $\varphi$ and all $i \in I$,*

  $$\frac{\varphi}{\Box_i\varphi}.$$

## *Multimodal logics*

- We use the same notation, $\boldsymbol{K}$, for the smallest normal multimodal logic.
- We define similarly $\boldsymbol{K}$-proofs and we also have that $\boldsymbol{K} = \{\varphi \mid \vdash_{\boldsymbol{K}} \varphi\}$.
- The multimodal logic generated by a set of formulas $\Gamma$ is also denoted by $\boldsymbol{K}\Gamma$. Furthermore, $\boldsymbol{K}\Gamma = \{\varphi \mid \vdash_{\boldsymbol{K}\Gamma} \varphi\}$.
- The definitions of $\Lambda$-deducibility, $\Lambda$-consistence, soundness and weak and strong completeness are unchanged.

# Epistemic Logics

Textbook:

R. Fagin, J.Y. Halpern, Y. Moses, M. Vardi, Reasoning About Knowledge, MIT Press, 2004

## *Reasoning about knowledge*

- Consider a multiagent system, in which multiple agents autonomously perform some joint action.
- The agents need to communicate with one another.
- Problems appear when the communication is error-prone.
- One could have scenarios like the following:
  - Agent $A$ sent the message to agent $B$.
  - The message may not arrive, and agent $A$ knows this.
  - Furthemore, this is common knowledge, so agent $A$ knows that agent $B$ knows that $A$ knows that if a message was sent it may not arrive.

Multiagent system = distributed system; agent = processor; action = computation

We use epistemic logic to make such reasoning precise.

## Epistemic logics

In epistemic logics, the multimodal language is used to reason about knowledge. Let $n \geq 1$ and $AG = \{1, \ldots, n\}$ be the set of agents.

- We consider the multimodal language $ML_{Ag}$.
- We write, for every $i = 1, \ldots, n$, $K_i\varphi$ instead of $\Box_i\varphi$.
- $K_i\varphi$ is read as the agent $i$ knows (that) $\varphi$.
- We denote by $\hat{K}_i$ the dual operator: $\hat{K}_i\varphi = \neg K_i \neg\varphi$.
- Then $\hat{K}_i\varphi$ is read as the agent $i$ considers possible (that) $\varphi$.

## Epistemic logics

*Definition 5.1*

*An epistemic logic is a set $\Lambda$ of formulas of $ML_{Ag}$ satisfying the following properties:*

- *$\Lambda$ contains all propositional tautologies and is closed under modus ponens.*
- *$\Lambda$ contains all formulas*

$$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi),$$

  *where $\varphi, \psi$ are formulas and $i \in Ag$.*

- *$\Lambda$ is closed under generalization: for any formula $\varphi$ and all $i \in Ag$,*

$$\frac{\varphi}{K_i\varphi}.$$

We denote by ***K*** the smallest epistemic logic.

## Epistemic logics

Recall the following axioms:

$$\begin{array}{rc} (T) & K_i\varphi \to \varphi \\ (D') & \neg K_i(\varphi \land \neg\varphi) \\ (B) & \varphi \to K_i\neg K_i\neg\varphi \end{array}$$

### *Properties of knowledge*

- Axiom $(T)$ is called the veridity or knowledge axiom: If an agent knows $\varphi$, then $\varphi$ must hold. What is known is true. This is often taken to be the property that distinguishes knowledge from other informational attitudes, such as belief.
- Axiom $(D')$ is the consistency axiom: an agent does not know both $\varphi$ and $\neg\varphi$. An agent cannot know a contradiction.
- Axiom $(B)$ says that if $\varphi$ holds, then an agent knows that it does not know $\neg\varphi$.

## Epistemic logics

Recall the following axioms:

$$\begin{array}{rl} (4) & K_i\varphi \to K_iK_i\varphi \\ (5') & \neg K_i\varphi \to K_i\neg K_i\varphi \end{array}$$

### *Properties of knowledge*

- Axiom (4) is positive introspection: if an agent knows $\varphi$, it knows that it knows $\varphi$. An agent knows what it knows.
- Axiom (5′) is negative introspection: if an agent does not know $\varphi$, it knows that it does not know $\varphi$. An agent is aware of what it doesn't know.
- Positive and negative introspection together imply that an agent has perfect knowledge about what it does and does not know.

## *Epistemic logics*

Let $\boldsymbol{S}5 = \boldsymbol{KD'B}4 = \boldsymbol{KD'B}5' = \boldsymbol{KT}5'$. $\boldsymbol{S}5$ is considered as the logic of idealised knowledge.

*Theorem 5.2*

*$\boldsymbol{S}5$ is sound and strongly complete with respect to the class of frames whose relations are equivalence relations.*

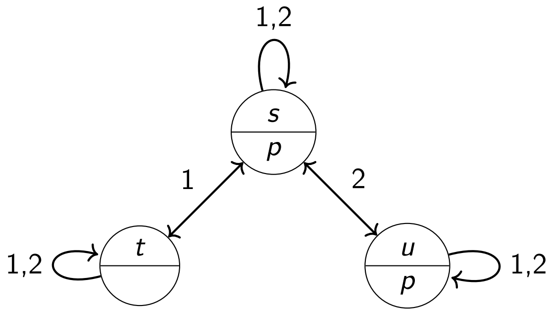
## *Models for epistemic logics*

A model $M = (W, \mathcal{K}_1, \ldots, \mathcal{K}_n, V)$ is represented as a labeled directed graph:

- the nodes of the graph are the states of the model;
- the label of each node $w \in W$ describes which atomic propositions are true at state $w$;
- we label edges by sets of agents; the label of the edge from node $w$ to node $v$ includes $i$ iff $\mathcal{K}_i wv$ holds.

## Models for epistemic logics

### Example

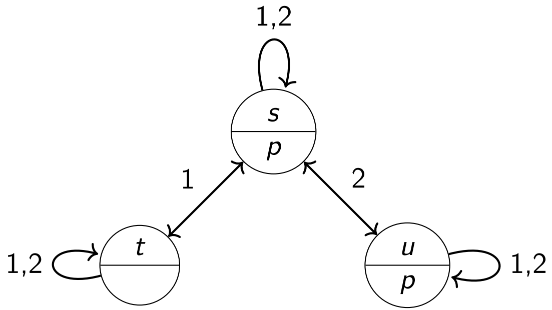


We have that $Ag = \{1, 2\}$, $PROP = \{p\}$ and $\mathcal{M} = (W, \mathcal{K}_1, \mathcal{K}_2, V)$, where

- $W = \{s, t, u\}$.
- $\mathcal{K}_1 = \{(s, s), (t, t), (u, u), (s, t), (t, s)\}$.
- $\mathcal{K}_2 = \{(s, s), (t, t), (u, u), (s, u), (u, s)\}$.
- $V(p) = \{s, u\}$.

## Models for epistemic logics

### Example



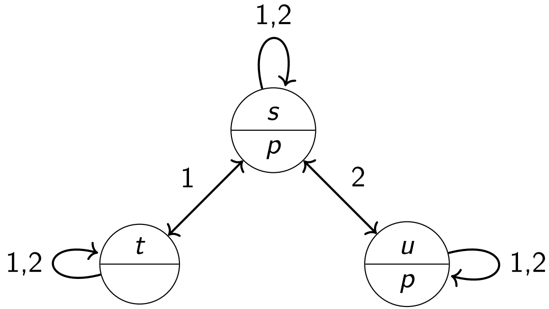
- $\mathcal{M}, s \Vdash p \wedge \neg K_1 p$.

**Proof:** We have that $s \in V(p)$, hence $\mathcal{M}, s \Vdash p$. Since $\mathcal{K}_1 st$ and $\mathcal{M}, t \nVdash p$, it follows that $\mathcal{M}, s \nVdash K_1 p$, hence $\mathcal{M}, s \Vdash \neg K_1 p$. Thus, $\mathcal{M}, s \Vdash p \wedge \neg K_1 p$. $\square$

In state $s$, $p$ is true, but agent 1 does not know it, since in state $s$ it considers both $s$ and $t$ possible. We say that agent 1 cannot distinguish $s$ from $t$. Agent 1's information is insufficient to enable it to distinguish whether the actual state is $s$ or $t$.

## *Models for epistemic logics*

*Example*



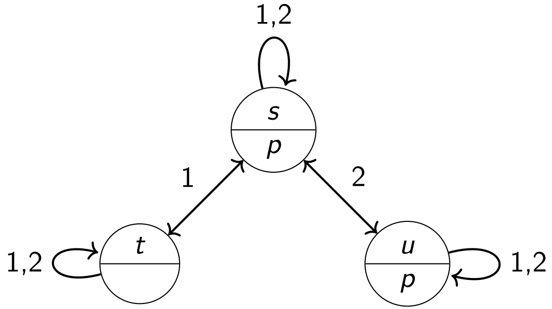
- $\mathcal{M}, s \Vdash K_2 p$.

**Proof:** We have that $\mathcal{M}, s \Vdash K_2 p$ iff for all $v \in W$, $\mathcal{K}_2 sv$ implies $\mathcal{M}, v \Vdash p$ iff $\mathcal{M}, s \Vdash p$ and $\mathcal{M}, u \Vdash p$ (as $\mathcal{K}_2 ss$, $\mathcal{K}_2 su$, but we don't have that $\mathcal{K}_2 st$), which is true. $\square$

In state $s$, agent 2 knows that $p$ is true, since $p$ is true in both states that agent 2 considers possible at $s$ (namely $s$ and $u$).

## *Epistemic logic* ***S*5**

*Example*

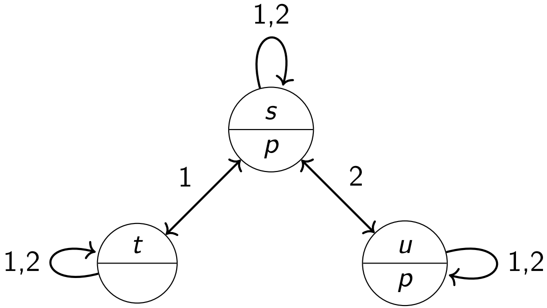


- $\mathcal{M}, s \Vdash \neg K_2 \neg K_1 p$.

**Proof:** We have that $\mathcal{M}, s \Vdash \neg K_2 \neg K_1 p$ iff $\mathcal{M}, s \nVdash K_2 \neg K_1 p$ iff there exists $v \in W$ such that $\mathcal{K}_2 sv$ and $\mathcal{M}, v \nVdash \neg K_1 p$ iff there exists $v \in W$ such that $\mathcal{K}_2 sv$ and $\mathcal{M}, v \Vdash K_1 p$. Take $v := u$. Then $\mathcal{K}_2 su$ and $\mathcal{M}, u \Vdash K_1 p$, since $\mathcal{M}, u \Vdash p$ and $\mathcal{K}_1 uw$ iff $w = u$. $\square$

## Models for epistemic logics

### Example



- $\mathcal{M}, s \Vdash \neg K_2 \neg K_1 p$.

Although agent 2 knows that $p$ is true in $s$, it does not know that agent 1 does not know this fact. Why? Because in a state that agent 2 considers possible, namely $u$, agent 1 does know that $p$ is true, while in another state agent 2 considers possible, namely $s$, agent 1 does not know this fact.

## A simple card game

$Ag = \{1, 2\}$

- Suppose that we have a deck consisting of three cards labeled $A$, $B$, and $C$. Agents 1 and 2 each get one of these cards; the third card is left face down.
- A possible world is characterized by describing the cards held by each agent. For example, in the world $(A, B)$, agent 1 holds card $A$ and agent 2 holds card $B$ (while card $C$ is face down).
- The set of possible worlds is
$$W = \{(A, B), (A, C), (B, A), (B, C), (C, A), (C, B)\}.$$
- In a world such as $(A, B)$, agent 1 thinks two worlds are possible: $(A, B)$ and $(A, C)$. Agent 1 knows that he has card $A$, but considers it possible that agent 2 could hold either card $B$ or card $C$.

## A simple card game

- Similarly, in world $(A, B)$, agent 2 also considers two worlds are possible: $(A, B)$ and $(C, B)$.
- In general, in a world $(X, Y)$, agent 1 considers $(X, Y)$ and $(X, Z)$ possible, while agent 2 considers $(X, Y)$ and $(Z, Y)$ possible, where $Z$ is different from both $X$ and $Y$.

- We can easily construct the $\mathcal{K}_1$ and $\mathcal{K}_2$ relations.
- It is easy to check that they are indeed equivalence relations.
- This is because an agent's possibility relation is determined by the information he has, namely, the card he is holding.

## A simple card game

We describe the frame $\mathcal{F}_c = (W, \mathcal{K}_1, \mathcal{K}_2)$ for the card game as a labeled graph. Since the relations are equivalence relations, we omit the self loops and the arrows on edges for simplicity (if there is an edge from state $w$ to state $v$, there must be an edge from $v$ to $w$ as well, by symmetry).

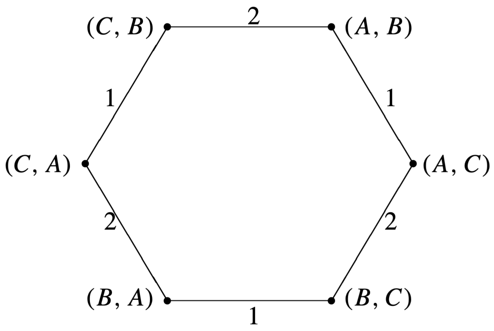
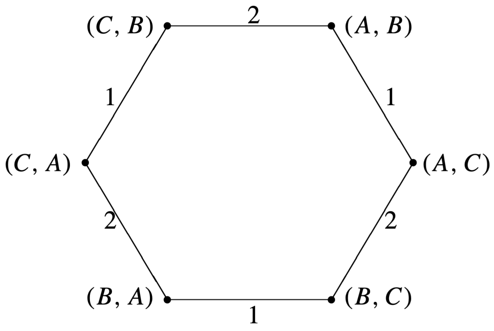


*Figure 7:* Frame describing a simple card game

## A simple card game



- In the world $(A, B)$, agent 1 knows that the world $(B, C)$ cannot be the case. This is captured by the fact that there is no edge from $(A, B)$ to $(B, C)$ labeled 1.
- Nevertheless, agent 1 considers it possible that agent 2 considers it possible that $(B, C)$ is the case. This is captured by the fact that there is an edge labeled 1 from $(A, B)$ to $(A, C)$, from which there is an edge labeled 2 to $(B, C)$.

## A simple card game

We still have not defined the model to be used in this example.

Define the set PROP of atomic propositions as

$$PROP = \{iX \mid i \in \{1,2\}, X \in \{A,B,C\}\}.$$

$iX$ will be interpreted as agent $i$ holds card $X$. Given this interpretation, we define the valuation $V$ in the obvious way:

$$V(iX) = \begin{cases} \{(X,Z) \mid Z \in \{A,B,C\} \setminus \{X\}\} & \text{if } i = 1 \\ \{(Z,X) \mid Z \in \{A,B,C\} \setminus \{X\}\} & \text{if } i = 2. \end{cases}$$

## *A simple card game*

Let $\mathcal{M}_c = (\mathcal{F}_c, V)$ be the model describing this card game.

*Examples*

- $\mathcal{M}_c, (A, B) \vDash 1A \wedge 2B$.
- $\mathcal{M}_c, (A, B) \vDash \mathsf{K}_1(2B \vee 2C)$: agent 1 knows that agent 2 holds either $B$ or $C$.
- $\mathcal{M}_c, (A, B) \vDash K_1\neg 2A$: agent 1 knows that agent 2 does not hold an $A$.
- $\mathcal{M}_c, (A, B) \vDash K_1\neg K_2 1A$: agent 1 knows that agent 2 does not know that agent 1 holds an $A$.

## Common and distributed knowledge

We need to reason about knowledge in a group and using this understanding to help us analyze multiagent systems.

- An agent in a group must take into account not only facts that are true about the world, but also the knowledge of other agents in the group.
- For example, in a bargaining situation, the seller of a car must consider what the potential buyer knows about the car's value. The buyer must also consider what the seller knows about what the buyer knows about the car's value, and so on.
- Such reasoning can get rather convoluted. Example: "Dean doesn't know whether Nixon knows that Dean knows that Nixon knows that McCord burgled O'Brien's office at Watergate."
- But this is precisely the type of reasoning that is needed when analyzing the knowledge of agents in a group.

## Common and distributed knowledge

We are often interested in situations in which everyone in the group knows a fact.

*Example*

A society certainly wants all drivers to know that a red light means stop and a green light means go. Suppose we assume that every driver in the society knows this fact and follows the rules. A driver does nor feel safe, unless she also knows that everyone else knows and is following the rules. Otherwise, a driver may consider it possible that, although she knows the rules, some other driver does not, and that driver may run a red light.

## Common and distributed knowledge

- In some cases we also need to consider the state in which simultaneously everyone knows a fact $\varphi$, everyone knows that everyone knows $\varphi$, everyone knows that everyone knows that everyone knows $\varphi$, and so on. We say that the group has common knowledge of $\varphi$.
- The notion of common knowledge was first studied by the philosopher David Lewis in the context of conventions: in order for something to be a convention, it must be common knowledge among the members of a group.
- John McCarthy, in the context of studying common-sense reasoning, characterized common knowledge as what any fool knows.

### Example

The convention that green means go and red means stop is presumably common knowledge among the drivers in our society.

## Common and distributed knowledge

- Common knowledge also arises in discourse understanding.
- Suppose that Ann asks Bob "What did you think of the movie?"" referring to the movie Star Wars they have just seen. Ann and Bob must both know that the movie refers to Star Wars, Ann must know that Bob knows (so that she can be sure that Bob will give a reasonable answer to her question), Bob must know that Ann knows that Bob knows (so that Bob knows that Ann will respond appropriately to his answer), and so on.
- There must be common knowledge of what movie is meant in order for Bob to answer the question appropriately.

- Common knowledge also turns out to be a prerequisite for achieving agreement.
- That is why common knowledge is a crucial notion in the analysis of interacting groups of agents.

## Common and distributed knowledge

A group has distributed knowledge of a fact $\varphi$ if the knowledge of $\varphi$ is distributed among its members, so that by using their knowledge together the members of the group can deduce $\varphi$, even though it may be the case that no member of the group individually knows $\varphi$.

### *Example*

Assume that Alice knows that Bob is in love with either Carol or Susan, and Charlie knows that Bob is not in love with Carol. Then together Alice and Charlie have distributed knowledge of the fact that Bob is in love with Susan, although neither Alice nor Charlie individually has this knowledge.

## Common and distributed knowledge

Let $\emptyset \neq G \subseteq Ag$ be a group of agents.

- Define, for every $\varphi$,

$$E_G\varphi = \bigwedge_{i\in G} K_i\varphi.$$

- $E_G\varphi$ is read as everyone in the group $G$ knows $\varphi$.
- For every model $\mathcal{M}$ and every $w \in \mathcal{M}$,

$$\mathcal{M}, w \Vdash E_G\varphi \text{ iff } \mathcal{M}, w \Vdash K_i\varphi \text{ for all } i \in G.$$

## Common and distributed knowledge

The language $ML_{Ag}$ does not allow us to express the notions of common knowledge and distributed knowledge.

Let $ML_{Ag}^{CD}$ be the language obtained by adding to $ML_{Ag}$ the following modal operators for any $\emptyset \neq G \subseteq Ag$:

- $C_G$, read as it is common knowledge among the agents in $G$;
- $D_G$, read as it is distributed knowledge among the agents in $G$.

Formulas of $ML_{Ag}^{CD}$ are defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid K_i\varphi \mid C_G\varphi \mid D_G\varphi,$$

where $p \in PROP$, $i \in Ag$ and $\emptyset \neq G \subseteq Ag$.

We omit the subscript $G$ when $G$ is the set of all agents.

## Common and distributed knowledge

Let $\emptyset \neq G \subseteq Ag$ be a group of agents.

We define $E_G^k\varphi$ $(k \geq 0)$ inductively:

$$E_G^0\varphi = \varphi, \qquad E_G^{k+1}\varphi = E_G E_G^k\varphi.$$

Let $\mathcal{M}$ be a model and $w \in \mathcal{M}$. We extend the definition of the satisfaction relation with the following clause:

$$\mathcal{M}, w \Vdash C_G\varphi \text{ iff } \mathcal{M}, w \Vdash E_G^k\varphi \text{ for all } k = 1, 2, \ldots.$$

Thus, the formula $C_G\varphi$ is true iff everyone in $G$ knows $\varphi$, everyone in $G$ knows that everyone in $G$ knows $\varphi$, etc.

## *Common and distributed knowledge*

Our definition of common knowledge has a graph-theoretical interpretation.
Let $\mathcal{M}$ be a model.

*Definition 5.3*

*Let $w, v$ be states in $\mathcal{M}$.*

- *We say that $v$ is G-reachable from $w$ in $k$ steps ($k \geq 1$) if there exist states $u_0, u_1, \ldots, u_k \in \mathcal{M}$ such that $u_0 = w$, $u_k = v$ and for all $j = 0, \ldots, k-1$, there exists $i \in G$ such that $\mathcal{K}_i u_j u_{j+1}$.*
- *$v$ is G-reachable from $w$ if $v$ is G-reachable from $w$ in $k$ steps for some $k \geq 1$.*

Thus, $v$ is $G$-reachable from $w$ iff there is a path in the graph from $w$ to $v$ whose edges are labeled by members of $G$.

## Common and distributed knowledge

*Proposition 5.4*

*Let $w$ be a state in $\mathcal{M}$.*

- *The following are equivalent for every $k \geq 1$:*
  - *$\mathcal{M}, w \Vdash E_G^k \varphi$;*
  - *$\mathcal{M}, v \Vdash \varphi$ for all states $v$ that are $G$-reachable from $w$ in $k$ steps.*
- *$\mathcal{M}, w \Vdash C_G \varphi$ iff $\mathcal{M}, v \Vdash \varphi$ for all states $v$ that are $G$-reachable from $w$.*

## *Common and distributed knowledge*

A group $G$ has distributed knowledge of $\varphi$ if the combined knowledge of the members of $G$ implies $\varphi$.

- The question is how can we capture the idea of combining knowledge in our framework.
- The answer is: we combine the knowledge of the agents in group $G$ by eliminating all worlds that some agent in $G$ considers impossible.

Let $\mathcal{M}$ be a model and $w \in \mathcal{M}$. We extend the definition of the satisfaction relation with the following clause:

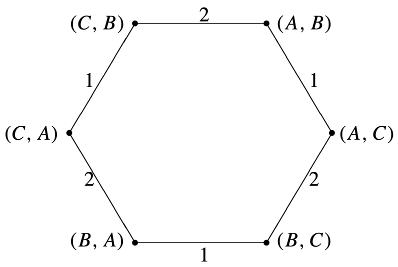
$$\begin{array}{lll} \mathcal{M}, w \Vdash D_G\varphi & \text{iff} & \mathcal{M}, v \Vdash \varphi \text{ for all } v \text{ such that } (w, v) \in \bigcap_{i \in G} \mathcal{K}_i \\ & \text{iff} & \mathcal{M}, v \Vdash \varphi \text{ for all } v \text{ such that } \mathcal{K}_i wv \text{ for all } i \in G. \end{array}$$

## Example - the card game

Let $\mathcal{M}_c = (\mathcal{F}_c, V)$ be the model describing the simple card game.

- $PROP = \{iX \mid i \in \{1,2\}, X \in \{A, B, C\}\}$.
- $iX$ read as agent $i$ holds card $X$
- $V(iX) = \begin{cases} \{(X,Z) \mid Z \in \{A,B,C\} \setminus \{X\}\} & \text{if } i = 1 \\ \{(Z,X) \mid Z \in \{A,B,C\} \setminus \{X\}\} & \text{if } i = 2. \end{cases}$

$\mathcal{F}_c$ is given by



## *Example - the card game*

Let $G = \{1, 2\}$.

- $\mathcal{M}_c \Vdash C_G(1A \vee 1B \vee 1C)$: it is common knowledge that agent 1 holds one of the cards $A$, $B$, and $C$.
- $\mathcal{M}_c \Vdash C_G(1B \rightarrow (2A \vee 2C))$: it is common knowledge that if agent 1 holds card $B$, then agent 2 holds either card $A$ or card $C$.
- $\mathcal{M}_c, (A, B) \Vdash D_G(1A \wedge 2B)$: if the agents could combine their knowledge, they would know that in world $(A, B)$, agent 1 holds card $A$ and agent 2 holds card $B$.

## Muddy children puzzle

- A group of $n$ children enters their house after having played in the mud outside. They are greeted in the hallway by their father, who notices that $k$ of the children have mud on their foreheads.
- He makes the following announcement, "At least one of you has mud on his forehead."
- The children can all see each other's foreheads, but not their own.
- The father then says, "Do any of you know that you have mud on your forehead? If you do, raise your hand now."
- No one raises his hand.
- The father repeats the question, and again no one moves.
- The father does not give up and keeps repeating the question.
- After exactly $k$ repetitions, all the children with muddy foreheads raise their hands simultaneously.

## *Muddy children puzzle*

For simplicity, let us call a child

- muddy if he has a muddy forehead;
- clean if he does not have a muddy forehead.

$k = 1$

- There exists only one muddy child.
- The muddy child knows the other children are clean.
- When the father says at least one is muddy, he concludes that it's him.
- None of the other children know at this point whether or not they are muddy.
- The muddy child raises his hand after the father's first question.
- After the muddy child raises his hand, the other children know that they are clean.

## *Muddy children puzzle*

$k = 2$

- There exist two muddy children.
- Imagine that you are one of the two muddy children.
- You see that one of the other children is muddy.
- After the father's first announcement, you do not have enough information to know whether you are muddy. You might be, but it could also be that the other child is the only muddy one.
- So, you do not raise the hand after the father's first question.
- You note that the other muddy child does not raise his hand.
- You realize then that you yourself must be muddy as well, or else that child would have raised his hand.
- So, after the father's second question, you raise your hand. Of course, so does the other muddy child.

## *Muddy children puzzle*

We shall analyse the muddy children puzzle using epistemic logic.

We assume that it is common knowledge that

- the father is truthful,
- all the children can and do hear the father,
- all the children can and do see which of the other children besides themselves have muddy foreheads,
- none of the children can see his own forehead,
- all the children are truthful and (extremely) intelligent.

## Muddy children puzzle

Suppose that there are $n$ children; we number them $1, \ldots, n$. Thus, we take $Ag = \{1, \ldots, n\}$.

- First consider the situation before the father speaks.
- Some of the children are muddy, while the rest are clean.
- We describe a possible situation by an $n$-tuple of 0's and 1's of the form $(x_1, \ldots, x_n)$, where $x_i = 1$ if child $i$ is muddy, and $x_i = 0$ otherwise.
- There are $2^n$ possible situations.

## Muddy children puzzle

$n = 3$

- Suppose that the actual situation is described by the tuple $(1, 0, 1)$, that says that child 1 and child 3 are muddy, while child 2 is clean.
- What situations does child 1 consider possible before the father speaks?
- Since child 1 can see the foreheads of all the children besides himself, his only doubt is about whether he is muddy or clean. Thus child 1 considers two situations possible: $(1, 0, 1)$ (the actual situation) and $(0, 0, 1)$. Similarly, child 2 considers two situations possible: $(1, 0, 1)$ and $(1, 1, 1)$.

In general, child *i* has the same information in two possible situations exactly if they agree in all components except possibly the *i*th component.

## Muddy children puzzle

We can capture the general situation by the frame

$$\mathcal{F} = (W, \mathcal{K}_1, \ldots, \mathcal{K}_n),$$

where

- $W = \{(x_1, \ldots, x_n) \mid x_i \in \{0,1\} \text{ for all } i = 1, \ldots, n\}$. Thus, $W$ has $2^n$ states.
- For every $i = 1, \ldots, n$,

  $\mathcal{K}_i wv$ iff $w$ and $v$ agree in all components except possibly the $i$th component.
- One can easily see that $\mathcal{K}_i$'s are equivalence relations.

Thus, $\mathcal{F}$ is a frame for the epistemic logic $\boldsymbol{S}5$.

## *Muddy children puzzle*

It remains to define $PROP$ and the valuation $V : PROP \to 2^W$.

- Since we want to reason about whether or not a given child is muddy, we take $PROP = \{p_1, \ldots, p_n, p\}$, where, intuitively, $p_i$ stands for child $i$ is muddy, while $p$ stands for at least one child is muddy.
- We define $V$ as follows:

$$V(p_i) = \{(x_1, \ldots, x_n) \in W \mid x_i = 1\},$$
$$V(p) = \{(x_1, \ldots, x_n) \in W \mid x_j = 1 \text{ for some } j = 1, \ldots, n\}.$$

- It follows that

$$\mathcal{M}, (x_1, \ldots, x_n) \Vdash p_i \text{ iff } x_i = 1,$$
$$\mathcal{M}, (x_1, \ldots, x_n) \Vdash p \text{ iff } x_j = 1 \text{ for some } j = 1, \ldots, n.$$

We have a model with $2^n$ nodes, each described by an $n$-tuple of 0's and 1's, such that two nodes are joined by an edge exactly if they differ in at most one component.

## Muddy children puzzle

Recall that we omit self-loops and the arrows on edges.

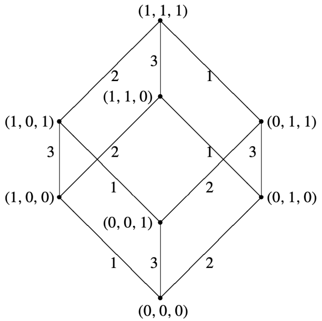
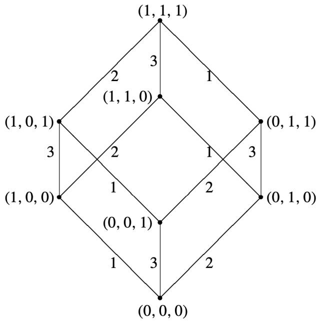


*Figure 8:* Frame for the muddy children puzzle with $n = 3$

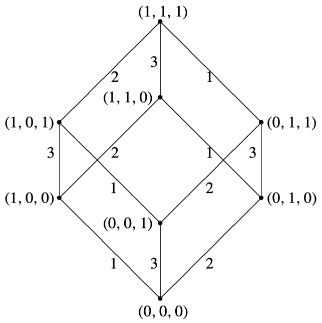
- $\mathcal{M}, (1,0,1) \Vdash K_1 \neg p_2$;
- $\mathcal{M}, (1,0,1) \Vdash K_1 p_3$;
- $\mathcal{M}, (1,0,1) \Vdash \neg K_1 p_1$;

## Muddy children puzzle



- $\mathcal{M} \Vdash C(p_2 \to K_1 p_2)$: it is common knowledge that if child 2 is muddy, then child 1 knows it.
- $\mathcal{M} \Vdash C(\neg p_2 \to K_1 \neg p_2)$: it is common knowledge that if child 2 is clean, then child 1 knows it.

## Muddy children puzzle



- $\mathcal{M}, (1,0,1) \Vdash Ep$ : in $(1,0,1)$, every child knows that at least one child is muddy even before the father speaks;
- $\mathcal{M}, (1,0,1) \Vdash \neg E^2 p$: $p$ is not true at the state $(0,0,0)$ that is reachable in two steps from $(1,0,1)$.

## *Muddy children puzzle*

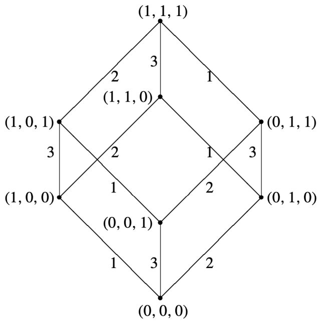
One can check that in the general case, if we have $n$ children of whom $k$ are muddy (so that the situation is described by an $n$-tuple exactly $k$ of whose components are 1's), then $E^{k-1}p$ is true, but $E^k p$ is not, since each state reachable in $k-1$ steps has at least one 1 (and so there is at least one muddy child), but the tuple $(0, \ldots, 0)$ is reachable in $k$ steps.

## *Muddy children puzzle*

Let us consider what happens after the father speaks.

- The father says $p$, which is already known to all the children if there are two or more muddy children.
- Nevertheless, the state of knowledge changes, even if all the children already know $p$.

## *Muddy children puzzle*



- In $(1,0,1)$, child 1 considers the situation $(0,0,1)$ possible and in $(0,0,1)$ child 3 considers $(0,0,0)$ possible.
- In $(1,0,1)$, before the father speaks, although everyone knows that at least one is muddy, child 1 thinks it possible that child 3 thinks it possible that none of the children is muddy.
- After the father speaks, it becomes common knowledge that at least one child is muddy.

## *Muddy children puzzle*

- In the general case, we can represent the change in the group's state of knowledge graphically by simply removing the point $(0, 0, \ldots, 0)$ from the cube.
- More accurately, what happens is that the node $(0, 0, \ldots, 0)$ remains, but all the edges between$(0, 0, \ldots, 0)$ and nodes with exactly one 1 disappear, since it is common knowledge that even if only one child is muddy, after the father speaks that child will not consider it possible that no one is muddy.

## Muddy children puzzle

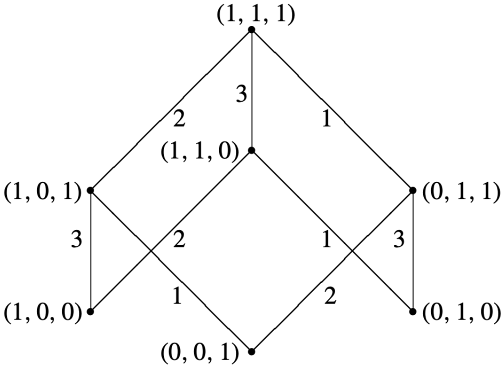


*Figure 9:* Frame for $n = 3$ after the father speaks

## Muddy children puzzle

Let us show that each time the children respond to the father's question with a No, the group's state of knowledge changes and the cube is further truncated.

- Consider what happens after the children respond No to the father's first question.
- Now all the nodes with exactly one 1 can be eliminated. (More accurately, the edges to these nodes from nodes with exactly two 1's all disappear from the graph.)
- Nodes with one or fewer 1's are no longer reachable from nodes with two or more 1's.

## Muddy children puzzle

- If the actual situation were described by, say, the tuple $(1, 0, \ldots, 0)$, then child 1 would initially consider two situations possible: $(1, 0, \ldots, 0)$, and $(0, 0, \ldots, 0)$.
- Since once the father speaks it is common knowledge that $(0, 0, \ldots, 0)$ is not possible, he would then know that the situation is described by $(1, 0, \ldots, 0)$, and thus would know that he is muddy.
- Once everyone answers No to the father's first question, it is common knowledge that the situation cannot be $(1, 0, \ldots, 0)$.
- Similar reasoning allows us to eliminate every situation with exactly one 1. Thus, after all the children have answered No to the father's first question, it is common knowledge that at least two children are muddy.

## *Muddy children puzzle*

- Further arguments in the same spirit can be used to show that after the children answer No $k$ times, we can eliminate all the nodes with at most $k$ 1's (or, more accurately, disconnect these nodes from the rest of the graph).
- We thus have a sequence of frames, describing the children's knowledge at every step in the process.
- Essentially, what is going on is that if, in some node $w$, it becomes common knowledge that a node $v$ is impossible, then for every node $u$ reachable from $w$, the edge from $u$ to $v$ (if there is one) is eliminated.

## Muddy children puzzle

- After $k$ rounds of questioning, it is common knowledge that at least $k+1$ children are muddy.
- If the true situation is described by a tuple with exactly $k+1$ 1's, then before the father asks the question for the $(k+1)$st time, the muddy children will know the exact situation, and in particular will know they are muddy, and consequently will answer Yes.
- Note that they could not answer Yes any earlier, since up to this point each muddy child considers it possible that he or she is clean.

## Muddy children puzzle

- According to the way we are modeling knowledge in this context, a child knows a fact if the fact follows from his or her current information.
- However, if one of the children were not particularly bright, then he might not be able to figure out that he knew that he is muddy, even though in principle he had enough information to do so.
- To answer Yes to the father's question, the child must actually be aware of the consequences of his information.
- Our definition implicitly assumes that (it is common knowledge that) all reasoners are logically omniscient, that is they are smart enough to compute all the consequences of the information that they have.
- Furthermore, this logical omniscience is common knowledge.

## Muddy children puzzle

- Consider now the situation in which the father does not initially say $p$.
- Before the father speaks the situation is described by the $n$-dimensional cube.
- When the father asks for the first time Does any of you know whether you have mud on your own forehead?, clearly all the children say No, since in every situation each child considers possible a situation in which he is clean.
- No information is gained from this answer, so the situation still can be represented by the $n$-dimensional cube.
- One shows by induction on $m$ that it is common knowledge that the father's $m$th question is also answered No and the state of knowledge after the father asks the $m$th question is still described by the cube.
- Hence, children's state of knowledge never changes, no matter how many times the father asks questions.

## *Partition model of knowledge*

Partition models of knowledge are defined in

Yoav Shoham, Kevin Leyton-Brown, Multiagents Systems, Cambridge University Press, 2009

Let $n \geq 1$ and $AG = \{1, \ldots, n\}$ be the set of agents.

### *Definition 5.5 (Partition frame)*

*A partition frame is a tuple* $\mathcal{P}_F = (W, I_1, \ldots, I_n)$*, where*

- $W$ *is a nonempty set of possible worlds.*
- *For every* $i = 1, \ldots, n$, $I_i$ *is a partition of* $W$*.*

The idea is that $I_i$ partitions $W$ into sets of possible worlds that are indistinguishable from the point of view of agent $i$.

## *Partition model of knowledge*

Recall: Let $A$ be a nonempty set. A partition of $A$ is a family $(A_j)_{j\in J}$ of nonempty subsets of $A$ satisfying the following properties:

$$A = \bigcup_{j\in J} A_j \text{ and } A_j \cap A_k = \emptyset \text{ for all } j \neq k.$$

Recall: Let $A$ be a nonempty set. There exists a bijection between the set of partitions of $A$ and the set of equivalence relations on $A$:

- $(A_j)_{j\in J}$ partition of $A \mapsto$ the equivalence relation on $A$ defined by $x \sim y \Leftrightarrow$ there exists $j \in J$ such that $x, y \in A_j$.
- $\sim$ equivalence relation on $A \mapsto$ the partition consisting of all the different equivalence classes of $\sim$.

## *Partition model of knowledge*

- For each $i = 1, \ldots, n$, let $R_{I_i}$ be the corresponding equivalence relation.
- Denote by $I_i(w)$ the equivalence class of $w$ in the relation $R_{I_i}$.
- If the actual world is $w$, then $I_i(w)$ is the set of possible worlds that agent $i$ cannot distinguish from $w$.
- $\mathcal{F} = (W, R_{I_i}, \ldots, R_{I_n})$ is a frame for the epistemic logic $\boldsymbol{S}5$.

Partition frame $=$ frame for the epistemic logic $\boldsymbol{S}5$

## Partition model of knowledge

*Definition 5.6 (Partition model)*

*A partition model over a language $\Sigma$ is a tuple $\mathcal{P}_M = (\mathcal{P}_F, \pi)$, where*

- $\mathcal{P}_F = (W, I_1, \ldots, I_n)$ *is a partition frame.*
- $\pi : \Sigma \to 2^W$ *is an interpretation function.*

For every statement $\varphi \in \Sigma$, we think of $\pi(\varphi)$ as the set of possible worlds in the partition model $\mathcal{P}_M$ where $\varphi$ is satisfied.

- Each possible world completely specifies the concrete state of affairs.
- We can take, for example, $\Sigma$ to be a set of formulas in propositional logic over some set of atomic propositions.

## *Partition model of knowledge*

We will use the notation $K_i\varphi$ as "agent $i$ knows that $\varphi$".

The following defines when a statement is true in a partition model.

*Definition 5.7 (Logical entailment for partition models)*

*Let $\mathcal{P}_M = (W, I_1, \ldots, I_n, \pi)$ be a partition model over $\Sigma$, and $w \in W$. We define the $\models$ (logical entailment) relation as follows:*

- *For any $\varphi \in \Sigma$, we say that $\mathcal{P}_M, w \models \varphi$ iff $w \in \pi(\varphi)$.*
- *$\mathcal{P}_M, w \models K_i\varphi$ iff for all worlds $v \in W$, if $v \in I_i(w)$, then $\mathcal{P}_M, v \models \varphi$.*

Partition model = model for the epistemic logic $\boldsymbol{S}5$

We can reason about knowledge rigorously in terms of partition models, hence using epistemic logic.

## Multiagent systems

Textbook:

R. Fagin, J.Y. Halpern, Y. Moses, M. Vardi, Reasoning About Knowledge, MIT Press, 2004

## Runs and systems

## Multiagent systems

- A multiagent system is any collection of interacting agents.
- One of the major application areas of reasoning about knowledge: multiagent systems.

### Examples

- The children and the father in the muddy children puzzle are agents in a multiagent system.
- A game such as poker is a multiagent system. We refer to agents as players.
- A distributed system consisting of processes in a computer network running a particular protocol is a multiagent system. We refer to agents as processes or sites.

## Multiagent systems

We define in the sequel a formal model of multiagent systems that is general enough to allow us to capture all the important features of multiagent systems.

### *Key assumptions*

- We look at the system at any point in time, each of the agents is in a unique state. We refer to this as the agent's local state.
- An agent's local state encapsulates all the information to which the agent has access.
- We do not make any additional assumptions about the local state.

## *Multiagent systems*

Assume that $n \geq 1$ and $AG = \{1, \ldots, n\}$ is the set of agents.

- As each agent has a state, we think of the whole system as being in some state.
- First idea: take the system's state to be a tuple of the form $(s_1, \ldots, s_n)$, where $s_i$ is the local state of agent $i$.
- However, in general, more than just the local states of the agents may be relevant when analyzing a system.

*Examples*

- Consider a message-passing system where processes send messages back and forth along communication lines. We would like to know about messages that are in transit or about whether a communication line is up or down.
- Consider a system of sensors observing some terrain. We need to include features of the terrain in a description of the state of the system.

## *Multiagent systems*

We conceptually divide a system into two components:

- the agents and
- the environment,

where we view the environment as everything else that is relevant. The environment can be viewed as just another agent, though it typically plays a special role in our analyses.

We define a global state of a system to be an $(n+1)$-tuple of the form $(s_e, s_1, \ldots, s_n)$, where $s_e$ is the state of the environment and $s_i$ is the local state of agent $i$.

## Multiagent systems

- A global state describes the system at a given point in time. But a system is not a static entity; it constantly changes.
- Since we are mainly interested in how systems change over time, we need to build time into our model.
- We define a run to be a function from time to global states.
- Intuitively, a run is a complete description of how the system's global state evolves over time.
- We take time to range over the natural numbers. Thus, time steps are discrete and time is infinite.
- The initial global state of the system in a possible execution $r$ is $r(0)$, the next global state is $r(1)$, and so on.
- The assumption that time is discrete is a natural one, as computers proceed in discrete time steps. Allowing time to be infinite makes it easier to model situations where there is no a priori time bound on how long the system will run.

## Multiagent systems

- We assume that time is measured on some clock external to the system.
- We do not assume that agents in the system necessarily have access to this clock; at time $m$ measured on the external clock, agent $i$ need not know it is time $m$.
- If an agent does know the time, then this information would be encoded in his local state.
- This external clock need not measure real time.
- We model the external clock in whatever way makes it easiest for us to analyze the system.
- In the case of the muddy children puzzle, there could be one tick of the clock for every round of questions by the father and every round of answers to the father's question.
- In a poker game, there could be one tick of the clock each time someone bets or discards.

## Multiagent systems

A system can have many possible runs, since the system's global state can evolve in many possible ways: there are a number of possible initial states and many things that could happen from each initial global state.

- The basic idea is to define a system to be a nonempty set of runs.
- Instead of trying to model the system directly, the definition models the possible behaviors of the system.
- The set of runs is nonempty, as the system we are modeling should have some behaviors.

We use the term system in two ways: as the real-life collection of interacting agents or as a set of runs.

## *Multiagent systems*

- Let $L_e$ be a set of possible states for the environment and let $L_i$ be a set of possible local states for agent $i$ for $i = 1, \dots, n$.
- We take $\mathcal{G} = L_e \times L_1 \times \dots \times L_n$ to be the set of global states.

### *Definition 6.1*

*A run over $\mathcal{G}$ is a function $r : \mathbb{N} \to \mathcal{G}$.*

A run $r$ over $\mathcal{G}$ can be identified with the sequence $(r(m))_{m\in\mathbb{N}}$ of global states in $\mathcal{G}$.

### *Definition 6.2*

*We refer to a pair $(r, m)$ consisting of a run $r$ and time $m$ as a point. We say that $r(m)$ is the global state at the point $(r, m)$.*

## Multiagent systems

A round takes place between two time points. We define round $m$ in run $r$ to take place between time $m-1$ and time $m$. We view an agent as performing an action during a round.

Let $(r, m)$ be a point and $r(m) = (s_e, s_1, \ldots, s_n)$ be the global state at $(r, m)$. We define

$$r_e(m) = s_e \text{ and } r_i(m) = s_i \text{ for all } i = 1, \ldots, n.$$

Thus, $r_i(m)$ is agent $i$'s local state at the point $(r, m)$.

### Definition 6.3

*A system $\mathcal{R}$ over $\mathcal{G}$ is a set of runs over $\mathcal{G}$. We say that $(r, m)$ is a point in system $\mathcal{R}$ if $r \in \mathcal{R}$. We denote by $\mathcal{P}_{\mathcal{R}}$ the set of points of $\mathcal{R}$.*

In practice, the appropriate set of runs will be chosen by the system designer or the person analyzing the system.

## *Bit-transmission problem*

- Imagine we have two processes, say a sender $S$ and a receiver $R$, that communicate over a communication line.
- The sender starts with one bit (either 0 or 1) that it wants to communicate to the receiver.
- The communication line is faulty. There is no guarantee that a message sent by either $S$ or $R$ will be received.
- We assume that a message is either received in the same round that it is sent, or lost altogether. Thus, rounds are long enough for a message to be sent and delivered.
- This type of message loss is the only possible faulty behavior in the system.

## *Bit-transmission problem*

- Because of the uncertainty regarding possible message loss, $S$ sends the bit to $R$ in every round, until $S$ receives a message from $R$ acknowledging receipt of the bit. We call this message from $R$ an ack message.
- $R$ starts sending the ack message in the round after it receives the bit. To allow $S$ to stop sending the bit, $R$ continues to send the ack repeatedly from then on.

This informal description gives what we call a protocol for $S$ and $R$: it is a specification of what they do at each step.

## Bit-transmission problem

- $S$ sends the bit to $R$ until $S$ receives the ack message; before it receives the ack message, $S$ does not know whether $R$ received the bit.
- $R$ knows perfectly well that $S$ stops sending messages after it receives an ack message, but $R$ never knows for certain that $S$ actually received its acknowledgment.
- Even if $R$ does not receive messages from $S$ for a while, $R$ does not know whether this is because $S$ received an ack message from $R$; it could be because the messages that $S$ sent were lost in the communication channel.

## Bit-transmission problem

- We could have $S$ send an ack-ack message (an acknowledgment to the acknowledgment) so that $R$ could stop sending the acknowledgment once it receives an ack-ack message from $S$.
- But this only pushes the problem up one level: $S$ will not be able to safely stop sending ack-ack messages, since $S$ has no way of knowing that $R$ has received an ack-ack message.
- This type of uncertainty is inherent in systems where communication is not guaranteed.

## *Bit-transmission problem*

We formalize the bit-transmission problem as a system.
To describe the set of runs that make up this system,

- we choose to have the local states of $S$ and $R$ include very little information; essentially, just enough to allow us to carry out our analysis.
- it is useful to have the environment's state record the events taking place in the system.

## Bit-transmission problem

Let $L_S$ be the set of possible local states of $S$. We take

$$L_S = \{0, 1, (0, ack), (1, ack)\}.$$

- $S$'s local state is $k \in \{0, 1\}$ if its initial bit is $k$ and it has not received an ack message from $R$.
- $S$'s local state is $(k, ack)$ if its initial bit is $k \in \{0, 1\}$ and it has received an ack message from $R$.

Let $L_R$ be the set of possible local states of $R$. We take

$$L_R = \{\lambda, 0, 1\}.$$

- $\lambda$ denotes the local state where $R$ has received no messages from $S$.
- $k \in \{0, 1\}$ denotes the local state where $R$ received the message $k$ from $S$.

## *Bit-transmission problem*

The environment's local state is used to record the history of events taking place in the system. At each round, we have the following possibilities:

- $S$ sends the bit to $R$ and $R$ does nothing.
  Notation: $(sendbit, \Lambda)$.
- $S$ does nothing and $R$ sends an ack to $S$.
  Notation: $(\Lambda, sendack)$.
- both $S$ and $R$ send messages. Notation: $(sendbit, sendack)$.

We let the environment's state be a finite sequence of elements from the set

$$\{(sendbit, \Lambda), (\Lambda, sendack), (sendbit, sendack)\}.$$

Here the $m^{\text{th}}$ member of the sequence describes the actions of the sender and receiver in round $m$.

## *Bit-transmission problem*

There are many possible runs in this system, but these runs must all satisfy certain constraints.

Initially, the system must start in a global state where nothing has been recorded in the environment's state, neither $S$ nor $R$ has received any messages, and $S$ has an initial bit of either 0 or 1. Thus, the initial global state of every run in the system has the form

$$((), k, \lambda),$$

where () is the empty sequence and $k \in \{0, 1\}$.

## *Bit-transmission problem*

Consecutive global states

$$r(m) = (s_e, s_S, s_R) \text{ and } r(m+1) = (s'_e, s'_S, s'_R) \text{ in a run } r$$

are related by the following conditions:

- If $s_R = \lambda$, then $s'_S = s_S$, $s'_e = s_e(sendbit, \Lambda)$ (where $s_e(sendbit, \Lambda)$ is the result of appending $(sendbit, \Lambda)$ to the sequence $s_e$), and either $s'_R = \lambda$ or $s'_R = s_S$.

  Before $R$ receives a message, it sends no messages; as a result, $S$ receives no message, so it continues to send the bit and its state does not change. $R$ may or may not receive the message sent by $S$ in round $m + 1$.

## *Bit-transmission problem*

Consecutive global states

$$r(m) = (s_e, s_S, s_R) \text{ and } r(m+1) = (s'_e, s'_S, s'_R) \text{ in a run } r$$

are related by the following conditions:

- If $s_S = s_R = k$, then $s'_R = k$, $s'_e = s_e(sendbit, sendack)$, and either $s'_S = k$ or $s'_S = (k, ack)$.

  After $R$ has received $S$'s bit, it starts sending acknowledgments, and its state undergoes no further changes. $S$ continues to send the bit, and it may or may not receive the acknowledgment sent by $R$ in round $m+1$.

## *Bit-transmission problem*

Consecutive global states

$$r(m) = (s_e, s_S, s_R) \text{ and } r(m+1) = (s'_e, s'_S, s'_R) \text{in a run } r$$

are related by the following conditions:

- If $s_S = (k, ack)$, then $s'_e = s_e(\Lambda, sendack)$, $s'_S = s_S$, and $s'_R = s_R$.

  Once $S$ has received $R$'s acknowledgment, $S$ stops sending the bit and $R$ continues to send acknowledgments. The local states of $S$ and $R$ do not change any more.

### *Definition 6.4*

*We take the system $\mathcal{R}^{bt}$ describing the bit-transmission problem to consist of all the runs meeting the constraints just described.*

## Incorporating knowledge and time

## *Incorporating knowledge*

- We already saw in the bit-transmission problem that we were making statements such as "R does not know for certain that S received its acknowledgment."
- An agent's actions depend on its knowledge.
- We shall see that knowledge can be incorporated in our framework in a straightforward way.
- The basic idea is that a statement such as R does not know $\varphi$ means that

  (*) as far as $R$ is concerned, the system could be at a point where $\varphi$ does not hold.
- We capture (*) using frames for epistemic logic.
- We think of $R$'s knowledge as being determined by its local state, so that $R$ cannot distinguish between two points of the system in which it has the same local state, and it can distinguish points in which its local state differs.

## Incorporating knowledge

Let $\mathcal{R}$ be a system over $\mathcal{G}$ and $i$ be an agent. Recall that $\mathcal{P}_{\mathcal{R}}$ denotes the set of points of $\mathcal{R}$.

*Definition 6.5*

*We define the binary relations $\sim_i$ on $\mathcal{G}$ and $\mathcal{K}_i$ on $\mathcal{P}_{\mathcal{R}}$ as follows:*

- *for all global states $s = (s_e, s_1, \ldots, s_n)$, $s' = (s'_e, t'_1, \ldots, t'_n)$,*

$$s \sim_i s' \quad \text{iff} \quad s_i = s'_i.$$

- *for all points $(r, m)$, $(r', m')$,*

$$(r, m)\,\mathcal{K}_i\,(r', m') \quad \text{iff} \quad r(m) \sim_i r'(m') \quad \text{iff} \quad r_i(m) = r'_i(m').$$

*If $s \sim_i s'$ or $(r, m)\,\mathcal{K}_i\,(r', m')$, we say that they are indistinguishable to agent $i$.*

## *Incorporating knowledge*

- For every agent $i$, $\sim_i$ and $\mathcal{K}_i$ are equivalence relations.
- $\mathcal{F}_{\mathcal{R}} = (\mathcal{P}_{\mathcal{R}}, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ is a frame for the epistemic logic ***S***5.

Thus, to every system $\mathcal{R}$ we associate a frame $\mathcal{F}_{\mathcal{R}}$ for the epistemic logic ***S***5.

Note that there is no relation $\mathcal{K}_e$ for the environment. This is because we are not usually interested in what the environment knows.

## Incorporating knowledge

Let $PROP$ be a set of atomic propositions.

### Definition 6.6

*An interpreted system is a pair $\mathcal{I} = (\mathcal{R}, \pi)$, where $\mathcal{R}$ is a system over a set $\mathcal{G}$ of global states and $\pi : \mathcal{G} \to (PROP \to \{0,1\})$ is an interpretation.*
*We also say that $\mathcal{I}$ is based on $\mathcal{R}$ or that $\mathcal{R}$ is the system underlying $\mathcal{I}$.*

Thus, $\pi$ assigns truth values to atomic propositions at global states: for every state $s \in \mathcal{G}$ and $p \in PROP$, $\pi(s)(p) \in \{0,1\}$.

### Remark 6.7

*$\pi$ induces also an interpretation over the points of $\mathcal{R}$. For every point $(r,m) \in \mathcal{R}$, take $\pi((r,m))$ to be $\pi(r(m))$. That is, for every $p \in PROP$,*

$$\pi((r,m))(p) = \pi(r(m))(p).$$

## Incorporating knowledge

We refer to the points and states of the system $\mathcal{R}$ as points and states, respectively, of the interpreted system $\mathcal{I}$. That is, we shall also use the notation $\mathcal{P}_{\mathcal{I}}$ for $\mathcal{P}_{\mathcal{R}}$.
Thus,

- $(r, m)$ is a point in the interpreted system $\mathcal{I} = (\mathcal{R}, \pi)$ iff $(r, m)$ is a point in $\mathcal{R}$ iff $r \in \mathcal{R}$.
- $\mathcal{I}$ is an interpreted system over $\mathcal{G}$ if $\mathcal{R}$ is a system over $\mathcal{G}$.

## Incorporating knowledge

Let $\mathcal{I} = (\mathcal{R}, \pi)$ be an interpreted system.

Define $V_{\mathcal{I}} : PROP \to 2^{\mathcal{P}_{\mathcal{I}}}$ as follows: for all $p \in PROP$,

$$\begin{aligned} V_{\mathcal{I}}(p) &= \{(r,m) \in \mathcal{P}_{\mathcal{I}} \mid \pi((r,m))(p) = 1\} \\ &= \{(r,m) \in \mathcal{P}_{\mathcal{I}} \mid \pi(r(m))(p) = 1\}. \end{aligned}$$

$\mathcal{M}_{\mathcal{I}} = (\mathcal{P}_{\mathcal{I}}, V_{\mathcal{I}}, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ is a model for the epistemic logic ***S***5.

Thus, to every interpreted system $\mathcal{I}$ we associate a model $\mathcal{M}_{\mathcal{I}}$ for the epistemic logic ***S***5.

## *Incorporating knowledge*

*Definition 6.8*

*For every formula $\varphi$ of $ML_{Ag}$, we say that*

$$\varphi \text{ is true in } \mathcal{I} \text{ at point } (r,m) \quad \textit{iff} \quad \mathcal{M}_{\mathcal{I}}, (r,m) \Vdash \varphi.$$

*Notation:* $(\mathcal{I}, r, m) \models \varphi$.

*Remark 6.9*

*For every $p \in PROP$, $(\mathcal{I}, r, m) \models p$ iff $\mathcal{M}_{\mathcal{I}}, (r,m) \Vdash p$ iff $(r,m) \in V_{\mathcal{I}}(p)$ iff $\pi((r,m))(p) = 1$ iff $\pi(r(m))(p) = 1$.*

As $\pi$ is a function on global states, the truth of an atomic proposition $p$ at a point $(r,m)$ depends only on the global state $r(m)$. This seems like a natural assumption; the global state is meant to capture everything that is relevant about the current situation.

## *Incorporating knowledge*

*Remark 6.10*

*For every $i \in AG$ and every formula $\varphi$, $(\mathcal{I}, r, m) \vDash K_i\varphi$*

iff $\mathcal{M}_\mathcal{I}, (r, m) \Vdash K_i\varphi$

iff $\mathcal{M}_\mathcal{I}, (r', m') \Vdash \varphi$ *for all* $(r', m')$ *such that* $(r, m)\, \mathcal{K}_i\, (r', m')$

iff $\mathcal{M}_\mathcal{I}, (r', m') \Vdash \varphi$ *for all* $(r', m')$ *such that* $r(m) \sim_i r'(m')$

iff $\mathcal{M}_\mathcal{I}, (r', m') \Vdash \varphi$ *for all* $(r', m')$ *such that* $r_i(m) = r'_i(m')$.

## *Incorporating knowledge*

*Proposition 6.11*

*Let $(r, m)$ and $(r', m')$ be points in $\mathcal{I}$ such that $r(m) = r'(m')$. Then for every formula $\varphi$,*

$$(\mathcal{I}, r, m) \vDash \varphi \textit{ iff } (\mathcal{I}, r', m') \vDash \varphi.$$

**Proof:** By induction on $\varphi$.

- $\varphi = p \in PROP$. Then $(\mathcal{I}, r, m) \vDash p$ iff $\pi(r(m))(p) = 1$ iff $\pi(r'(m'))(p) = 1$ iff $(\mathcal{I}, r', m') \vDash \varphi$.
- The cases $\varphi = \neg\psi$ and $\varphi = \psi \to \chi$ are obvious.
- $\varphi = K_i\psi$. Then

$$\begin{aligned}(\mathcal{I}, r, m) \vDash K_i\varphi \quad & \text{iff } \mathcal{M}_{\mathcal{I}}, (r^*, m^*) \Vdash \varphi \text{ for all } (r^*, m^*) \\ & \text{such that } r_i(m) = r_i^*(m^*) \\ & \text{iff } \mathcal{M}_{\mathcal{I}}, (r^*, m^*) \Vdash \varphi \text{ for all } (r^*, m^*) \\ & \text{such that } r_i'(m') = r_i^*(m^*) \\ & \text{iff } (\mathcal{I}, r', m') \vDash K_i\varphi. \qquad \Box\end{aligned}$$

## *Incorporating knowledge*

*Definition 6.12*

*We say that $\varphi$ is true in an interpreted system $\mathcal{I}$ if $(\mathcal{I}, r, m) \models \varphi$ for all points $(r, m)$ in $\mathcal{I}$.*
*Notation:* $\mathcal{I} \models \varphi$.

*Definition 6.13*

*Let $\mathcal{M}$ be a class of interpreted systems. We say that $\varphi$ is true in $\mathcal{M}$ if $\mathcal{I} \models \varphi$ for every interpreted system $\mathcal{I} \in \mathcal{M}$.*
*Notation:* $\mathcal{M} \models \varphi$.

## *Incorporating knowledge*

*Definition 6.14*

*We say that $\varphi$ is valid in a system $\mathcal{R}$ if $\mathcal{I} \models \varphi$ for every interpreted system $\mathcal{I}$ based on $\mathcal{R}$.*
*Notation:* $\mathcal{I} \models \varphi$.

*Definition 6.15*

*Let $\mathcal{F}$ be a class of systems. We say that $\varphi$ is valid in $\mathcal{F}$ if $\mathcal{R} \models \varphi$ for every system $\mathcal{R} \in \mathcal{F}$.*
*Notation:* $\mathcal{F} \models \varphi$.

## *Bit-transmission problem - again*

Consider the bit-transmission problem again.

We take *PROP* to consist of six atomic propositions:

- $bit = 0$ representing the assertion that the value of $S$'s initial bit is 0;
- $bit = 1$ representing the assertion that the value of $S$'s initial bit is 1;
- *recbit* representing the assertion that $R$ has received $S$'s message;
- *recack* representing the assertion that $S$ has received $R$'s acknowledgment;
- *sentbit* representing the assertion that $S$ has just sent a message;
- *sentack* representing the assertion that $R$ has just sent a message.

## Bit-transmission problem - again

### Definition 6.16

*Define the interpreted system* $\mathcal{I}^{bt} = (\mathcal{R}^{bt}, \pi^{bt})$*, where* $\mathcal{R}^{bt}$ *is the system defined by Definition 6.4 and* $\pi^{bt}$ *is an interpretation such that for all points* $(r, m)$*,*

- $(\mathcal{I}^{bt}, r, m) \vDash bit = k$ *iff* $r_S(m)$ *is either* $k$ *or* $(k, ack)$ *for* $k = 0, 1$*;*
- $(\mathcal{I}^{bt}, r, m) \vDash recbit$ *iff* $r_R(m) \neq \lambda$*;*
- $(\mathcal{I}^{bt}, r, m) \vDash recack$ *iff* $r_S(m) = (k, ack)$ *for some* $k = 0, 1$*;*
- $(\mathcal{I}^{bt}, r, m) \vDash sentbit$ *iff the last tuple in* $r_e(m)$ *is* $(sendbit, sendack)$ *or* $(sendbit, \Lambda)$*;*
- $(\mathcal{I}^{bt}, r, m) \vDash sentack$ *iff the last tuple in* $r_e(m)$ *is* $(sendbit, sendack)$ *or* $(\Lambda, sendack)$*.*

The truth value of all the atomic propositions is completely determined by the global state, since the environment's state records the events taking place in the system.

## *Bit-transmission problem - again*

After $R$ receives $S$'s bit, then $R$ knows the value of the bit.

*Proposition 6.17*

*Let $(r, m)$ be a point such that $r_R(m) = k$ for $k = 0, 1$. Then $(\mathcal{I}^{bt}, r, m) \vDash K_R(bit = k)$.*

**Proof:** Let $(r', m')$ be a point s.t. $r_R(m) = r'_R(m') = k$. Then $S$ must have initial bit $k$ at $(r', m')$, so $(\mathcal{I}^{bt}, r', m') \vDash bit = k$. $\square$

When $S$ receives $R$'s ack message, then $S$ knows that $R$ knows the initial bit.

*Proposition 6.18*

*Let $(r, m)$ be a point such that $r_S(m) = (k, ack)$ for $k = 0, 1$. Then $(\mathcal{I}^{bt}, r, m) \vDash K_S K_R(bit = k)$.*

**Proof:** Let $(r', m')$ be a point s.t. $r_S(m) = r'_S(m') = (k, ack)$. We have to prove that $(\mathcal{I}^{bt}, r', m') \vDash K_R(bit = k)$. This follows from Proposition 6.17, using the fact that $r'_R(m') = k$. $\square$

## Incorporating time

- The temporal language $ML_{temp}$ is obtained by adding to $ML_{Ag}$ the following temporal operators: $\Box$ (always), its dual $\Diamond$ (eventually), $\bigcirc$ (next time), and $U$ (until).
- Formulas of $ML_{temp}$ are defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid K_i\varphi \mid \Box\varphi \mid \Diamond\varphi \mid \bigcirc\varphi \mid \varphi U \varphi,$$

  where $p \in PROP$ and $i \in Ag$.
- We sometimes refer to formulas of $ML_{temp}$ as temporal formulas and to formulas of $ML_{Ag}$ as knowledge formulas.

Intuition:

- $\Box\varphi$ is true if $\varphi$ is true now and at all later points;
- $\Diamond\varphi$ is true if $\varphi$ is true at some point in the future;
- $\bigcirc\varphi$ is true if $\varphi$ is true at the next step;
- $\varphi U \psi$ is true if $\varphi$ is true until $\psi$ is true.

## *Incorporating time*

Let $\mathcal{I} = (\mathcal{R}, \pi)$ be an interpreted system. Then for all points $(r, m)$ in $\mathcal{I}$ and every formulas $\varphi$, $\psi$,

$$\begin{array}{lll}
(\mathcal{I}, r, m) \models \Box\varphi & \text{iff} & (\mathcal{I}, r, m') \models \varphi \text{ for all } m' \geq m, \\
(\mathcal{I}, r, m) \models \Diamond\varphi & \text{iff} & (\mathcal{I}, r, m') \models \varphi \text{ for some } m' \geq m, \\
(\mathcal{I}, r, m) \models \bigcirc\varphi & \text{iff} & (\mathcal{I}, r, m+1) \models \varphi, \\
(\mathcal{I}, r, m) \models \varphi U \psi & \text{iff} & (\mathcal{I}, r, m') \models \psi \text{ for some } m' \geq m \text{ and} \\
 & & (\mathcal{I}, r, m'') \models \varphi \text{ for all } m'' \text{ with } m \leq m'' < m'.
\end{array}$$

- The interpretation of $\bigcirc\varphi$ as "$\varphi$ is true at the next step" makes sense because our notion of time is discrete.
- All the other temporal operators make perfect sense even for continuous notions of time.

## *Incorporating time*

### *Proposition 6.19*

*For every interpreted system $\mathcal{I}$,*

$$\mathcal{I} \models \Diamond\varphi \leftrightarrow \top U \varphi \text{ and } \mathcal{I} \models \Box\varphi \leftrightarrow \neg\Diamond\neg\varphi.$$

Thus, we can take $\bigcirc$ and $U$ as our basic temporal operators, and define $\Diamond$ and $\Box$ in terms of $U$.

- The truth of a temporal formula depends only on the run.
- The truth of $\varphi$ at a point $(r, m)$ in $\mathcal{I} = (\mathcal{R}, \pi)$ does not depend on $\mathcal{R}$ at all, but only on $\pi$, so we can write $(\pi, r, m) \models \varphi$.
- In general, temporal operators are used for reasoning about events that happen along a single run.

### *Definition 6.20*

*We say that $r$ satisfies $\varphi$ if $(\pi, r, 0) \models \varphi$ holds.*

## *Incorporating time*

Once we have temporal operators, there are a number of important notions that we can express.

- The formula $\Box\Diamond\varphi$ is true iff $\varphi$ occurs infinitely often; that is, $(\mathcal{I}, r, m) \models \Box\Diamond\varphi$ exactly if the set $\{m' \mid (\mathcal{I}, r, m') \models \varphi\}$ is infinite.
- The formula $\Diamond\Box\varphi$ is true iff $\varphi$ is true almost everywhere; that is, $(\mathcal{I}, r, m) \models \Diamond\Box\varphi$ iff for some $m'$ and all $m'' \geq m'$, we have $(\mathcal{I}, r, m'') \models \varphi$.

- The temporal operators that we have defined can talk about events that happen only in the present or future.
- We can add temporal operators for reasoning about the past, for example, an analogue to $\Diamond$ that says at some time in the past.

## *Bit-transmission problem - again*

Consider the bit-transmission problem again.

The formula $\Box(\mathit{recbit} \rightarrow \Diamond \mathit{recack})$ says that if at some point along a run the receiver receives the bit sent by the sender, then at some point in the future the sender will receive the acknowledgment sent by the receiver.

The formula $\Diamond(K_R(\mathit{bit} = 0) \vee K_R(\mathit{bit} = 1))$ says that the receiver eventually knows the sender's initial bit.