Canonical models and filtrations in three-valued propositional modal logic

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In this paper we present a new and simpler axiomatization for the three-valued modal propositional logic defined by P. Ostermann in [9]. We define the canonical model and the filtration for this logic, and we prove the three-valued versions of the Truth and Filtration Lemmas, well-known results in the propositional modal logic. We obtain a new proof for the completeness of the smallest three-valued normal logic **3-K**. Also, **3-K** has the finite model property.
These results can be easily extended to three-valued versions of other systems of propositional modal logic.

Keywords: modal logic, three-valued Lukasiewicz logic, canonical model, filtration.

1 Introduction

P. Ostermann defined in [9] a system of propositional modal logic based on the n-valued Lukasiewicz logic. He proved the completeness of this logic following the method used for modal logic by S. Kripke in [8].
For the three-valued case we present a very simple and natural axiomatization. Our proof of the completeness theorem is achieved combining the method used to demonstrate the completeness of three-valued Lukasiewicz logic ([5]) with a technique that has been standard in modal logic for many years- the canonical model/filtration technique (see [6, 2, 7]).

2 Syntax and Semantics

In this section we give syntactical and semantical definitions for our logic. We use Lukasiewicz's three-valued logic as defined in [1] and [4]. The basic reference for modal logic is [6](see also [2, 3, 7].

2.1 Three-valued modal logics

The *language* of the three-valued propositional modal logic is determined by:
(i) a countable set AF of *atomic formulas*, denoted by $v_0, v_1, \cdots, etc.$

(ii) the propositional connectives $\neg$, $\rightarrow$,
(iii) the modal operator $\Box$.
The set $Fmla$ of *formulas* is defined inductively as follows:
(i) $AF \subseteq Fmla$,
(ii) if $p \in Fmla$ and $q \in Fmla$, then $p \rightarrow q$ and $\neg p \in Fmla$,
(iii) if $p \in Fmla$, then $\Box p \in Fmla$.
We use the standard abbreviations $\vee$, $\wedge$, $\leftrightarrow$, $\oplus$ and $\odot$ from the three-valued Lukasiewicz logic:
$p \vee q := ((p \rightarrow q) \rightarrow q)$,
$p \wedge q := \neg(\neg p \vee \neg q)$,
$p \leftrightarrow q := (p \rightarrow q) \wedge (q \rightarrow p)$,
$p \oplus q := \neg p \rightarrow q$,
$p \odot q := \neg(\neg p \oplus \neg q)$.
We also define:
$\sim p := p \rightarrow \neg p$,
$\Diamond p := \neg \Box \neg p$.
By a *three-valued modal logic* over the given language we understand any set $\mathcal{L}$ of formulas that satisfies the following conditions:
(i) $\mathcal{L}$ contains every tautology of the three-valued Lukasiewicz logic,
(ii) $\mathcal{L}$ is closed under *modus ponens*,
(iii) $\mathcal{L}$ is closed under *uniform substitution*.
The members of a three-valued modal logic $\mathcal{L}$ are called its *theorems*. We write $\vdash_{\mathcal{L}} p$ to mean that p is a theorem of $\mathcal{L}$.
An example of a three-valued modal logic is $Fmla$.
Let $\mathcal{L}$ be a three-valued modal logic and $\Sigma \cup \{p\} \subseteq Fmla$. We say that p is *$\mathcal{L}$-deducible from* Σ and we write $\Sigma \vdash_{\mathcal{L}} p$ if there exist $p_0, \cdots, p_{n-1} \in \Sigma$ such that

$$\vdash_{\mathcal{L}} (p_0 \rightarrow (p_1 \rightarrow (\cdots \rightarrow (p_{n-1} \rightarrow p) \cdots))).$$

(in the case $n = 0$, this means that $\vdash_{\mathcal{L}} p$).
A set Σ of formulas is *$\mathcal{L}$-consistent* if there is no formula p such that $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} \neg p$; if not, Σ is *$\mathcal{L}$-inconsistent*. A $\mathcal{L}$-consistent set Σ is *maximal $\mathcal{L}$-consistent* if $p \in \Sigma$ for any formula p such that $\Sigma \cup \{p\}$ is $\mathcal{L}$-consistent.
In the sequel we shall prove some syntactical properties.
We remind some tautologies of the three-valued Lukasiewicz logic, which will be used in our proofs ([1, 4]):
(t1) $p \rightarrow (q \rightarrow p)$,
(t2) $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$,
(t3) $p \rightarrow p$,
(t4) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$,
(t5) $p \leftrightarrow \neg\neg p$,
(t6) $\neg p \rightarrow (p \rightarrow q)$,
(t7) $(p \rightarrow (p \rightarrow (q \rightarrow r))) \rightarrow ((p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow (p \rightarrow r)))$,
(t8) $\sim\sim p \rightarrow p$,
(t9) $(p \rightarrow \sim p) \rightarrow \sim p$,

(t10) $(p \wedge (q \wedge r)) \leftrightarrow ((p \wedge q) \wedge r)$,
(t11) $(p \wedge q) \leftrightarrow (q \wedge p)$,
(t12) $(p \wedge q) \rightarrow p$,
(t13) $p \rightarrow (q \rightarrow (p \wedge q))$,
(t14) $(p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r)$,
(t15) $(p \vee q) \leftrightarrow (q \vee p)$,
(t16) $p \rightarrow (p \vee q)$,
(t17) $(p \odot (q \odot r)) \leftrightarrow ((p \odot q) \odot r)$,
(t18) $(p \odot q) \leftrightarrow (q \odot p)$,
(t19) $p \odot q \rightarrow p$,
(t20) $p \rightarrow (q \rightarrow p \odot q)$,
(t21) $(p \rightarrow (q \rightarrow r)) \leftrightarrow (p \odot q \rightarrow r)$,
(t22) $(p \rightarrow q) \rightarrow (p \odot r \rightarrow q \odot r)$,
(t23) $(p \oplus (q \oplus r)) \leftrightarrow ((p \oplus q) \oplus r)$,
(t24) $(p \oplus q) \leftrightarrow (q \oplus p)$,
(t25) $p \rightarrow p \oplus q$,
(t26) $(p \vee q) \rightarrow (p \oplus q)$.

Let $\mathcal{L}$ be a three-valued modal logic.

Remark 2.1 By (t21) and (t17), $\vdash_{\mathcal{L}} p$ iff there exist $p_0, \cdots, p_{n-1} \in \Sigma$ such that $\vdash_{\mathcal{L}} p_0 \odot p_1 \odot \cdots \odot p_{n-1} \rightarrow p$.

Proposition 2.2 Let $\Sigma \subseteq Fmla$ and $p, q \in Fmla$.
(i) $\vdash_{\mathcal{L}} p$ iff $\emptyset \vdash_{\mathcal{L}} p$.
(ii) If $\vdash_{\mathcal{L}} p$, then $\Sigma \vdash_{\mathcal{L}} p$.
(iii) If $\mathcal{L}^{'}$ is another three-valued modal logic such that $\mathcal{L} \subseteq \mathcal{L}^{'}$, then $\Sigma \vdash_{\mathcal{L}} p$ implies $\Sigma \vdash_{\mathcal{L}'} p$.
(iv) If $\Sigma \vdash_{\mathcal{L}} p$, then $\Gamma \vdash_{\mathcal{L}} p$ for any superset Γ of Σ.
(v) If $\Sigma \vdash_{\mathcal{L}} p$, then there exists a finite subset Γ of Σ such that $\Gamma \vdash_{\mathcal{L}} p$.
(vi) If $p \in \Sigma$, then $\Sigma \vdash_{\mathcal{L}} p$.
(vii) If $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} p \rightarrow q$, then $\Sigma \vdash_{\mathcal{L}} q$.
(viii) $\Sigma \vdash_{\mathcal{L}} p$ iff there exists a finite sequence $q_0, \cdots, q_m = p$ such that for any $i \leq m$, we have one of the following possibilities:
   (a) $q_i \in \mathcal{L} \cup \Sigma$,
   (b) there exist $j, k < i$ such that $q_k = q_j \rightarrow q_i$.

Proof: The proof is immediate by the definitions. See, for example, [1], Lemma 1.7, pag. 465-466 and [6], Exercises 2.2, pag. 18. $\square$

A sequence $q_0, \cdots, q_m = p$ as in Proposition 2.2.(viii) is called a *formal proof* of p from Σ.
A Deduction Theorem is also true.

Theorem 2.3 Let $\Sigma \subseteq Fmla$ and $p, q \in Fmla$. Then
   $\Sigma \cup \{p\} \vdash_{\mathcal{L}} q$ iff $\Sigma \vdash_{\mathcal{L}} p \rightarrow (p \rightarrow q)$.

Proof: Similar to [1], Lemma 1.7.(d), pag. 466. $\Box$

We give now some properties of $\mathcal{L}$-consistent and maximal $\mathcal{L}$-consistent sets of formulas.

Proposition 2.4 Let $\Sigma \subseteq Fmla$ and $p \in Fmla$.
(i) Σ is $\mathcal{L}$-inconsistent iff $\Sigma \vdash_{\mathcal{L}} r$ for any formula r.
(ii) $\Sigma \cup \{p\}$ is $\mathcal{L}$-inconsistent iff $\Sigma \vdash_{\mathcal{L}} \sim p$.
(iii) $\Sigma \cup \{\sim p\}$ is $\mathcal{L}$-inconsistent iff $\Sigma \vdash_{\mathcal{L}} p$.
(iv) Σ is $\mathcal{L}$-consistent iff every finite subset of Σ is $\mathcal{L}$-consistent.
(v) If Σ is $\mathcal{L}$-consistent, then for any formula p, at least one of $\Sigma \cup \{p\}$ and $\Sigma \cup \{\sim p\}$ is $\mathcal{L}$-consistent.
(vi) If $\Sigma \subseteq \mathcal{L}$, then Σ is $\mathcal{L}$-consistent iff $\mathcal{L} \neq Fmla$.

Proof: (i) ($\Rightarrow$) There exists $p \in Fmla$ such that $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} \neg p$. Apply (t6) and Proposition 2.2.(vii) to get $\Sigma \vdash_{\mathcal{L}} r$.
($\Leftarrow$) Obviously.
(ii) ($\Rightarrow$) Suppose that $\Sigma \cup \{p\}$ is $\mathcal{L}$-inconsistent. Then $\Sigma \cup \{p\} \vdash_{\mathcal{L}} \neg p$, by (i). Applying Deduction Theorem we obtain $\Sigma \vdash_{\mathcal{L}} p \to \sim p$. Using (t9) and Proposition 2.2.(ii) and (vii), it follows that $\Sigma \vdash_{\mathcal{L}} \sim p$.
($\Leftarrow$) By $\Sigma \vdash_{\mathcal{L}} \sim p$ and Proposition 2.2.(vi) and (iv) we get $\Sigma \cup \{p\} \vdash_{\mathcal{L}} p$ and $\Sigma \cup \{p\} \vdash_{\mathcal{L}} \sim p$. Using the definition of $\sim p$ it follows that $\Sigma \cup \{p\} \vdash_{\mathcal{L}} p$ and $\Sigma \cup \{p\} \vdash_{\mathcal{L}} \neg p$, hence $\Sigma \cup \{p\}$ is $\mathcal{L}$-inconsistent.
(iii) ($\Rightarrow$) By the fact that $\Sigma \cup \{\sim p\}$ is $\mathcal{L}$-inconsistent, using (ii) we obtain $\Sigma \vdash_{\mathcal{L}} \sim\sim p$. Applying (t8), it follows that $\Sigma \vdash_{\mathcal{L}} p$.
($\Leftarrow$) Similar to (ii).
(iv) ($\Rightarrow$) Obviously.
($\Leftarrow$) Suppose that Σ is $\mathcal{L}$-inconsistent and let p be any formula. Then, $\Sigma \vdash_{\mathcal{L}} \neg(p \to p)$. Applying Proposition 2.2.(v), there exists a finite subset Γ of Σ such that $\Gamma \vdash_{\mathcal{L}} \neg(p \to p)$. But, by (t3), $\Gamma \vdash_{\mathcal{L}} (p \to p)$. We get Γ is $\mathcal{L}$-inconsistent, so we contradict the hypothesis.
(v) Suppose that $\Sigma \cup \{p\}$ and $\Sigma \cup \{\sim p\}$ are $\mathcal{L}$-inconsistent. By (ii) and (iii) we have $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} \sim p$, hence Σ is $\mathcal{L}$-inconsistent.
(vi) ($\Rightarrow$) Obviously.
($\Leftarrow$) Suppose that Σ is $\mathcal{L}$-inconsistent, hence $\Sigma \vdash_{\mathcal{L}} p$ for any formula p. Because $\Sigma \subseteq \mathcal{L}$, we get $\mathcal{L} \vdash_{\mathcal{L}} p$ for any formula $p \in Fmla$. Using the fact that $\mathcal{L}$ is closed under modus ponens it follows that $\mathcal{L} = Fmla$. $\Box$

Proposition 2.5 Let Σ be a maximal $\mathcal{L}$-consistent set and $p, q \in Fmla$.
(i) $\Sigma \vdash_{\mathcal{L}} p$ implies $p \in \Sigma$.
(ii) $\mathcal{L} \subseteq \Sigma$.
(iii) If $\Sigma \subseteq \Gamma$ and Γ is $\mathcal{L}$- consistent, then $\Sigma = \Gamma$.
(iv) $p \in \Sigma$ iff $\sim p \notin \Sigma$.
(v) $p \vee q \in \Sigma$ iff ($p \in \Sigma$ or $q \in \Sigma$).
(vi) $p \wedge q \in \Sigma$ iff ($p \in \Sigma$ and $q \in \Sigma$).

(vii) $p \odot q \in \Sigma$ iff ($p \in \Sigma$ and $q \in \Sigma$).
(viii) If $p \in \Sigma$ or $q \in \Sigma$, then $p \oplus q \in \Sigma$.
(ix) If $(p \to q) \in \Sigma$, then $p \in \Sigma$ implies $q \in \Sigma$.
(x) If $(p \leftrightarrow q) \in \Sigma$, then $p \in \Sigma$ iff $q \in \Sigma$.

Proof: (i) Suppose that $p \notin \Sigma$. It follows that $\Sigma \cup \{p\}$ is $\mathcal{L}$-inconsistent, hence, using Lemma 2.4.(ii), $\Sigma \vdash_{\mathcal{L}} \sim p$. By $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} \sim p$ we get $\Sigma \vdash_{\mathcal{L}} \neg p$, so Σ is $\mathcal{L}$-inconsistent.
(ii) Immediately using (i).
(iii) Let $p \in \Gamma$. Then $\Sigma \cup \{p\} \subseteq \Gamma$ and Γ is $\mathcal{L}$-consistent. Hence, $\Sigma \cup \{p\}$ is $\mathcal{L}$-consistent. It follows that $p \in \Sigma$.
(iv) ($\Leftarrow$) Let $p \in \Sigma$ and suppose that $\sim p \in \Sigma$. It follows that Σ is $\mathcal{L}$-inconsistent.
($\Rightarrow$) Suppose that $\sim p \notin \Sigma$. Then $\Sigma \cup \{\sim p\}$ is $\mathcal{L}$-inconsistent and, by Proposition 2.4.(vi), it follows that $\Sigma \cup \{p\}$ is $\mathcal{L}$-consistent. Hence, $p \in \Sigma$.
(v) ($\Rightarrow$) Suppose that $p \vee q \in \Sigma$ and $p \notin \Sigma$. By (iv) it follows that $\sim p \in \Sigma$. By $p \vee q \in \Sigma$, (t26) and the definition of $\oplus$ we get $\neg p \to q \in \Sigma$. Using (t2), $p \to \neg p \in \Sigma$ and $\neg p \to q \in \Sigma$ we obtain $p \to q \in \Sigma$. So, we have $p \vee q = (p \to q) \to q \in \Sigma$ and $p \to q \in \Sigma$. Hence, $q \in \Sigma$.
($\Leftarrow$) It follows immediately by (t16).
(vi) ($\Rightarrow$) Apply (t12)
($\Leftarrow$) Apply (t13).
(vii) ($\Rightarrow$) Apply (t19)
($\Leftarrow$) Apply (t20).
(viii) Apply (t25).
(ix) Apply (i) and Proposition 2.2.(vii).
(x) Apply (vi) and (i). □

Lemma 2.6 (**Lindebaum's Lemma**) Every $\mathcal{L}$-consistent set of formulas is contained in a maximal $\mathcal{L}$-consistent set.

Proof: Similar to [1], Chapter 9, Proposition 1.9, pag. 467. □

Let $W_{\mathcal{L}}$ be the set of all maximal $\mathcal{L}$-consistent sets of formulas.

Remark 2.7 By Proposition 2.4.(vi) and Lindebaum's Lemma we get
$W_{\mathcal{L}} \neq \emptyset$ iff $\mathcal{L}$ is $\mathcal{L}$-consistent iff $\mathcal{L} \neq Fmla$.

Proposition 2.8 Let $\mathcal{L}$ be an $\mathcal{L}$-consistent three-valued modal logic.
(i) $\{p \in Fmla \mid \Gamma \vdash_{\mathcal{L}} p\} = \bigcap\{\Sigma \in W_{\mathcal{L}} \mid \Gamma \subseteq \Sigma\}$, i. e. $\Gamma \vdash_{\mathcal{L}} p$ iff p belongs to every maximal $\mathcal{L}$-consistent set that includes Γ.
(ii) $\mathcal{L} = \bigcap\{\Sigma \in W_{\mathcal{L}}\}$, i. e. $\vdash_{\mathcal{L}} p$ iff p belongs to every maximal $\mathcal{L}$-consistent set.

Proof: (i) ($\Rightarrow$)Suppose that $\Gamma \vdash_{\mathcal{L}} p$ and let $\Sigma \in W_{\mathcal{L}}$ be such that $\Gamma \subseteq \Sigma$. Then $\Sigma \vdash_{\mathcal{L}} p$ and, by Proposition 2.5.(i), $p \in \Sigma$.
($\Leftarrow$) Suppose that p belongs to every maximal $\mathcal{L}$-consistent set that includes Γ and that we don't have $\Gamma \vdash_{\mathcal{L}} p$. Then, by Proposition 2.4.(iii), $\Gamma \cup \{\sim p\}$ is

$\mathcal{L}$-consistent. Applying Lindebaum's Lemma we obtain a maximal $\mathcal{L}$-consistent set Θ such that $\Gamma \cup \{\sim p\} \subseteq \Theta$. Hence $\Gamma \subseteq \Theta$, therefore $p \in \Gamma$. It folllows that $\sim p, p \in \Theta$, which contradicts Proposition 2.5.(iv).
(ii) It follows by (i).$\Box$

2.2 Three-valued normal logics

By a *three-valued normal logic* we understand a three-valued modal logic $\mathcal{L}$ that also satisfies the following conditions:
(i) $\mathcal{L}$ contains every formula of the form
(K) $\Box(p \to q) \to (\Box p \to \Box q)$,
$(A1)$ $\Box(p \oplus p) \leftrightarrow (\Box p \oplus \Box p)$,
(ii) $\mathcal{L}$ is closed under *necessitation*; that is, if $p \in \mathcal{L}$, then also $\Box p \in \mathcal{L}$.

Proposition 2.9 Let $\mathcal{L}$ be a three-valued normal logic. For any formulas p, q,
(i) If $\vdash_{\mathcal{L}} p \to q$, then $\vdash_{\mathcal{L}} \Box p \to \Box q$.
(ii) If $\vdash_{\mathcal{L}} p \leftrightarrow q$, then $\vdash_{\mathcal{L}} \Box p \leftrightarrow \Box q$.
(iii) $\vdash_{\mathcal{L}} \Diamond\neg p \leftrightarrow \neg\Box p$.
(iv) $\sim\Diamond p \leftrightarrow \Box\sim p$.

Proof: (i) Apply (K) and the closure of $\mathcal{L}$ under necessitation.
(ii) By the definition of $\leftrightarrow$, $\vdash_{\mathcal{L}} p \leftrightarrow q$ is equivalent to $\vdash_{\mathcal{L}} (p \to q) \wedge (q \to p)$. By (t12) we get $\vdash_{\mathcal{L}} p \to q$ and $\vdash_{\mathcal{L}} q \to p$. Using (i) twice we obtain $\vdash_{\mathcal{L}} \Box p \to \Box q$ and $\vdash_{\mathcal{L}} \Box q \to \Box p$. Now, (t13) gives us $\vdash_{\mathcal{L}} (\Box p \to \Box q) \wedge (\vdash_{\mathcal{L}} \Box q \to \Box p)$, hence $\vdash_{\mathcal{L}} \Box p \leftrightarrow \Box q$.
(iii) By (t5) and (i) we have $\vdash_{\mathcal{L}} \Box p \leftrightarrow \Box\neg\neg p$. By (t4) we get $\vdash_{\mathcal{L}} \neg\Box\neg\neg p \leftrightarrow \neg\Box p$, hence $\vdash_{\mathcal{L}} \Diamond\neg p \leftrightarrow \neg\Box p$.
(iv) By the definition of $\oplus$ and (t5) we obtain $\vdash_{\mathcal{L}} \Box(\neg p \oplus \neg p) \leftrightarrow \Box\sim p$ and $\vdash_{\mathcal{L}} (\Box\neg p \oplus \Box\neg p) \leftrightarrow \sim\neg\Box\neg p$. Using $(A1)$ it follows that $\vdash_{\mathcal{L}} \Box\sim p \leftrightarrow \sim\neg\Box\neg p$, hence $\vdash_{\mathcal{L}} \Box\sim p \leftrightarrow \sim\Diamond p$. $\Box$

Proposition 2.10 Let $\mathcal{L}$ be a three-valued normal logic. For any $\Sigma \subseteq Fmla$ and any formula p,
$\Sigma \vdash_{\mathcal{L}} p$ implies $\{\Box q \,|\, q \in \Sigma\} \vdash_{\mathcal{L}} \Box p$.

Proof: There exist $p_0, \cdots, p_{n-1} \in \Sigma$ such that $\vdash_{\mathcal{L}} (p_0 \to (p_1 \to (\cdots \to (p_{n-1} \to p) \cdots)))$. Applying (K) and the fact that $\mathcal{L}$ is closed under modus ponens, we get $\vdash_{\mathcal{L}} (\Box p_0 \to (\Box p_1 \to (\cdots \to (\Box p_{n-1} \to \Box p) \cdots)))$. Hence, $\{\Box q \,|\, q \in \Sigma\} \vdash_{\mathcal{L}} \Box p$. $\Box$

It is clear that the intersection of any collection of three-valued normal logics is also a three-valued normal logic. In particular, the intersection of all three-valued normal logics is the *smallest* three-valued normal logic, denoted **3-K**.

2.3 Semantics

The semantics of our system is defined as in [9]. A *frame* is a pair $\mathcal{F} = \langle W, R\rangle$, where W is a non-empty set, and R is a binary relation on W. A *model* based on a frame $\mathcal{F} = \langle W, R\rangle$ is a triple $\mathcal{M} = \langle W, R, V\rangle$, where $V : Fmla \times W \to L_3 = \{0, \frac{1}{2}, 1\}$ is a function called *valuation* that have the properties:
(i) $V(\neg p, s) = 1 - V(p, s)$,
(ii) $V(p \to q, s) = min\{1, 1 - V(p, s) + V(q, s)\}$,
(iii) $V(\Box p, s) = min\{V(p, t) \,|\, sRt\}$,
for all p, $q \in Fmla$, $s \in W$.
Let p be a formula. We define the next notions:
- p is *true at point* s in the model $\mathcal{M}$ and we write $\mathcal{M}, s \models p$ if $V(p, s) = 1$,
-p is *true* in model $\mathcal{M}$ and we write $\mathcal{M} \models p$ if $\mathcal{M}, s \models p$ for any $s \in W$,
- p is *valid* in a frame $\mathcal{F}$ and we write $\mathcal{F} \models p$ if $\mathcal{M} \models p$ for any model $\mathcal{M}$ based on $\mathcal{F}$.
If $\mathcal{C}$ is a class of models (frames), then p is *true* (respectively *valid*) in $\mathcal{C}$, $\mathcal{C} \models p$, if p is true (respectively valid) in all members of $\mathcal{C}$.
Let $\mathcal{C}$ be a class of frames, or of models and $\mathcal{L}$ be a three-valued modal logic. We say that $\mathcal{L}$ is *sound* with respect to $\mathcal{C}$ if for all formulas p,
if $\vdash_{\mathcal{L}} p$, then $\mathcal{C} \models p$.
$\mathcal{L}$ is *complete* with respect to $\mathcal{C}$ if for any p,
if $\mathcal{C} \models p$, then $\vdash_{\mathcal{L}} p$.
$\mathcal{L}$ is *determined* by $\mathcal{C}$ if $\mathcal{L}$ is both sound and complete with respect to $\mathcal{C}$.

3 Canonical models and completeness

Let $\mathcal{L}$ be an $\mathcal{L}$-consistent three-valued normal logic.
The *canonical model* of $\mathcal{L}$ is the model $\mathcal{M}_{\mathcal{L}} = \langle W_{\mathcal{L}}, R_{\mathcal{L}}, V_{\mathcal{L}}\rangle$, where:
(i) $W_{\mathcal{L}}$ is the set of all maximal $\mathcal{L}$-consistent sets of formulas,
(ii) for all $s, t \in W_{\mathcal{L}}$,
$sR_{\mathcal{L}}t$ iff $\{p \in Fmla \,|\, \Box p \in s\} \subseteq t$,
(iii) for every atomic formula p and every $s \in W_{\mathcal{L}}$,

$$V_{\mathcal{L}}(p, s) = \begin{cases} 1, & \text{if } p \in s \\ 0, & \text{if } \neg p \in s \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

The *canonical frame* for $\mathcal{L}$ is $\mathcal{F}_{\mathcal{L}} = \langle W_{\mathcal{L}}, R_{\mathcal{L}}\rangle$.

Proposition 3.1 For any $s \in W_{\mathcal{L}}$, and any $p \in Fmla$,
$\Box p \in s$ iff for all $t \in W_{\mathcal{L}}$, $sR_{\mathcal{L}}t$ implies $p \in t$.

Proof: ($\Rightarrow$) By the definition of $R_{\mathcal{L}}$.
($\Leftarrow$) Suppose that for all $t \in W_{\mathcal{L}}$,

$sR_{\mathcal{L}}t$ implies $p \in t$,
i. e.

$\{q \in Fmla \,|\, \Box q \in s\} \subseteq t$ implies $p \in t$.

Let $\Gamma = \{q \in Fmla \mid \Box q \in s\}$. We get $p \in \bigcap\{t \in W_{\mathcal{L}} \mid \Gamma \subseteq t\}$, hence, using Proposition 2.8.(i), $\Gamma \vdash_{\mathcal{L}} p$. Applying Proposition 2.10 it follows that $\{\Box q \,|\, q \in \Gamma\} \vdash_{\mathcal{L}} \Box p$, so $s \vdash_{\mathcal{L}} \Box p$. By the fact that s is a maximal $\mathcal{L}$ -consistent set and Proposition 2.5.(i), we obtain $\Box p \in s$. $\Box$

Corollary 3.2 For any $s \in W_{\mathcal{L}}$, and any $p \in Fmla$,

$\Box p \notin s$ iff there exists $t \in W_{\mathcal{L}}$ such that $sR_{\mathcal{L}}t$ and $p \notin t$.

Theorem 3.3 For every formula p and every $s \in W_{\mathcal{L}}$,

$$V_{\mathcal{L}}(p, s) = \begin{cases} 1, & \text{if } p \in s \\ 0, & \text{if } \neg p \in s \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

Proof: We prove by induction on p that for any formula p:
(i) $p \in s$ iff $V_{\mathcal{L}}(p, s) = 1$,
(ii) $\neg p \in s$ iff $V_{\mathcal{L}}(p, s) = 0$.
The intended result is then obvious.
The cases $p \in AF$, p is $\neg q$ and p is $q \rightarrow r$ are similar to [1], Chapter 9, Proposition 1.9 and [5].
Consider now that p is $\Box q$, where q satisfies the inductive hypothesis. Then
(i) By Proposition 3.1, we have that

$p \in s$ iff $\Box q \in s$ iff for all $t \in W_{\mathcal{L}}$, $sR_{\mathcal{L}}t$ implies $q \in t$.

By the inductive hypothesis for q, it follows that

$p \in s$ iff for all $t \in W_{\mathcal{L}}$, $sR_{\mathcal{L}}t$ implies $V_{\mathcal{L}}(q, t) = 1$
iff $min\{V_{\mathcal{L}}(q, t) \,|\, sR_{\mathcal{L}}t\} = 1$ iff $V_{\mathcal{L}}(\Box q, s) = 1$.

Hence, $p \in s$ iff $V_{\mathcal{L}}(p, s) = 1$.
(ii) By Proposition 2.5.(iv), we have that

$\neg p \in s$ iff $\neg\Box q \in s$ iff $\sim\neg\Box q \notin s$.

By Proposition 2.9.(iii), Proposition 2.5.(i) and (x), we get

$\sim\neg\Box q \notin s$ iff $\sim\Diamond\neg q \notin s$.

Apply now Proposition 2.9.(iv) to get

$\sim\Diamond\neg q \notin s$ iff $\Box\sim\neg q \notin s$.

By Corollary 3.2, Proposition 2.5.(iv), and the inductive hypothesis we have

$\Box\sim\neg q \notin s$ iff there exist $t \in W_{\mathcal{L}}$ such that $sR_{\mathcal{L}}t$ and $\sim\neg q \notin t$ iff there exists $t \in W_{\mathcal{L}}$ such that $sR_{\mathcal{L}}t$ and $\neg q \in t$ iff there exists $t \in W_{\mathcal{L}}$ such that $sR_{\mathcal{L}}t$ and $V_{\mathcal{L}}(q, t) = 0$ iff $V_{\mathcal{L}}(\Box q, s) = 0$.

Hence, $\neg p \in s$ iff $V_{\mathcal{L}}(p, s) = 0$. $\Box$

By this theorem we get a three-valued version of the Truth Lemma, a well-known result in the classical modal logic.

Proposition 3.4 (the Truth Lemma) Let p be a formula. Then, for all $s \in W_{\mathcal{L}}$,

$\mathcal{M}_{\mathcal{L}}, s \models p$ iff $p \in s$.

Corollary 3.5 $\mathcal{M}_{\mathcal{L}}$ determines $\mathcal{L}$, i. e. for all formulas p,

$\mathcal{M}_{\mathcal{L}} \models p$ iff $\vdash_{\mathcal{L}} p$.

Proof: Applying the Truth Lemma and Proposition 2.8.(ii) we get $\mathcal{M}_{\mathcal{L}} \models p$ iff for all $s \in W_{\mathcal{L}}$, $\mathcal{M}_{\mathcal{L}}, s \models p$ iff for all $s \in W_{\mathcal{L}}$, $p \in s$ iff $p \in \bigcap\{s \,|\, s \in W_{\mathcal{L}}\}$ iff $p \in \mathcal{L}$ iff $\vdash_{\mathcal{L}} p$. □

Thus we obtain the completeness theorem for **3-K**.

Theorem 3.6 **3-K** is determined by the class of all frames.

Proof: (Soundness) We prove easily that for any frame $\mathcal{F}$, $\mathcal{L}_{\mathcal{F}} = \{p \in Fmla \,|\, \mathcal{F} \models p\}$ is a three-valued normal logic, so $\mathbf{3-K} \subseteq \mathcal{L}_{\mathcal{F}}$. Hence, $\vdash_{\mathbf{3-K}} p$ implies $\mathcal{F} \models p$ for any frame $\mathcal{F}$.
(Completeness) If we don't have $\vdash_{\mathbf{3-K}} p$, then, by Corolarry 3.5, p is false in $\mathcal{M}_{\mathbf{3-K}}$, and so is not valid in the frame $\mathcal{F}_{\mathbf{3-K}}$. □

4 Filtrations and Decidability

In this section we extend the filtration technique from the classical modal logic [6] to the three-valued modal logic.
Fix a model $\mathcal{M} = \langle W, R, V \rangle$ and a set $\Sigma \subseteq Fmla$ that is closed under subformulas. For every $s \in W$, let
$\Sigma_s^1 = \{p \in \Sigma \,|\, V(p,s) = 1\}$, $\Sigma_s^{\frac{1}{2}} = \{p \in \Sigma \,|\, V(p,s) = \frac{1}{2}\}$, $\Sigma_s^0 = \{p \in \Sigma \,|\, V(p,s) = 0\}$, and $\Sigma_s = (\{1\} \times \Sigma_s^1) \cup (\{\frac{1}{2}\} \times \Sigma_s^{\frac{1}{2}}) \cup (\{0\} \times \Sigma_s^0)$.
We define the relation $\sim_\Sigma$ on W by :

$s \sim_\Sigma t$ iff $V(p,s) = V(p,t)$ for all $p \in \Sigma$.

Then it is clear that $\sim_\Sigma$ is an equivalence relation and that for any $s, t \in W$,

$s \sim_\Sigma t$ iff $\Sigma_s = \Sigma_t$.

For every $s \in W$, let $[s]$ be its $\sim_\Sigma$-equivalence class and W^Σ the set of all such equivalence classes.

Lemma 4.1 If Σ is finite, then W^Σ is finite and has at most $3 \cdot 2^n$ elements, where n is the number of elements of Σ.

Proof: Since $[s] = [t]$ iff $s \sim_\Sigma t$ iff $\Sigma_s = \Sigma_t$, putting

$f([s]) = \Sigma_s$

gives a well-defined one-to-one mapping

$f : W^\Sigma \to (\{1\} \times \mathcal{P}(\Sigma)) \cup (\{\frac{1}{2}\} \times \mathcal{P}(\Sigma)) \cup (\{0\} \times \mathcal{P}(\Sigma))$.

Hence, if $|\Sigma| = n$, then $|W^\Sigma| \leq |(\{1\} \times \mathcal{P}(\Sigma)) \cup (\{\frac{1}{2}\} \times \mathcal{P}(\Sigma)) \cup (\{0\} \times \mathcal{P}(\Sigma))| = 3 \cdot 2^n$.
□

A binary relation R' on W^{Σ} is called a Σ-*filtration* of R if it satisfies:
(F1) if sRt, then $[s]R'[t]$,
(F2) if $[s]R'[t]$, then for all $p \in Fmla$,
$\Box p \in \Sigma$ implies $V(\Box p, s) \leq V(p, t)$.
Now let $AF^{\Sigma} = AF \cap \Sigma$ be the set of atomic formulas that belong to Σ, and define $V^{\Sigma}_{AF} : AF^{\Sigma} \times W^{\Sigma} \to L_3$ by $V^{\Sigma}_{AF}(p, [s]) = V(p, s)$. It follows immediately that V^{Σ}_{AF} is well-defined.
A model $\mathcal{M}' = \langle W^{\Sigma}, R', V^{\Sigma} \rangle$ in which R' is a Σ-filtration of R, and $V^{\Sigma} : AF \times W^{\Sigma} \to L_3$ is an arbitrary extension of V^{Σ}_{AF} is called a Σ-*filtration* of $\mathcal{M}$.

Theorem 4.2 (Filtration Lemma) If $p \in \Sigma$, then for any $s \in W$,
$V(p, s) = V^{\Sigma}(p, [s])$.

Proof: By induction on the formula p.
-$p \in AF$.
Then, by the hypothesis, $p \in AF^{\Sigma}$. Hence, $V^{\Sigma}(p, [s]) = V^{\Sigma}_{AF}(p, [s]) = V(p, s)$.
-p is $\neg q$.
Since Σ is closed under subformulas, by $\neg q \in \Sigma$ we get $q \in \Sigma$. So, we can apply the induction hypothesis for q to obtain $V(q, s) = V^{\Sigma}(q, [s])$. It follows that $V(p, s) = 1 - V(q, s) = 1 - V^{\Sigma}(q, [s]) = V^{\Sigma}(p, [s])$.
-p is $q \to r$.
By the fact that Σ is closed under subformulas, we similarly obtain that $V(q, s) = V^{\Sigma}(q, [s])$ and $V(r, s) = V^{\Sigma}(r, [s])$. Hence,
$V(p, s) = min\{1, 1 - V(q, s) + V(r, s)\} = min\{1, 1 - V^{\Sigma}(q, [s]) + V^{\Sigma}(r, [s])\} = V^{\Sigma}(p, [s])$.
-p is $\Box q$.
By the fact that Σ is closed under subformulas and $\Box q \in \Sigma$ we get $q \in \Sigma$, hence, by the induction hypothesis, $V(q, s) = V^{\Sigma}(q, [s])$.
We prove that $V(p, s) \leq V^{\Sigma}(q, [s])$. Let $t \in W$ be such that $[s]R'[t]$. Since $\Box q \in \Sigma$, we have by (F2) that $V(\Box q, s) \leq V(q, t)$. Hence,
$V^{\Sigma}(p, [s]) = min\{V^{\Sigma}(q, [t]) \mid [s]R'[t]\} = min\{V(q, t) \mid [s]R'[t]\} \geq V(\Box q, s) = V(p, s)$.
The fact that $V(p, s) \geq V^{\Sigma}(q, [s])$ follows immediately by (F1). $\Box$

Theorem 4.3 **3-K** is determined by the class of all finite frames. Moreover, if a formula p has n subformulas, then
$\vdash_{\mathbf{3-K}} p$ iff p is valid in all frames having at most $3 \cdot 2^n$ elements.

Proof: Similar to [6], Theorem 4.6, pag. 34. $\Box$

Using this theorem we obtain

Corollary 4.4 **3-K** has the finite model property. Hence, **3-K** is decidable.

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