Ultraproducts and uniform rates of asymptotic regularity and metastability

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1 Introduction

A lot of theorems of (functional) analysis provide the convergence or related properties of various sequences defined by formulas. The most elementary example is the Banach fixed point theorem for the Picard iteration of a contraction in a metric space. What is often required in order for this kind of results to be practical is that the results obtained be "effective". For example, we do not only need to know that a sequence is convergent – "for any ε there exists an N_ε" – but also how fast, i.e. obtain a formula or at least a computable bound for that N_ε. Proof mining is a branch of mathematics that attempts to do exactly that: namely, it provides algorithms that can then be applied to the actual analysis proofs (formalized in a suitable proof system) in order to obtain such computational information. (This area of study was first proposed by Kreisel in the 1960s, under the name "unwinding of proofs", and given maturity by Kohlenbach in the 1990s.) Another goal that is achieved by this method is that the formulas that are such obtained "work" for the general class of objects for which the original theorems applied – in other words, the realizers or bounds are independent of non-numerical parameters such as the actual space involved and only depend on data like its diameter.

It turns out that this second desideratum can be often more easily achieved by so-called "semantic" or "model-theoretic" methods, i.e. using a modification of the famous ultraproduct method from model theory. Some early results in this direction can be found in [4]. Recently, Avigad and Iovino [1] have obtained similar (but non-effective) results to those obtained by proof theorists regarding a property of sequences called metastability. Metastability is an equivalent formulation (due to Tao) of the property of a sequence to be Cauchy that has the advantage that it is expressed in a more suitable logical form (an $\forall\exists$-sentence). That way, it is more amenable to both model-theoretic and proof-theoretic analysis of bounds. It is this method of Avigad and Iovino that we seek to describe here in extreme detail.

After two preliminary sections, in Section 4, we define the main concept that is used – the metric ultraproduct – and prove its basic properties, as well as more elaborate (and peripheral) ones like its metric completeness. Section 5 presents the main result from the paper, i.e. the one that obtains a bound for the rate of metastability of a large class of sequences in metric spaces. Section 6 presents in detail two applications from the original paper which show how at least the independence of parameters shown by proof theorists for a number of well-known iterations can be obtained as special cases of Avigad and Iovino's general theorem from the previous section.

The last section presents an original result, namely a modification of the main theorem to obtain in a restricted case a property known as "asymptotic regularity". We first prove the analogue to the main theorem and then proceed to apply it in a case already tackled by proof theorists, in order to obtain the same independence of parameters.

2 Limits relative to ultrafilters

This section presents in detail how one may work with limits relative to ultrafilters (called "ultralimits" by Tao).

2.1 Topological phenomena

Definition 2.1. If X is a topological space, F is a filter in the Boolean algebra $\mathcal{P}(X)$ and $l \in X$, then l is called a **limit** for F if any open set that contains l is an element of F.

Remark 2.2. *In the above definition, one can restrict zirself to checking the open sets that are in a given base of X (and in the case of a metric space, only on the l-centered open – or closed! – balls).*

Theorem 2.3. *A topological space X is compact iff any ultrafilter in $\mathcal{P}(X)$ has at least one limit.*

Proof. Suppose X is compact and D is an ultrafilter such that any $l \in X$ is contained in some open A_l that is not in D. Then the union of all D_l's is X, so, by compactness, there is a finite $F \subseteq X$ s.t. the union of the D_l's with $l \in F$ is X, which is in U. But since U is also prime, there is an l s.t. $D_l \in U$, contradicting our hypothesis.

Suppose, now, that the ultrafilter property holds and that X is not compact. Then there is some family of opens $\{X_i\}_{i \in I}$ that covers X s.t. any finite subfamily of it leaves a point outside its union. For each finite subset J of I denote by P_J the set of points outside the union of the subfamily $\{X_i\}_{i \in J}$. Since each P_J is nonempty and $P_J \cap P_K = P_{J \cup K}$, there is an ultrafilter D containing all the P_J's. Let l be a limit for D. Then, since $\{X_i\}_{i \in I}$ covers X, l is in some X_i, so $X_i \in D$. But the complement of X_i is $P_{\{i\}}$ which must also be in D by the construction of D. So the empty set is in D, which is a contradiction. □

Theorem 2.4. *A topological space X is Hausdorff iff any ultrafilter in $\mathcal{P}(X)$ has at most one limit.*

Proof. Suppose X is Hausdorff and D is an ultrafilter which has two distinct limits l_1 and l_2. Let V and W be two disjoint open neighborhoods for l_1 and l_2, respectively. But since l_1 and l_2 are limits for D, we have that $V, W \in D$, so $\emptyset = V \cap W \in D$, contradiction.

Suppose, now, that any ultrafilter has at most one limit. If X is not Hausdorff, there are $x, y \in X$ such that any two open neighborhoods U and V of x and y, respectively, have non-empty intersection. Take the set F containing all subsets of X which are supersets of such intersections. Then F is a proper filter, and any ultrafilter D that extends F has both x and y as limits, contradicting our hypothesis. □

If $f : A \to B$ is a function, one can "extend" it to a function $f_* : Ult(\mathcal{P}(A)) \to Ult(\mathcal{P}(B))$, defined by $f_*(D) = \{M \subseteq B \mid f^{-1}(M) \in D\}$, for any ultrafilter D in $\mathcal{P}(A)$.

2.2 Metric phenomena

Putting all this together, in the case of a sequence $\{a_i\}_{i \in \mathbb{N}}$ in a metric space (X, d), if D is a non-principal ultrafilter in $\mathcal{P}(\mathbb{N})$, then $a_*(D)$ has at most one limit, which, if exists, is the only element l of X that satisfies the following: for any $\varepsilon > 0$, the set $\{i \in \mathbb{N} \mid d(a_i, l) \le \varepsilon\}$ is in D. We will denote this situation by:

$$\lim_{i \to D} a_i = l$$

Lemma 2.5. *If D is a non-principal ultrafilter in $\mathcal{P}(\mathbb{N})$ and $\{a_i\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}}$ are real-valued sequences such that $\lim_{i \to D} a_i$ and $\lim_{i \to D} b_i$ exist, then $\lim_{i \to D}(a_i + b_i)$ also exists and:*

$$\lim_{i \to D}(a_i + b_i) = \lim_{i \to D} a_i + \lim_{i \to D} b_i$$

Proof. We denote by l and m, respectively, the two limits in the right hand side of the above. Let $\varepsilon > 0$ be arbitrary and take η to be equal to $\frac{\varepsilon}{2}$. Then, by the limit property, the sets:

$$A_\eta = \{i \in \mathbb{N} \mid |a_i - l| \le \eta\}$$

$$B_\eta = \{i \in \mathbb{N} \mid |b_i - m| \le \eta\}$$

are in D, so $A_\eta \cap B_\eta$ is in D. Take an $i \in A_\eta \cap B_\eta$. Then:

$$\begin{aligned} |a_i + b_i - (l + m)| &\le |a_i - l| + |b_i - m| \\ &\le 2\eta \\ &= \varepsilon. \end{aligned}$$

We have, then, that $A_\eta \cap B_\eta \subseteq \{i \in \mathbb{N} \mid |a_i + b_i - (l + m)| \le \varepsilon\}$, so the latter set is in D. Since ε was chosen arbitrarily, we have that $\lim_{i \to D}(a_i + b_i)$ exists and is equal to $l + m$, which was what we sought to show. □

Lemma 2.6. *If D is a non-principal ultrafilter in $\mathcal{P}(\mathbb{N})$ and $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ are real-valued sequences such that $\lim_{i\to D} a_i$ and $\lim_{i\to D} b_i$ exist, then $\lim_{i\to D}(a_i \cdot b_i)$ also exists and:*

$$\lim_{i\to D}(a_i \cdot b_i) = \lim_{i\to D} a_i \cdot \lim_{i\to D} b_i$$

Proof. We denote by l and m, respectively, the two limits in the right hand side of the above. Let $\varepsilon > 0$ be arbitrary and take $\eta > 0$ be such that $\eta^2 + |l|\eta + |m|\eta \leq \varepsilon$. Then, by the limit property, the sets:

$$A_\eta = \{i \in \mathbb{N} \mid |a_i - l| \leq \eta\}$$

$$B_\eta = \{i \in \mathbb{N} \mid |b_i - m| \leq \eta\}$$

are in D, so $A_\eta \cap B_\eta$ is in D. Take an $i \in A_\eta \cap B_\eta$. Then:

$$\begin{aligned}
|a_ib_i - lm| &= |a_ib_i - ma_i + ma_i - lm| \\
&= |a_i(b_i - m) + m(a_i - l)| \\
&\leq |a_i(b_i - m)| + |m(a_i - l)| \\
&\leq |a_i|\eta + |m|\eta \\
&= |a_i - l + l|\eta + |m|\eta \\
&\leq |a_i - l|\eta + |l|\eta + |m|\eta \\
&\leq \eta^2 + |l|\eta + |m|\eta \\
&\leq \varepsilon.
\end{aligned}$$

We have, then, that $A_\eta \cap B_\eta \subseteq \{i \in \mathbb{N} \mid |a_ib_i - lm| \leq \varepsilon\}$, so the latter set is in D. Since ε was chosen arbitrarily, we have that $\lim_{i\to D}(a_i \cdot b_i)$ exists and is equal to $l \cdot m$, which was what we sought to show. □

Lemma 2.7. *If D is a non-principal ultrafilter in $\mathcal{P}(\mathbb{N})$ and $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ are real-valued sequences such that $\lim_{i\to D} a_i$ and $\lim_{i\to D} b_i$ exist and for any $i \in \mathbb{N}$, $a_i \leq b_i$, then:*

$$\lim_{i\to D} a_i \leq \lim_{i\to D} b_i$$

Proof. We denote by l and m, respectively, the two limits. Let $\varepsilon > 0$ be arbitrary. Then, by the limit property, the sets:

$$A_\varepsilon = \{i \in \mathbb{N} \mid |a_i - l| \leq \varepsilon\}$$

$$B_\varepsilon = \{i \in \mathbb{N} \mid |b_i - m| \leq \varepsilon\}$$

are in D, so $A_\varepsilon \cap B_\varepsilon$ is nonempty. Take an $i \in A_\varepsilon \cap B_\varepsilon$. Then:

$$l \leq a_i + \varepsilon$$

$$-m \leq -b_i + \varepsilon$$

Summing up, we have that:

$$l - m \leq a_i - b_i + 2\varepsilon$$

Given that $a_i - b_i \leq 0$, we have $l - m \leq 2\varepsilon$, and since ε was chosen arbitrarily, $l - m \leq 0$, which was what we sought to show. □

3 Background on analysis

This section recalls elementary results from real and functional analysis in a way that may be useful for a reader outside the field.

Theorem 3.1. *Let (X,d) be a metric space and $\{a_i\}_{i\in\mathbb{N}}$ be a sequence in X. TFAE:*

1) $\{a_i\}_{i\in\mathbb{N}}$ is Cauchy, i.e. for any $\varepsilon>0$ there is an N_ε s.t. for any $m,n\geq N_\varepsilon$, $d(a_m,a_n)\leq\varepsilon$;
2) for any $\varepsilon>0$ and any $F:\mathbb{N}\to\mathbb{N}$ there is an $N_{F,\varepsilon}$ s.t. for any $m,n\in[N_{F,\varepsilon},F(N_{F,\varepsilon})]$, $d(a_m,a_n)\leq\varepsilon$.

Proof. (1) $\Rightarrow$ (2) Let $\varepsilon>0$ and $F:\mathbb{N}\to\mathbb{N}$ be as in (2). We take $N_{F,\varepsilon}$ to be the N_ε from (1) and we are done.

(2) $\Rightarrow$ (1) Suppose that there is an $\varepsilon>0$ s.t. for any natural N there are $m_N,n_N\geq N$ s.t. $d(a_{m_N},a_{n_N})>\varepsilon$. Define the function $F:\mathbb{N}\to\mathbb{N}$ by setting for any $N\in\mathbb{N}$, $F(N)=\max\{m_N,n_N\}$. Consider the $N_{F,\varepsilon}$ from (2) associated to our F and ε. Then, by (2), for any $m,n\in[N_{F,\varepsilon},F(N_{F,\varepsilon})]$, $d(a_m,a_n)\leq\varepsilon$. But our m_N and n_N are in $[N_{F,\varepsilon},F(N_{F,\varepsilon})]$ and $d(a_{m_N},a_{n_N})>\varepsilon$, which is a contradiction. □

Note that the proof above works exactly the same for the "$<\varepsilon$" characterizations.

Definition 3.2. Let (X,d) be a metric space and $\{a_i\}_{i\in\mathbb{N}}$ be a sequence in X. A function $\Phi:\mathbb{N}^\mathbb{N}\times(0,a)\to\mathbb{N}$, where $a\in\mathbb{R}_+$, such that $N_{F,\varepsilon}$ can be chosen in Theorem 3.1, (2), for all $\varepsilon<a$, to be equal to $\Phi(F,\varepsilon)$, is called a **rate of metastability** for the sequence $\{a_i\}_{i\in\mathbb{N}}$. For this reason, we can call a Cauchy sequence a **metastable** sequence.

Definition 3.3. If (X,d) and (Y,d') are metric spaces, a function $f:X\to Y$ is called **nonexpansive** if for all $x,y\in X$ we have that:

$$d'(f(x),f(y))\leq d(x,y)$$

3.1 Convexity

Definition 3.4. An **W-hyperbolic space** is a triple (X,d,W) where (X,d) is a metric space and $W:X^2\times[0,1]\to X$ is a function such that:

1) for any $x,y,z\in X$ and $\lambda\in[0,1]$, $d(z,W(x,y,\lambda))\leq(1-\lambda)d(z,x)+\lambda d(z,y)$;
2) for any $x,y\in X$ and $\lambda,\mu\in[0,1]$, $d(W(x,y,\lambda),W(x,y,\mu))=|\lambda-\mu|d(x,y)$;
3) for any $x,y,z,w\in X$ and $\lambda\in[0,1]$, $d(W(x,z,\lambda),W(y,w,\lambda))\leq(1-\lambda)d(x,y)+\lambda d(z,w)$.

Definition 3.5. If (X,d,W) is a W-hyperbolic space and $x,y\in X$, we define the set $[x,y]$ to be the set of all $z\in X$ such that there is some $\lambda\in[0,1]$ with $z=W(x,y,\lambda)$.

Definition 3.6. If (X,d,W) is a W-hyperbolic space and $C\subseteq X$, C is called convex if for any $x,y\in C$ we have that $[x,y]\subseteq C$.

Definition 3.7. An W-hyperbolic space (X,d,W) is called a **CAT(0) space** if for any $x,y,m,z\in X$ such that $d(x,m)=d(y,m)=\frac{1}{2}d(x,y)$, we have that:

$$d(z,m)^2\leq\frac{1}{2}d(z,x)^2+\frac{1}{2}d(z,y)^2-\frac{1}{4}d(x,y)^2$$

3.2 Banach spaces

Definition 3.8. A **normed real vector space** is a pair $(X, \|\|)$, where X is a real vector space and $\|\| : V \to \mathbb{R}$ is a **norm** on X, i.e. it has the following properties:

1) for any $\alpha \in \mathbb{R}$ and any $v \in X$, $\|\alpha v\| = |\alpha|\|v\|$;
2) for any $v, w \in X$, $\|v + w\| \leq \|v\| + \|w\|$;
3) for any $v \in X$ s.t. $\|v\| = 0$, we have that $v = 0$.

Remark 3.9. *Any normed real vector space $(X, \|\|)$ has a canonical structure of a metric space, given by the distance function $d_{\|\|} : X \times X \to \mathbb{R}$ defined by $d_{\|\|}(v, w) = \|v - w\|$, for any $v, w \in X$.*

Definition 3.10. A **Banach space** is a normed real vector space such that its canonical distance function makes it a complete metric space.

Remark 3.11. *Let X and Y be two normed real vector spaces and $f : X \to Y$ be a linear map. TFAE:*

1) f is continuous, i.e. for any $\varepsilon > 0$ there is a δ_ε s.t. for any $v \in X$ with $\|v\| \leq \delta_\varepsilon$ we have that $\|f(v)\| \leq \varepsilon$;
2) there is a $c \in \mathbb{R}$ s.t. for any $v \in X$ we have that $\|f(v)\| \leq c\|v\|$.

Proof. (1) $\Rightarrow$ (2) Take $c := \frac{1}{\delta_1}$.

(2) $\Rightarrow$ (1) Take $\delta_\varepsilon := \frac{\varepsilon}{c}$. □

Definition 3.12. Let X and Y be two normed real vector spaces and $f : X \to Y$ be a linear and continuous map. We define the **norm** of f to be the number:

$$\|f\| = \sup_{v \in X \setminus \{0\}} \frac{\|f(v)\|}{\|v\|}$$

(well-defined because of the remark above). This makes the set of continuous linear maps from X to Y into a normed real vector space. If $Y = \mathbb{R}$ the resulting space is denoted by X' and is called the **continuous dual** of X.

Definition 3.13. A normed real vector space is called **reflexive** if for any $\phi \in X''$ there is a $v \in X$ s.t. for any $f \in X'$ we have that $\phi(f) = f(x)$.

Definition 3.14. Let X be a normed real vector space and $T : X \to X$ a linear continuous map. We call T **power-bounded** if there is a M s.t. for any $n \in \mathbb{N}$, $\|T^n\| \leq M$.

4 Metric ultraproducts

Let $\mathcal{X} = \{(X_i, d_i)\}_{i \in \mathbb{N}}$ be a countable family of metric spaces. For each i fix a "basepoint" $b_i \in X_i$. Let D be a non-principal ultrafilter in $\mathcal{P}(\mathbb{N})$. Consider the following set:

$$\mathcal{X}_{\{b_i\}_{i \in \mathbb{N}}} = \{\{x_i\}_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} X_i \mid \sup_{i \in \mathbb{N}} d_i(x_i, b_i) < \infty\}$$

(Note that if there is an $M > 0$ such that all the spaces in $\mathcal{X}$ have diameters less than M, the set above does not depend on the choice of basepoints.)

Let $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ be two elements of $\mathcal{X}_{\{b_i\}_{i \in \mathbb{N}}}$. Say $A = \sup_{i \in \mathbb{N}} d_i(x_i, b_i)$ and $B = \sup_{i \in \mathbb{N}} d_i(y_i, b_i)$. Then, for any i:

$$0 \leq d_i(x_i, y_i) \leq d_i(x_i, b_i) + d_i(y_i, b_i) \leq A + B$$

So the sequence $\{d_i(x_i, y_i)\}_{i\in\mathbb{N}}$ takes values in the interval $[0, A+B]$, which is compact.

That means we can define the function $\widetilde{d}_D : \mathcal{X}_{\{b_i\}_{i\in\mathbb{N}}} \times \mathcal{X}_{\{b_i\}_{i\in\mathbb{N}}} \to \mathbb{R}_+$ by setting, for any $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$:

$$\widetilde{d}_D(\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}}) = \lim_{i\to D} d_i(x_i, y_i),$$

so for any $\varepsilon > 0$ and for any $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$, the set

$$\{i \in \mathbb{N} \mid |d_i(x_i, y_i) - \widetilde{d}_D(\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}})| \le \varepsilon\}$$

is in D.

Consider now the relation $\sim_D$ on $\mathcal{X}_{\{b_i\}_{i\in\mathbb{N}}}$ such that for any $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$:

$$\{x_i\}_{i\in\mathbb{N}} \sim_D \{y_i\}_{i\in\mathbb{N}} \Leftrightarrow \widetilde{d}_D(\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}}) = 0$$

The relation $\sim_D$ is easily seen to be reflexive and symmetric. To prove transitivity, consider three elements of $\mathcal{X}_{\{b_i\}_{i\in\mathbb{N}}}$, namely $\{x_i\}_{i\in\mathbb{N}}$, $\{y_i\}_{i\in\mathbb{N}}$ and $\{z_i\}_{i\in\mathbb{N}}$, such that $\{x_i\}_{i\in\mathbb{N}} \sim_D \{y_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}} \sim_D \{z_i\}_{i\in\mathbb{N}}$. Let $\varepsilon > 0$ be arbitrary and η to be $\frac{\varepsilon}{2}$. Then the sets:

$$A_\eta = \{i \in \mathbb{N} \mid d_i(x_i, y_i) \le \eta\}$$

$$B_\eta = \{i \in \mathbb{N} \mid d_i(y_i, z_i) \le \eta\}$$

are in D, so $A_\eta \cap B_\eta$ is also in D. Take $i \in A_\eta \cap B_\eta$. Then:

$$\begin{aligned} d_i(x_i, z_i) &\le d_i(x_i, y_i) + d_i(y_i, z_i) \\ &\le 2\eta \\ &= \varepsilon. \end{aligned}$$

We have, then, that $A_\eta \cap B_\eta \subseteq \{i \in \mathbb{N} \mid d_i(x_i, z_i) \le \varepsilon\}$, so the latter set is in D. Since ε was chosen arbitrarily, we have that $\lim_{i\to D} d_i(x_i, z_i)$ exists and is equal to 0, which was what we sought to show.

Lemma 4.1. *If $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ are such that the set $A = \{i \in \mathbb{N} \mid x_i = y_i\}$ is in D, then:*

$$\{x_i\}_{i\in\mathbb{N}} \sim_D \{y_i\}_{i\in\mathbb{N}}$$

Proof. Take $\varepsilon > 0$. We need to show that the set

$$A_\varepsilon = \{i \in \mathbb{N} \mid d_i(x_i, y_i) \le \varepsilon\}$$

is in D. Take $i \in A$. Then $x_i = y_i$, so $d_i(x_i, y_i) = 0 \le \varepsilon$ and $i \in A_\varepsilon$. So $A \subseteq A_\varepsilon$ and since $A \in D$, A_ε is also in D. □

We denote by $\mathcal{X}_D$ the quotient set of $\mathcal{X}_{\{b_i\}_{i\in\mathbb{N}}}$ by $\sim_D$. We define the function $d_D : \mathcal{X}_D \times \mathcal{X}_D \to \mathbb{R}_+$ by setting, for any $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$:

$$d_D(\widehat{\{x_i\}_{i\in\mathbb{N}}}, \widehat{\{y_i\}_{i\in\mathbb{N}}}) = \widetilde{d}_D(\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}})$$

Let us check that this is indeed well defined. Consider four sequences $\{x_i\}_{i\in\mathbb{N}}$, $\{x_i'\}_{i\in\mathbb{N}}$, $\{y_i\}_{i\in\mathbb{N}}$ and $\{y_i'\}_{i\in\mathbb{N}}$, such that $\{x_i\}_{i\in\mathbb{N}} \sim_D \{x_i'\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}} \sim_D \{y_i'\}_{i\in\mathbb{N}}$.

We denote by l the number $\widetilde{d}_D(\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}})$ and by l' the number $\widetilde{d}_D(\{x_i'\}_{i\in\mathbb{N}}, \{y_i'\}_{i\in\mathbb{N}})$. We want to show $l = l'$. Let $\varepsilon > 0$ be arbitrary. Then the sets:

$$A_\varepsilon = \{i \in \mathbb{N} \mid d_i(x_i, x_i') \le \varepsilon\}$$

$$B_\varepsilon = \{i \in \mathbb{N} \mid d_i(y_i, y_i') \le \varepsilon\}$$
$$C_\varepsilon = \{i \in \mathbb{N} \mid |d_i(x_i, y_i) - l| \le \varepsilon\}$$
$$F_\varepsilon = \{i \in \mathbb{N} \mid |d_i(x_i', y_i') - l'| \le \varepsilon\}$$

are in D. Then $A_\varepsilon \cap B_\varepsilon \cap C_\varepsilon \cap F_\varepsilon$ is in D, so it is nonempty. Take $i \in A_\varepsilon \cap B_\varepsilon \cap C_\varepsilon \cap F_\varepsilon$. Then, by the triangle inequality for d_i:

$$d_i(x_i', y_i') \le d_i(x_i', x_i) + d_i(x_i, y_i) + d_i(y_i, y_i')$$

so

$$d_i(x_i', y_i') - d_i(x_i, y_i) \le d_i(x_i', x_i) + d_i(y_i, y_i')$$

Similarly, we have that:

$$d_i(x_i, y_i) - d_i(x_i', y_i') \le d_i(x_i', x_i) + d_i(y_i, y_i')$$

so we can write

$$\begin{aligned} |d_i(x_i', y_i') - d_i(x_i, y_i)| &\le d_i(x_i', x_i) + d_i(y_i, y_i') \\ &\le 2\varepsilon. \end{aligned}$$

Then:

$$\begin{aligned} |l - l'| &\le |l - d_i(x_i, y_i)| + |d_i(x_i', y_i') - d_i(x_i, y_i)| + |d_i(x_i', y_i') - l'| \\ &\le \varepsilon + 2\varepsilon + \varepsilon \\ &= 4\varepsilon. \end{aligned}$$

Since ε was chosen arbitrarily, $|l - l'| = 0$, so $l = l'$.

We claim that d_D is a metric on $\mathcal{X}_D$. Clearly $d_D(\widehat{\{x_i\}_{i\in\mathbb{N}}}, \widehat{\{y_i\}_{i\in\mathbb{N}}}) = 0$ iff $\widehat{\{x_i\}_{i\in\mathbb{N}}} = \widehat{\{y_i\}_{i\in\mathbb{N}}}$, by the very definition of $\sim_D$. It is symmetric since the d_i's are symmetric. The triangle inequality follows immediately by applying Lemmas 2.5 and 2.7.

The metric space $(\mathcal{X}_D, d_D)$ just constructed is called the **metric ultraproduct** of the family $\mathcal{X}$ with respect to the family of basepoints $\{b_i\}_{i\in\mathbb{N}}$ and the (non-principal) ultrafilter D.

Lemma 4.2. *Let $\widehat{\{x_i\}_{i\in\mathbb{N}}}, \widehat{\{y_i\}_{i\in\mathbb{N}}} \in \mathcal{X}_D$ and $a \in \mathbb{R}$. Let B be the set $\{i \in \mathbb{N} \mid d_i(x_i, y_i) \le a\}$. Then:*

1) if $d_D(\widehat{\{x_i\}_{i\in\mathbb{N}}}, \widehat{\{y_i\}_{i\in\mathbb{N}}}) < a$, then $B \in D$;
2) if $B \in D$, then $d_D(\widehat{\{x_i\}_{i\in\mathbb{N}}}, \widehat{\{y_i\}_{i\in\mathbb{N}}}) \le a$.

Proof. We denote by l the number $d_D(\widehat{\{x_i\}_{i\in\mathbb{N}}}, \widehat{\{y_i\}_{i\in\mathbb{N}}})$, so, for any $\varepsilon > 0$, the set

$$\{i \in \mathbb{N} \mid |d_i(x_i, y_i) - l| \le \varepsilon\}$$

is in D.

1) We know that $l < a$. Let $\varepsilon > 0$ be such that $l + \varepsilon < a$. So the set:

$$A = \{i \in \mathbb{N} \mid |d_i(x_i, y_i) - l| \le \varepsilon\}$$

is in D. Take an $i \in A$. Then $d_i(x_i, y_i) \le l + \varepsilon < a$ and $i \in B$. So $A \subseteq B$ and $B \in D$.

2) Take an $\varepsilon > 0$ and an i in the intersection of

$$\{i \in \mathbb{N} \mid |d_i(x_i, y_i) - l| \le \varepsilon\}$$

and B, which is necessarily nonempty, since both sets are in D. Then

$$l - \varepsilon \le d_i(x_i, y_i) \le a$$

so

$$l - \varepsilon \le a$$

and since ε was chosen arbitrarily we have that $l \le a$.

□

Theorem 4.3. *The metric space $(\mathcal{X}_D, d_D)$ is complete.*

Proof. Consider a Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}_D$ that we want to prove convergent.

Let $\{b_{jn}\}_{j,n\in\mathbb{N}}$ be such that for any $n \in \mathbb{N}$, $a_n = \widehat{\{b_{jn}\}_{j\in\mathbb{N}}}$. By the Cauchyness of the sequence, we have that for any $m \in \mathbb{N}$ there is an N_m such that for any $n \geq N_m$, $d_D(a_n, a_{N_m}) < \frac{1}{2^m}$. Moreover, we can choose the N_m's such that for $m < m'$ we have $N_m < N_{m'}$ (by a simple transformation like $N'_0 := N_0$, $N'_{k+1} := \max\{N_{k+1}, N'_k + 1\}$).

Using Lemma 4.2, (1), we have that for any m and any $n \leq N_m$, the set $\{j \in \mathbb{N} \mid d_j(b_{jn}, b_{jN_m}) \leq \frac{1}{2^m}\}$ is in D.

We shall now define a family $\{f_{jn}\}_{n,j\in\mathbb{N}}$ such that:

- for any n, the set $\{j \in \mathbb{N} \mid f_{jn} = b_{jn}\}$ is in D;
- for any m, any $n > N_m$ and any $j \in \mathbb{N}$, $d_j(f_{jn}, f_{jN_m}) \leq \frac{1}{2^{m-1}}$.

From the first property, one would conclude, applying Lemma 4.1, that, for any n, $\widehat{\{b_{jn}\}_{j\in\mathbb{N}}} = \widehat{\{f_{jn}\}_{j\in\mathbb{N}}}$ and so the family $\{f_{jn}\}_{j,n\in\mathbb{N}}$ can "represent" the sequence $\{a_n\}_{n\in\mathbb{N}}$.

We will define the family inductively. For $n \leq N_0$ and $j \in \mathbb{N}$, set f_{jn} to be equal to b_{jn}. If $n > N_0$, then the set $\{l \mid N_l < n\}$ is nonempty and bounded above (by n itself), so it has a maximum which we will denote by k_n. We can see that if $p \in \mathbb{N}$, then $k_{N_p} = p - 1$. For our n, we have that the set $B = \{j \in \mathbb{N} \mid d_j(b_{jn}, b_{jN_{k_n}}) \leq \frac{1}{2^{k_n}}\}$ is in D. Also, the elements $\{f_{jN_{k_n}}\}_{j\in\mathbb{N}}$ are already defined, so the set $\{j \in \mathbb{N} \mid f_{jN_{k_n}} = b_{jN_{k_n}}\}$ is in D. Take a j in the intersection of that set and B, which is nonempty (since both sets are in D). Then:

$$d_j(b_{jn}, f_{jN_{k_n}}) = d_j(b_{jn}, b_{jN_{k_n}}) \leq \frac{1}{2^{k_n}}$$

So the set:

$$C = \{j \in \mathbb{N} \mid d_j(b_{jn}, f_{jN_{k_n}}) \leq \frac{1}{2^{k_n}}\}$$

is in D. We are now able to define:

$$f_{jn} = \begin{cases} b_{jn}, & \text{if } j \in C. \\ f_{jN_{k_n}}, & \text{otherwise.} \end{cases}$$

We can see that the set $\{j \in \mathbb{N} \mid f_{jn} = b_{jn}\}$ contains C, so it is in D and the first property is checked for this n.

Consider now a $j \in \mathbb{N}$. If $j \in C$, then $d_j(f_{jn}, f_{jN_{k_n}}) = d_j(b_{jn}, f_{jN_{k_n}}) \leq \frac{1}{2^{k_n}}$. If $j \notin C$, then $d_j(f_{jn}, f_{jN_{k_n}}) = d_j(f_{jN_{k_n}}, f_{jN_{k_n}}) = 0 \leq \frac{1}{2^{k_n}}$. Anyway, for any j we see that $d_j(f_{jn}, f_{jN_{k_n}}) \leq \frac{1}{2^{k_n}}$.

We have now finished constructing the family. Note that in particular for any $j, p \in \mathbb{N}$, since $k_{N_p} = p - 1$ and so $N_{k_{N_p}} = N_{p-1}$, we have that $d_j(f_{jN_p}, f_{jN_{p-1}}) \leq \frac{1}{2^{p-1}}$.

It remains to check the second property that is required of the family. Take $m \in \mathbb{N}$, $n \geq N_m$ and $j \in \mathbb{N}$. This means that also $k_n \geq m$. Then:

$$\begin{aligned} d_j(f_{jn}, f_{jN_m}) &\leq d_j(f_{jn}, f_{jN_{k_n}}) + d_j(f_{jN_{k_n}}, f_{jN_{k_n-1}}) + \cdots + d_j(f_{jN_{m+1}}, f_{jN_m}) \\ &\leq \frac{1}{2^{k_n}} + \frac{1}{2^{k_n-1}} + \cdots + \frac{1}{2^m} \\ &\leq \sum_{i=m}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2^{m-1}}. \end{aligned}$$

We can "rewrite" this second property in the following way: for any $\varepsilon > 0$ there is an N_ε such that for any $m, n \geq N_\varepsilon$ and any $j \in \mathbb{N}$, $d_j(f_{jn}, f_{jm}) \leq \varepsilon$.

Define now, for any $j \in \mathbb{N}$, $c_j := f_{jj}$ (this is the first time we use that the family $\mathcal{X}$ is countable!) and $c = \widehat{\{c_j\}_{j\in\mathbb{N}}}$. We will prove that c is the limit of the sequence $\{a_n\}_{n\in\mathbb{N}}$, i.e. that for any $\varepsilon > 0$ there is an M_ε such that for any $n \geq M_\varepsilon$ we have $d_D(a_n, c) \leq \varepsilon$.

Take an $\varepsilon > 0$. We will take the M_ε to be the N_ε from two paragraphs above. Take j from the set $G = \{k \mid k \geq N_\varepsilon\}$ which is in D since it is cofinite and D is non-principal (this is the first time we use this!). Then $j \geq N_\varepsilon$ and so $d_j(f_{jn}, c_j) = d_j(f_{jn}, f_{jj}) \leq \varepsilon$. So the set $\{j \mid d_j(f_{jn}, c_j) \leq \varepsilon\}$ contains G, so it is in D, and since the family $\{f_{jn}\}_{j,n\in\mathbb{N}}$ represents our sequence, then, by Lemma 4.2, (2), $d_D(a_n, c) \leq \varepsilon$. □

5 The main result

From now on, frequently, but not always, if X is a metric space, then X will also denote (by abuse of language) its underlying set and d_X will be its metric. That is, $X = (X, d_X)$.

Theorem 5.1. *(Avigad, Iovino [1]) Let $\mathcal{C}$ be a (possibly proper) class (whose elements will be used as indices). Let $\{A_i\}_{i\in\mathcal{C}}$ be a family of metric spaces. For each $i \in \mathcal{C}$ choose $b_i \in A_i$ a basepoint and a sequence $\{a_{in}\}_{n\in\mathbb{N}} \subseteq A_i$ such that for any $n \in \mathbb{N}$, the boundedness condition $\sup_{i\in\mathcal{C}} d_{A_i}(a_{in}, b_i) < \infty$ holds. Let D be a non-principal ultrafilter in $\mathcal{P}(\mathbb{N})$. TFAE:*

1) for any $\varepsilon > 0$ and any $F : \mathbb{N} \to \mathbb{N}$ there exists a $B_{F,\varepsilon} \in \mathbb{N}$ (a "bound on the rate of metastability") such that for any $i \in \mathcal{C}$ there exists an $N_{F,\varepsilon,i} \leq B_{F,\varepsilon}$ s.t. for any $m, n \in [N_{F,\varepsilon,i}, F(N_{F,\varepsilon,i})]$ we have that $d_{A_i}(a_{im}, a_{in}) < \varepsilon$.

2) for any $g : \mathbb{N} \to \mathcal{C}$, if we denote by $(\mathcal{X}_D, d_D)$ the metric ultraproduct of the (countable) family $\{A_{g(k)}\}_{k\in\mathbb{N}}$ w.r.t. the family of basepoints $\{b_{g(k)}\}_{k\in\mathbb{N}}$ and the ultrafilter D and if we denote, for any $n \in \mathbb{N}$, c_n to be the element $\widehat{\{a_{g(k)n}\}_{k\in\mathbb{N}}}$ of $\mathcal{X}_D$ (well-defined because of the boundedness condition), then for any $\varepsilon > 0$ and any $F : \mathbb{N} \to \mathbb{N}$ there is an $N_{F,\varepsilon}$ s.t. for any $m, n \in [N_{F,\varepsilon}, F(N_{F,\varepsilon})]$, we have that $d_D(c_m, c_n) < \varepsilon$ – that is, by Theorem 3.1, the sequence $\{c_n\}_{n\in\mathbb{N}}$ is Cauchy, or, equivalently, it is convergent, since we have previously proved that $(\mathcal{X}_D, d_D)$ is complete.

Note that to get from "$\{c_n\}_{n\in\mathbb{N}}$ is convergent" to the statement in (1), which will be the direction most commonly used in applications, we do not need the fact that $(\mathcal{X}_D, d_D)$ is complete.

Proof. (1) ⇒ (2) Let g, $(\mathcal{X}_D, d_D)$ and $\{c_n\}_{n\in\mathbb{N}}$ be as in the hypothesis of (2). Take an $\varepsilon > 0$ and an $F : \mathbb{N} \to \mathbb{N}$. Applying (1), we get that for any $k \in \mathbb{N}$ there exists an $N_k \leq B_{F,\frac{\varepsilon}{2}}$ such that for any $m, n \in [N_k, F(N_k)]$ we have that $d_{A_{g(k)}}(a_{g(k)m}, a_{g(k)n}) \leq \frac{\varepsilon}{2}$.

We seek an $N_{F,\varepsilon}$ s.t. for any $m, n \in [N_{F,\varepsilon}, F(N_{F,\varepsilon})]$, we have that $d_D(c_m, c_n) < \varepsilon$.

For any $j \in \overline{1,b}$ denote by R_j the set $\{k \in \mathbb{N} \mid N_k = j\}$. Since for any $k \in \mathbb{N}$, $N_k \in \overline{1,b}$, we have that:

$$\bigcup_{j\in\overline{1,b}} R_j = \mathbb{N} \in D$$

so, by the fact that D is a prime filter, there is an $N \in \overline{1,b}$ such that $R_N \in D$. We put $N_{F,\varepsilon}$ to be this N.

Let now m, n be in $[N, F(N)]$. Let k be an arbitrary element of R_N. Then $N_k = N$, so

$$d_{A_{g(k)}}(a_{g(k)m}, a_{g(k)n}) \leq \frac{\varepsilon}{2}$$

Since k was chosen arbitrarily, we have that:

$$R_N \subseteq \{k \in \mathbb{N} \mid d_{A_{g(k)}}(a_{g(k)m}, a_{g(k)n}) \leq \frac{\varepsilon}{2}\}$$

so the latter set is also in D. By Lemma 4.2, (2), $d_D(c_m, c_n) \leq \frac{\varepsilon}{2} < \varepsilon$.

(2) $\Rightarrow$ (1) Suppose that the negation of (1) holds, i.e. that there exist $\varepsilon > 0$ and $F : \mathbb{N} \to \mathbb{N}$ s.t. for any B there is an $i_B \in \mathcal{C}$ such that for any $N \leq B$ there are $m, n \in [N, F(N)]$ with $d_{A_{i_B}}(a_{i_B m}, a_{i_B n}) \geq \varepsilon$.

Now for any $N \in \mathbb{N}$, denote by F_N the set:

$$\{B \in \mathbb{N} \mid \exists m, n \in [N, F(N)], d_{A_{i_B}}(a_{i_B m}, a_{i_B n}) \geq \varepsilon\}$$

Consider $N \in \mathbb{N}$, and let l be greater than N. Then $N \leq l$ so by the negation of (1), $l \in F_N$. So the set F_N contains the set of naturals greater than N and since the latter is cofinite, it is in D, so F_N is also in D.

Take $g : \mathbb{N} \to \mathcal{C}$ to be the function defined by $g(B) = i_B$ for any $B \in \mathbb{N}$, where i_B is the one from the negation of (1). Using the notation from (2), take N to be $N_{F,\frac{\varepsilon}{2}}$. Then for any $m, n \in [N, F(N)]$, we have that $d_D(c_m, c_n) < \frac{\varepsilon}{2}$. By applying Lemma 4.2, (1), we get that for any $m, n \in [N, F(N)]$, the set $C_{m,n} = \{B \in \mathbb{N} \mid d_{A_{i_B}}(a_{i_B m}, a_{i_B n}) \leq \frac{\varepsilon}{2}\}$ is in D.

On the other hand, from the definition of F_N we have that:

$$\bigcup_{(m,n)\in[N,F(N)]^2} \{B \in \mathbb{N} \mid d_{A_{i_B}}(a_{i_B m}, a_{i_B n}) \geq \varepsilon\} = F_N \in D$$

so there are $m_0, n_0 \in [N, F(N)]$ such that $C' = \{B \in \mathbb{N} \mid d_{A_{i_B}}(a_{i_B m_0}, a_{i_B n_0}) \geq \varepsilon\} \in D$.

Since both C_{m_0,n_0} and C' are in D, we have that $C_{m_0,n_0} \cap C' \in D$, so it is nonempty. Take a $B \in C_{m_0,n_0} \cap C'$. Then:

$$\varepsilon \leq d_{A_{i_B}}(a_{i_B m_0}, a_{i_B n_0}) \leq \frac{\varepsilon}{2}$$

so $\varepsilon \leq \frac{\varepsilon}{2}$, which is clearly absurd. □

6 Applications

6.1 Halpern iterates on CAT(0) spaces

We will put a bound on the following result of Saejung.

Theorem 6.1. *(Saejung [7]) Let (X, d, W) be a CAT(0) space, C a closed, convex subset of X of finite diameter, $T : C \to C$ a nonexpansive map with at least one fixed point and $f, u \in C$. Let $\{\lambda_k\}_{k\geq 1}$ be a real-valued sequence that satisfies the following three conditions:*

1) $\lim_{k\to\infty} \lambda_k = 0$;
2) $\sum_{k=1}^{\infty} |\lambda_{k+1} - \lambda_k| = 0$;
3) $\sum_{k=1}^{\infty} \lambda_k = \infty$.

Define recursively the **Halpern iteration** *corresponding to these data as:*

$$H(f, u, 0, \{\lambda_k\}_{k\geq 1}, T, W) = f$$
$$H(f, u, n+1, \{\lambda_k\}_{k\geq 1}, T, W) = W(u, T(H(f, u, n, \{\lambda_k\}_{k\geq 1}, T, W)), \lambda_{n+1})$$

Then the sequence $\{H(f, u, n, \{\lambda_k\}_{k\geq 1}, T, W)\}_{n\in\mathbb{N}}$ is convergent.

Proof. Omitted. □

Kohlenbach and Leuştean, using proof mining techniques, have obtained the following result establishing an effective (i.e. computable) uniform bound on the rate of metastability for the Halpern iteration in Saejung's Theorem 6.1 from above, provided that the diameter of C is bounded by a fixed constant M. We will reprove the result using Avigad and Iovino's Theorem 5.1, although losing in the process the exact effective formula obtained for the bound.

Theorem 6.2. *(Kohlenbach, Leuştean [3]) For any $M \geq 0$, any sequence $\{\lambda_k\}_{k\geq 1}$ that satisfies the three conditions in Saejung's result, any $\varepsilon > 0$ and any $F : \mathbb{N} \to \mathbb{N}$, there exists a $B_{F,\varepsilon,M,\{\lambda_k\}_{k\geq 1}}$ (actually, the original bound depends on $\{\lambda_k\}_{k\geq 1}$ in the sense of the convergence or divergence rates of the three sequences from the three conditions being involved in the effective formula) such that for any CAT(0) space (X, d, W), any closed, convex subset C of X of diameter less than M, any nonexpansive map $T : C \to C$ with at least one fixed point and any $f, u \in C$, we have that there is an $N \leq B_{F,\varepsilon,M,\{\lambda_k\}_{k\geq 1}}$ such that for any $m, n \in [N, F(N)]$, $d(H(f, u, n, \{\lambda_k\}_{k\geq 1}, T, W), H(f, u, m, \{\lambda_k\}_{k\geq 1}, T, W)) < \varepsilon$.*

Proof. (Avigad, Iovino [1]) We first fix the M and the $\{\lambda_k\}_{k\geq 1}$.

We seek to apply Theorem 5.1. Let $\mathcal{C}$ be the proper class containing all tuples (X, d, W, C, T, f, u) containing combinations of objects satisfying the stated conditions above. For any such tuple, we fix $A_{(X,d,W,C,T,f,u)}$ to be (X, d), $b_{(X,d,W,C,T,f,u)}$ to be f and for any $n \in \mathbb{N}$, $a_{(X,d,W,C,T,f,u)n}$ to be $H(f, u, n, \{\lambda_k\}_{k\geq 1}, T, W)$. We also fix a non-principal ultrafilter D in $\mathcal{P}(\mathbb{N})$. If $(X, d, W, C, T, f, u) \in \mathcal{C}$, then $b_{(X,d,W,C,T,f,u)} \in C$ and for any $n \in \mathbb{N}$, $a_{(X,d,W,C,T,f,u)n} \in C$, so $d(a_{(X,d,W,C,T,f,u)n}, b_{(X,d,W,C,T,f,u)}) \leq diam(C) \leq M$, so the boundedness condition holds.

Suppose now that (2) in Theorem 5.1 holds for our data. Then, by the theorem, (1) also holds. This solves our problem, because: (i) the bound from (1) will also depend on the M and $\{\lambda_k\}_{k\geq 1}$ that we have already fixed and (ii) the phrase "for any $i \in \mathcal{C}$" translates into the "for any CAT(0) space (X, d, W)..." et. al. from our conclusion.

It remains, therefore, to prove (2). Let $g : \mathbb{N} \to \mathcal{C}$ be a function. We denote, for any $i \in \mathbb{N}$, the components of $g(i)$ in such a way that $g(i)$ will be equal to $(X_i, d_i, W_i, C_i, T_i, f_i, u_i)$. Also, for any $i \in \mathbb{N}$, we write $a_{(X_i,d_i,W_i,C_i,T_i,f_i,u_i)n} = H(f_i, u_i, n, \{\lambda_k\}_{k\geq 1}, T_i, W_i)$ as a_{in}. We will also use the notations $\mathcal{X}_D$, d_D and $\{c_n\}_{n\in\mathbb{N}}$ from (2).

We need $\{c_n\}_{n\in\mathbb{N}}$ to be convergent and we seek to apply Saejung's Theorem 6.1. For that, we first need to endow the metric space $(\mathcal{X}_D, d_D)$ with additional structure.

Define $W : \mathcal{X}_D^2 \times [0, 1] \to \mathcal{X}_D$ by setting for each $\widehat{\{x_i\}}_{i\in\mathbb{N}}$ and $\widehat{\{y_i\}}_{i\in\mathbb{N}}$ in $\mathcal{X}_D$, $W(\widehat{\{x_i\}}_{i\in\mathbb{N}}, \widehat{\{y_i\}}_{i\in\mathbb{N}}, \lambda)$ to be equal to $\widehat{\{W_i(x_i, y_i, \lambda)\}}_{i\in\mathbb{N}}$. Let us see that this is indeed well defined. Suppose we have $\widehat{\{x_i\}}_{i\in\mathbb{N}} = \widehat{\{x'_i\}}_{i\in\mathbb{N}}$ and $\widehat{\{y_i\}}_{i\in\mathbb{N}} = \widehat{\{y'_i\}}_{i\in\mathbb{N}}$. We want to show that $\widehat{\{W_i(x_i, y_i, \lambda)\}}_{i\in\mathbb{N}} = \widehat{\{W_i(x'_i, y'_i, \lambda)\}}_{i\in\mathbb{N}}$, i.e. that for any $\varepsilon > 0$, the set $F_\varepsilon = \{i \in \mathbb{N} \mid d_i(W_i(x_i, y_i, \lambda), W_i(x'_i, y'_i, \lambda)) \leq \varepsilon\}$ is in D. Take an $\varepsilon > 0$. Then the sets:

$$A_\varepsilon = \{i \in \mathbb{N} \mid d_i(x_i, x'_i) \leq \varepsilon\}$$

$$B_\varepsilon = \{i \in \mathbb{N} \mid d_i(y_i, y'_i) \leq \varepsilon\}$$

are in D and also $A_\varepsilon \cap B_\varepsilon \in D$. Take an $i \in A_\varepsilon \cap B_\varepsilon$. Then:

$$\begin{aligned} d_i(W_i(x_i, y_i, \lambda), W_i(x'_i, y'_i, \lambda)) &\leq (1-\lambda)d(x_i, x'_i) + \lambda d(y_i, y'_i) \quad \text{(by the property 3 of W-hyperbolic spaces)} \\ &\leq (1-\lambda)\varepsilon + \lambda\varepsilon \quad \text{(by } i \in A_\varepsilon \cap B_\varepsilon\text{)} \\ &= \varepsilon. \end{aligned}$$

So $i \in F_\varepsilon$ and given that i was chosen arbitrarily, $A_\varepsilon \cap B_\varepsilon \subseteq F_\varepsilon$ and $F_\varepsilon \in D$.

The fact that $(\mathcal{X}_D, d_D, W)$ is a W-hyperbolic, and even a CAT(0) space, follows easily from Lemmas 2.5, 2.6 and 2.7.

Take C to be the subset of $\mathcal{X}_D$ containing all the elements $\widehat{\{x_i\}_{i\in\mathbb{N}}}$ such that there is some valid family $\{y_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} C_i$ with $\widehat{\{x_i\}_{i\in\mathbb{N}}} = \widehat{\{y_i\}_{i\in\mathbb{N}}}$. C is easily seen to be convex and of diameter less than M because all the C_i's are. We will now prove that C is closed.

Suppose there is some $z = \widehat{\{z_i\}_{i\in\mathbb{N}}} \in \mathcal{X}_D$ such that for any $n \in \mathbb{N}$ there is a $y_n \in C$ (taking $y_n = \widehat{\{y_{ni}\}_{i\in\mathbb{N}}}$ s.t. for any i, $y_{ni} \in C_i$) with $d_D(y_n, z) < \frac{1}{n}$ (taking $\frac{1}{0}$ to be $3M$). We want to show that $z \in C$. Define A_n to be the set $\{k \in \mathbb{N} \mid d_k(y_{nk}, z_k) \leq \frac{1}{n}\}$ (so $A_0 = \mathbb{N}$). By Lemma 4.2, (1), $A_n \in D$ for any n. Setting $A'_0 = A_0 = \mathbb{N}$ and for any n, $A'_{n+1} = A_{n+1} \cap A'_n$ we obtain a decreasing sequence $\{A'_n\}_{n\in\mathbb{N}}$ of subsets in D. Put now, for any $i \in \mathbb{N}$, B_i to be the set of all n such that $i \in A'_n$. By construction, each B_i is a non-empty (since $A'_0 = \mathbb{N}$) initial segment of $\mathbb{N}$ (possibly the whole $\mathbb{N}$). Put n_i to be the greatest number smaller than i which is in B_i. We will prove that $z = \widehat{\{z_i\}_{i\in\mathbb{N}}} = \widehat{\{y_{n_i i}\}_{i\in\mathbb{N}}}$, thereby proving that $z \in C$, since all the y_{ni}'s are in their respective C_i's.

For that, we want to show that for any $m \in \mathbb{N}$, the set

$$G_m = \{j \in \mathbb{N} \mid d_j(y_{n_j j}, z_j) \leq \frac{1}{m}\}$$

is in D. Take a $m \in \mathbb{N}$. We have that $A'_m \in D$ and also that $\{l \mid l \geq m\} \in D$ (since it is cofinite). Suppose j is in their intersection. Then, since $j \in A'_m$, $m \in B_j$. Also, since $j \geq m$, $m \leq j$, so m is smallest or equal to n_j (which is the greatest number smaller than j which is in B_j). Since $n_j \in B_j$, $j \in A'_{n_j} \subseteq A_{n_j}$, so $d_k(y_{n_j j}, z_j) \leq \frac{1}{n_j} \leq \frac{1}{m}$. We have shown that $j \in G_m$, therefore, since j is arbitrarily chosen from $A'_m \cap \{l \mid l \geq m\}$, that $A'_m \cap \{l \mid l \geq m\} \subseteq G_m$. So $G_m \in D$, which was what we sought to prove.

Define now $T : C \to C$ by setting $T(\widehat{\{x_i\}_{i\in\mathbb{N}}}) = \widehat{\{T_i(x_i)\}_{i\in\mathbb{N}}}$. This can be seen to be well-defined by using roughly the same argument as for the W but using the fact that the T_i's are non-expansive. Also, the T just constructed can be easily seen to also be non-expansive and to have a fixed point.

Finally, set $f = \widehat{\{f_i\}_{i\in\mathbb{N}}}$ and $u = \widehat{\{u_i\}_{i\in\mathbb{N}}}$. In order to apply Saejung's theorem and see that $\{c_n\}_{n\in\mathbb{N}}$ converges, it remains only to prove that for any n,

$$c_n = H(f, u, n, \{\lambda_k\}_{k\geq 1}, T, W)$$

For $n = 0$, we see that:

$$\begin{aligned} c_0 &= \widehat{\{a_{i0}\}_{i\in\mathbb{N}}} \\ &= \widehat{\{H(f_i, u_i, 0, \{\lambda_k\}_{k\geq 1}, T_i, W_i)\}_{i\in\mathbb{N}}} \\ &= \widehat{\{f_i\}_{i\in\mathbb{N}}} \\ &= f \\ &= H(f, u, 0, \{\lambda_k\}_{k\geq 1}, T, W). \end{aligned}$$

Suppose now that for some n, $c_n = H(f, u, n, \{\lambda_k\}_{k\geq 1}, T, W)$. Then:

$$\begin{aligned} c_{n+1} &= \widehat{\{a_{in+1}\}_{i\in\mathbb{N}}} \\ &= \widehat{\{H(f_i, u_i, n+1, \{\lambda_k\}_{k\geq 1}, T_i, W_i)\}_{i\in\mathbb{N}}} \\ &= \widehat{\{W_i(u_i, T_i(H(f_i, u_i, n, \{\lambda_k\}_{k\geq 1}, T_i, W_i)), \lambda_{n+1})\}_{i\in\mathbb{N}}} \\ &= W(\widehat{\{u_i\}_{i\in\mathbb{N}}}, \widehat{\{T_i(H(f_i, u_i, n, \{\lambda_k\}_{k\geq 1}, T_i, W_i))\}_{i\in\mathbb{N}}}, \lambda_{n+1}) \\ &= W(u, T(\widehat{\{H(f_i, u_i, n, \{\lambda_k\}_{k\geq 1}, T_i, W_i)\}_{i\in\mathbb{N}}}), \lambda_{n+1}) \\ &= W(u, T(\widehat{\{a_{in}\}_{i\in\mathbb{N}}}), \lambda_{n+1}) \\ &= W(u, T(c_n), \lambda_{n+1}) \\ &= W(u, T(H(f, u, n, \{\lambda_k\}_{k\geq 1}, T, W)), \lambda_{n+1}) \qquad \text{(by the induction hypothesis)} \\ &= H(f, u, n+1, \{\lambda_k\}_{k\geq 1}, T, W). \end{aligned}$$

The proof is finished. $\square$

6.2 Ergodic averages of power-bounded operators on Banach spaces

We will now focus on bounding the following result of Lorch.

Theorem 6.3. *(Lorch [6]) Let X be a reflexive Banach space, $T : X \to X$ a power-bounded map and $f \in B$. We define, for any $n \geq 1$, $L(T, n, f)$ to be equal to $\frac{1}{n}\sum_{s=0}^{n-1} T^s(f)$ (the "ergodic average" of T). Then the sequence $\{L(T, n, f)\}_{n\geq 1}$ is convergent.*

Proof. Omitted. $\square$

As we extended in the previous section the metric ultraproduct construction to encompass CAT(0) structures, so will we now extend it to construct "ultraproducts of Banach spaces", such that the distance induced by the ultraproduct norm will be the ultraproduct distance of the distances given by the individual norms. Since the proofs are analogous to the CAT(0) case, we will not give the details – they work by repeated applications of Lemmas 2.5, 2.6 and 2.7. We only note that the basepoints used are the zero vectors of the spaces.

The theorem establishing the bounding is the following:

Theorem 6.4. *Let $\mathcal{D}$ be a class of Banach spaces such that any ultraproduct of any countable family of them is a reflexive Banach space. Then for any $M > 0$, any $\varepsilon > 0$ and any $F : \mathbb{N} \to \mathbb{N}$ there is a $B_{F,\varepsilon,M}$ s.t. for any $X \in \mathcal{D}$, any $T : X \to X$ satisfying for any $n \in \mathbb{N}$, $\|T^n\| \leq M$ and for any $f \in X$ of norm less than 1 there is an $N \leq B_{F,\varepsilon,M}$ such that for any $m, n \in [N, F(N)]$, $\|L(T, n, f) - L(T, m, f)\| < \varepsilon$.*

Proof. (Avigad, Iovino [1]) Like in the last bounding, we first fix the M. Let $\mathcal{C}$ be the proper class containing all triples (X, T, f) such that $X \in \mathcal{D}$, $T : X \to X$ is s.t. for any $n \in \mathbb{N}$, $\|T^n\| \leq M$ and $f \in X$ and its norm is less than 1. We take $A_{(X,T,f)}$ to be X, $b_{(X,T,f)}$ to be 0_X and $a_{(X,T,f)n}$ to be $L(T, n, f)$. We also fix a non-principal ultrafilter D in $\mathcal{P}(\mathbb{N})$.

Let us check the boundedness condition. The distance between an $a_{(X,T,f)n}$ and 0_X will be the norm of $L(T, n, f)$. But then:

$$\begin{aligned}
\|L(T, n, f)\| &= \|\frac{1}{n}\sum_{s=0}^{n-1} T^s(f)\| \\
&\leq \frac{1}{n}\sum_{s=0}^{n-1} \|T^s(f)\| \\
&\leq \frac{1}{n}\sum_{s=0}^{n-1} \|T^s\|\|f\| \\
&\leq \frac{1}{n}\sum_{s=0}^{n-1} M \cdot 1 \\
&= \frac{1}{n} \cdot n \cdot M \\
&= M.
\end{aligned}$$

Take now, a class function $g : \mathbb{N} \to \mathcal{C}$. We denote, for any $i \in \mathbb{N}$, the components of $g(i)$ in such a way that $g(i)$ will be equal to (X_i, T_i, f_i). Also, for any $i \in \mathbb{N}$, we write $a_{(X_i,T_i,f_i)n} = L(T_i, n, f_i)$ as a_{in}. We will also use the notations $\mathcal{X}_D$, $\|\|_D$ and $\{c_n\}_{n\in\mathbb{N}}$.

Since, by our hypothesis, $(\mathcal{X}_D, \|\|_D)$ is reflexive, in order to apply Lorch's result to get that $\{c_n\}_{n\in\mathbb{N}}$ is convergent we only need to construct a power-bounded operator $T : \mathcal{X}_D \to \mathcal{X}_D$ such that $c_n = L(T, n, \widehat{\{f_i\}_{i\in\mathbb{N}}})$. This is done like in the previous example, setting $T(\widehat{\{x_i\}_{i\in\mathbb{N}}}) = \widehat{\{T_i(x_i)\}_{i\in\mathbb{N}}}$. Note that we only need that for any $i \in \mathbb{N}$, $\|T^i\| \leq M$ in order to establish that T is well-defined. The conclusion of our theorem follows immediately, after applying Avigad and Iovino's Theorem 5.1. □

Note that in order to get a bounding like the above, we had to make sure that the initial point of the iteration is in the closed unit ball so that the boundedness condition could be verified. The following "unrestricted" corollary is the most we can get out of this method.

Corollary 6.5. *Let $\mathcal{D}$ be a class of Banach spaces such that any ultraproduct of any countable family of them is a reflexive Banach space. Then for any $M > 0$, any $\rho > 0$ and any $F : \mathbb{N} \to \mathbb{N}$ there is a $B'_{F,\rho,M}$ s.t. for any $X \in \mathcal{D}$, any $T : X \to X$ satisfying for any $n \in \mathbb{N}$, $\|T^n\| \leq M$, for any $f \in X$ and for any $\varepsilon > 0$ with $\frac{\|f\|}{\rho} < \varepsilon$, there is an $N \leq B'_{F,\rho,M}$ such that for any $m, n \in [N, F(N)]$, $\|L(T, n, f) - L(T, m, f)\| < \varepsilon$.*

Proof. Take an M, a ρ and an F like in our hypothesis and set $B'_{F,\rho,M} := B_{F,\frac{1}{\rho},M}$ (the bound from the previous theorem). Take X, T, f, ε like needed and set $g := \frac{1}{\|f\|}f$, so that $f = \|f\|g$ and $\|g\| \leq 1$. Then, applying the last theorem for X, T and g we get that there is an $N \leq B'_{F,\rho,M}$ $(= B_{F,\frac{1}{\rho},M})$ such that for any $m, n \in [N, F(N)]$, $\|L(T, n, g) - L(T, m, g)\| < \frac{1}{\rho}$. But then:

$$\begin{aligned}\|L(T, n, f) - L(T, m, f)\| &= \|L(T, n, \|f\|g) - L(T, m, \|f\|g)\| \\ &= \|\|f\|L(T, n, g) - \|f\|L(T, m, g)\| \\ &= \|f\|\|L(T, n, g) - L(T, m, g)\| \\ &\leq \|f\| \cdot \frac{1}{\rho} \\ &< \varepsilon.\end{aligned}$$

The proof is finished. □

7 Rates of asymptotic regularity

In this section, we will modify the method from above to obtain uniform rates of asymptotic regularity.

7.1 The general result

Lemma 7.1. *Let $\{m_n\}_{n\in\mathbb{N}}$ be a nonincreasing sequence of positive real numbers such that for any ε there is an N_ε s.t. $x_{N_\varepsilon} < \varepsilon$. Then $\{m_n\}_{n\in\mathbb{N}}$ converges to 0.*

Proof. Obvious. □

Theorem 7.2. *Let $\mathcal{C}$ be a (possibly proper) class (whose elements will be used as indices). Let $\{A_i\}_{i\in\mathcal{C}}$ be a family of metric spaces. For each $i \in \mathcal{C}$ choose $b_i \in A_i$ a basepoint and two sequences $\{a_{in}\}_{n\in\mathbb{N}} \subseteq A_i$, $\{a'_{in}\}_{n\in\mathbb{N}} \subseteq A_i$ such that for any $n \in \mathbb{N}$, the boundedness conditions $\sup_{i\in\mathcal{C}} d_{A_i}(a_{in}, b_i) < \infty$ and $\sup_{i\in\mathcal{C}} d_{A_i}(a'_{in}, b_i) < \infty$ hold. Suppose, in addition, that for any $i \in \mathcal{C}$, $\{d_{A_i}(a_{in}, a'_{in})\}_{n\in\mathbb{N}}$ is a nonincreasing sequence of real numbers. Let D be a non-principal ultrafilter in $\mathcal{P}(\mathbb{N})$. Then:*

(a) for any $g : \mathbb{N} \to \mathcal{C}$*, if we denote by* $(\mathcal{X}_D, d_D)$ *the metric ultraproduct of the (countable) family* $\{A_{g(k)}\}_{k\in\mathbb{N}}$ *w.r.t. the family of basepoints* $\{b_{g(k)}\}_{k\in\mathbb{N}}$ *and the ultrafilter* D *and if we denote, for any* $n \in \mathbb{N}$*,* c_n *to be the element* $\widehat{\{a_{g(k)n}\}_{k\in\mathbb{N}}}$ *of* $\mathcal{X}_D$ *and* c'_n *to be the element* $\widehat{\{a'_{g(k)n}\}_{k\in\mathbb{N}}}$ *of* $\mathcal{X}_D$ *(well-defined because of the boundedness conditions), then* $\{d_D(c_n, c'_n)\}_{n\in\mathbb{N}}$ *is a nonincreasing sequence of real numbers.*

(b) TFAE:

1) *for any* $\varepsilon > 0$ *there exists a* $B_\varepsilon \in \mathbb{N}$ *such that for any* $i \in \mathcal{C}$ *and any* $n \geq B_\varepsilon$ *we have that* $d_{A_i}(a_{in}, a'_{in}) < \varepsilon$*;*
2) *for any* $\varepsilon > 0$ *there exists a* $B_\varepsilon \in \mathbb{N}$ *such that for any* $i \in \mathcal{C}$ *we have that* $d_{A_i}(a_{iB_\varepsilon}, a'_{iB_\varepsilon}) < \varepsilon$*;*
3) *for any* $g : \mathbb{N} \to \mathcal{C}$*, if we denote by* $(\mathcal{X}_D, d_D)$ *the metric ultraproduct of the (countable) family* $\{A_{g(k)}\}_{k\in\mathbb{N}}$ *w.r.t. the family of basepoints* $\{b_{g(k)}\}_{k\in\mathbb{N}}$ *and the ultrafilter* D *and if we denote, for any* $n \in \mathbb{N}$*,* c_n *to be the element* $\widehat{\{a_{g(k)n}\}_{k\in\mathbb{N}}}$ *of* $\mathcal{X}_D$ *and* c'_n *to be the element* $\widehat{\{a'_{g(k)n}\}_{k\in\mathbb{N}}}$ *of* $\mathcal{X}_D$*, then for any* $\varepsilon > 0$ *there exists a* $N_\varepsilon \in \mathbb{N}$ *such that for any* $n \geq N_\varepsilon$ *we have that* $d_D(c_n, c'_n) < \varepsilon$*;*
4) *for any* $g : \mathbb{N} \to \mathcal{C}$*, if we denote by* $(\mathcal{X}_D, d_D)$ *the metric ultraproduct of the (countable) family* $\{A_{g(k)}\}_{k\in\mathbb{N}}$ *w.r.t. the family of basepoints* $\{b_{g(k)}\}_{k\in\mathbb{N}}$ *and the ultrafilter* D *and if we denote, for any* $n \in \mathbb{N}$*,* c_n *to be the element* $\widehat{\{a_{g(k)n}\}_{k\in\mathbb{N}}}$ *of* $\mathcal{X}_D$ *and* c'_n *to be the element* $\widehat{\{a'_{g(k)n}\}_{k\in\mathbb{N}}}$ *of* $\mathcal{X}_D$*, then for any* $\varepsilon > 0$ *there exists a* $N_\varepsilon \in \mathbb{N}$ *such that we have that* $d_D(c_{N_\varepsilon}, c'_{N_\varepsilon}) < \varepsilon$*.*

Proof. (a) Follows immediately by Lemma 2.7.

(b) (1) $\Leftrightarrow$ (2) Follows immediately by Lemma 7.1.

(3) $\Leftrightarrow$ (4) Follows immediately by (a) and Lemma 7.1.

(2) $\Rightarrow$ (4) Let g, $(\mathcal{X}_D, d_D)$, $\{c_n\}_{n\in\mathbb{N}}$ and $\{c'_n\}_{n\in\mathbb{N}}$ be as in the hypothesis of (3). Take an $\varepsilon > 0$. We take N_ε to be $B_{\frac{\varepsilon}{2}}$, using the notation from (2), and we need to show that $d_D(c_{N_\varepsilon}, c'_{N_\varepsilon}) < \varepsilon$.

By (2), we have that for any $k \in \mathbb{N}$, $d_{A_{g(k)}}(a_{g(k)N_\varepsilon}, a'_{g(k)N_\varepsilon}) < \frac{\varepsilon}{2}$. So the set:

$$\{k \in \mathbb{N} \mid d_{A_{g(k)}}(a_{g(k)N_\varepsilon}, a'_{g(k)N_\varepsilon}) \leq \frac{\varepsilon}{2}\}$$

is the whole $\mathbb{N}$, which is in D. By Lemma 4.2, (2), $d_D(c_{N_\varepsilon}, c'_{N_\varepsilon}) \leq \frac{\varepsilon}{2} < \varepsilon$.

(4) $\Rightarrow$ (2) Suppose that the negation of (2) holds, i.e. that there exists an $\varepsilon > 0$ s.t. for any B there is an $i_B \in \mathcal{C}$ such that $d_{A_{i_B}}(a_{i_B B}, a'_{i_B B}) \geq \varepsilon$.

Take $g : \mathbb{N} \to \mathcal{C}$ to be the function defined by $g(B) = i_B$ for any $B \in \mathbb{N}$, where i_B is the one from the negation of (2). Using the notation from (4), denote by N the number $N_{\frac{\varepsilon}{2}}$. Then, from (4), we have that $d_D(c_N, c'_N) \leq \frac{\varepsilon}{2}$, so, by Lemma 4.2, (1), the set:

$$\{B \in \mathbb{N} \mid d_{A_{i_B}}(a_{i_B N}, a'_{i_B N}) \leq \frac{\varepsilon}{2}\}$$

belongs to D. In order to get a contradiction, we only need to show that the set:

$$\{B \in \mathbb{N} \mid d_{A_{i_B}}(a_{i_B N}, a'_{i_B N}) \geq \varepsilon\}$$

disjoint from the one above, also belongs to D. We will show that by proving that it is cofinite. Take a $B' \geq N$. Then $N \leq B'$, so:

$$\begin{aligned} d_{A_{i_{B'}}}(a_{i_{B'}N}, a'_{i_{B'}N}) &\geq d_{A_{i_{B'}}}(a_{i_{B'}B'}, a'_{i_{B'}B'}) && \text{(by nonincreasingness)} \\ &\geq \varepsilon. && \text{(by the negation of (2))} \end{aligned}$$

and we are done. □

7.2 Application

Let (X, d, W) be a CAT(0) space, $C \subseteq X$ a nonempty convex subset, $T : C \to C$ a nonexpansive mapping, $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0, 1)$ and $f \in C$. Define the **Krasnoselski-Mann iteration** corresponding to these data by:

$$m_0 := f$$

$$m_{n+1} := W(m_n, T(m_n), \lambda_n)$$

Lemma 7.3. *With the above notations, we have that:*

(a) the sequence $\{d(m_n, T(m_n)\}_{n\in\mathbb{N}}$ is nonincreasing;
(b) for any $n \in \mathbb{N}$, $d(m_n, x) \leq (\sum_{i=0}^{n-1} \lambda_i) \cdot d(f, T(f))$.

Proof. (a) [5], Lemma 3.1, (1).

(b) [5], Lemma 3.1, (3), for $y := f$. □

Corollary 7.4. *Let $b > 0$. If in addition we suppose that for any $\delta > 0$ there is an $y_\delta \in C$ with $d(f, y_\delta) \leq b$ and $d(y_\delta, T(y_\delta)) < \delta$ (the approximate fixed point property), then:*

(a) for any $n \in \mathbb{N}$, $d(m_n, f) \leq (\sum_{i=0}^{n-1} \lambda_i) \cdot 2b$;
(b) for any $n \in \mathbb{N}$, $d(T(m_n), f) \leq (1 + \sum_{i=0}^{n-1} \lambda_i) \cdot 2b$.

Proof. (a) By Lemma 7.3, (b), we only need to prove that $d(f, T(f)) \leq 2b$. Take $\delta > 0$. Then:

$$\begin{aligned} d(f, T(f)) &\leq d(f, y_\delta) + d(y_\delta, T(y_\delta)) + d(T(y_\delta), T(f)) && \text{(by the triangle inequality)} \\ &\leq d(f, y_\delta) + d(y_\delta, T(y_\delta)) + d(f, y_\delta) && \text{(by the nonexpansiveness of } T\text{)} \\ &\leq 2b + \delta. \end{aligned}$$

Since δ was chosen arbitrarily, the conclusion follows.

(b) We have that:

$$\begin{aligned} d(T(m_n), f) &\leq d(T(m_n), T(f)) + d(T(f), f) && \text{(by the triangle inequality)} \\ &\leq d(m_n, f) + d(f, T(f)) && \text{(by the nonexpansiveness of } T\text{)} \\ &\leq (1 + \sum_{i=0}^{n-1} \lambda_i) \cdot 2b. \end{aligned}$$

□

The following Groetsch-like result was proved and effectively "uniformized" by Leuştean [5]. (A first result of this kind had been obtained by Kohlenbach [2] in the context of uniformly convex normed spaces.)

Theorem 7.5. *Let (X, d, W) be a CAT(0) space, $C \subseteq X$ a nonempty convex subset, $f \in C$, $b > 0$, $T : C \to C$ a nonexpansive mapping and $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (0, 1)$ with the property that:*

$$\sum_{k=0}^{\infty} \lambda_k(1 - \lambda_k) = \infty.$$

Suppose also that for any $\delta > 0$ there is an $y_\delta \in C$ with $d(f, y_\delta) \leq b$ and $d(y_\delta, T(y_\delta)) < \delta$. Denote by $\{m_n\}_{n\in\mathbb{N}}$ the Krasnoselski-Mann iteration corresponding to these data. Then:

$$\lim_{n\to\infty} d(m_n, T(m_n)) = 0.$$

Proof. Omitted. □

We shall combine it with the general result from the last section in order to obtain the following uniform version of it.

Theorem 7.6. *For any $b \geq 0$, any sequence $\{\lambda_k\}_{k\geq 1}$ such that $\sum_{k=0}^{\infty} \lambda_k(1-\lambda_k) = \infty$ and any $\varepsilon > 0$, there exists a $N_{\varepsilon,b,\{\lambda_k\}_{k\geq 1}}$ (actually, the original bound depends on $\{\lambda_k\}_{k\geq 1}$ in the sense that the divergence rate of the sequence from the condition is involved in the effective formula) such that for any CAT(0) space (X, d, W), any nonempty, convex subset C of X, any nonexpansive map $T : C \to C$ and any $f \in C$ such that for any $\delta > 0$ there is an $y_\delta \in C$ with $d(f, y_\delta) \leq b$ and $d(y_\delta, T(y_\delta)) < \delta$, denoting by $\{m_n\}_{n\in\mathbb{N}}$ the Krasnoselski-Mann iteration corresponding to these data, we have that for any $n \geq N_{\varepsilon,b,\{\lambda_k\}_{k\geq 1}}$, $d(m_n, T(m_n)) < \varepsilon$.*

Proof. We first fix the b and the $\{\lambda_k\}_{k\geq 1}$.

We seek to apply Theorem 7.2. Let $\mathcal{C}$ be the proper class containing all tuples (X, d, W, C, T, f) containing combinations of objects satisfying the stated conditions above. For any such tuple, we fix $A_{(X,d,W,C,T,f)}$ to be (X, d), $b_{(X,d,W,C,T,f)}$ to be f and for any $n \in \mathbb{N}$, $a_{(X,d,W,C,T,f)n}$ to be the m_n of these data and $a'_{(X,d,W,C,T,f)n}$ to be $T(m_n)$. We also fix a non-principal ultrafilter D in $\mathcal{P}(\mathbb{N})$. The boundedness conditions hold because of the results in Corollary 7.4. The nondecreasingness condition holds because of Lemma 7.3, (a).

Suppose now that (3) in Theorem 7.2 holds for our data. Then, by the theorem, (1) also holds. This solves our problem, because: (i) the bound from (1) will also depend on the b and the $\{\lambda_k\}_{k\geq 1}$ that we have already fixed and (ii) the phrase "for any $i \in \mathcal{C}$" translates into the "for any CAT(0) space (X, d, W)..." et. al. from our conclusion.

It remains, therefore, to prove (3). Let $g : \mathbb{N} \to \mathcal{C}$ be a function. We denote, for any $i \in \mathbb{N}$, the components of $g(i)$ in such a way that $g(i)$ will be equal to $(X_i, d_i, W_i, C_i, T_i, f_i)$. Also, for any $i \in \mathbb{N}$, we write $a_{(X_i,d_i,W_i,C_i,T_i,f_i)n}$ as a_{in} and $a'_{(X_i,d_i,W_i,C_i,T_i,f_i)n}$ as a'_{in}. We will also use the notations $\mathcal{X}_D$, d_D, $\{c_n\}_{n\in\mathbb{N}}$ and $\{c'_n\}_{n\in\mathbb{N}}$ from (3).

We need that $\lim_{n\to\infty} d_D(c_n, c'_n) = 0$ and we seek to apply the Groetsch-like result. For that, we first need to endow the metric space $(\mathcal{X}_D, d_D)$ with additional structure.

Define $W : \mathcal{X}_D^2 \times [0, 1] \to \mathcal{X}_D$ by setting for each $\widehat{\{x_i\}_{i\in\mathbb{N}}}$ and $\widehat{\{y_i\}_{i\in\mathbb{N}}}$ in $\mathcal{X}_D$, $W(\widehat{\{x_i\}_{i\in\mathbb{N}}}, \widehat{\{y_i\}_{i\in\mathbb{N}}}, \lambda)$ to be equal to $\widehat{\{W_i(x_i, y_i, \lambda)\}_{i\in\mathbb{N}}}$. Let us see that this is indeed well defined. Suppose we have $\widehat{\{x_i\}_{i\in\mathbb{N}}} = \widehat{\{x'_i\}_{i\in\mathbb{N}}}$ and $\widehat{\{y_i\}_{i\in\mathbb{N}}} = \widehat{\{y'_i\}_{i\in\mathbb{N}}}$. We want to show that $\widehat{\{W_i(x_i, y_i, \lambda)\}_{i\in\mathbb{N}}} = \widehat{\{W_i(x'_i, y'_i, \lambda)\}_{i\in\mathbb{N}}}$, i.e. that for any $\varepsilon > 0$, the set $F_\varepsilon = \{i \in \mathbb{N} \mid d_i(W_i(x_i, y_i, \lambda), W_i(x'_i, y'_i, \lambda)) \leq \varepsilon\}$ is in D. Take an $\varepsilon > 0$. Then the sets:

$$A_\varepsilon = \{i \in \mathbb{N} \mid d_i(x_i, x'_i) \leq \varepsilon\}$$

$$B_\varepsilon = \{i \in \mathbb{N} \mid d_i(y_i, y'_i) \leq \varepsilon\}$$

are in D and also $A_\varepsilon \cap B_\varepsilon \in D$. Take an $i \in A_\varepsilon \cap B_\varepsilon$. Then:

$$\begin{aligned} d_i(W_i(x_i, y_i, \lambda), W_i(x'_i, y'_i, \lambda)) &\leq (1-\lambda)d(x_i, x'_i) + \lambda d(y_i, y'_i) && \text{(by the W-hyperbolic axioms)} \\ &\leq (1-\lambda)\varepsilon + \lambda\varepsilon && \text{(by } i \in A_\varepsilon \cap B_\varepsilon\text{)} \\ &= \varepsilon. \end{aligned}$$

So $i \in F_\varepsilon$ and given that i was chosen arbitrarily, $A_\varepsilon \cap B_\varepsilon \subseteq F_\varepsilon$ and $F_\varepsilon \in D$.

The fact that $(\mathcal{X}_D, d_D, W)$ is a W-hyperbolic space, and even a CAT(0) space, follows easily from Lemmas 2.5, 2.6 and 2.7.

Take C to be the subset of $\mathcal{X}_D$ containing all the elements $\widehat{\{x_i\}_{i\in\mathbb{N}}}$ such that there is some valid family $\{y_i\}_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} C_i$ with $\widehat{\{x_i\}_{i\in\mathbb{N}}} = \widehat{\{y_i\}_{i\in\mathbb{N}}}$. C is easily seen to be convex and nonempty because all the C_i's are.

Define now $T : C \to C$ by setting $T(\widehat{\{x_i\}_{i\in\mathbb{N}}}) = \widehat{\{T_i(x_i)\}_{i\in\mathbb{N}}}$. This can be seen to be well-defined by using roughly the same argument as for the W but using the fact that the T_i's are non-expansive. Also, the T just constructed can be easily seen to also be non-expansive and for any n, $T(c_n) = c'_n$.

Set $f := \widehat{\{f_i\}_{i\in\mathbb{N}}}$. We will now prove the approximate fixed point property with regard to this f. Take a $\delta > 0$. Then, for every $i \in \mathbb{N}$, choose a y_i such that $d_i(f_i, y_i) \le b$ and $d_i(y_i, T(y_i)) < \frac{\delta}{2}$. Set $y_\delta := \widehat{\{y_i\}_{i\in\mathbb{N}}}$. Then, applying Lemma 2.7, we get that $d_D(f, y_\delta) \le b$ and $d_D(y_\delta, T(y_\delta)) \le \frac{\delta}{2} \le \delta$.

It remains for us to prove that the sequence $\{c_n\}_{n\in\mathbb{N}}$ is the Krasnoselski-Mann iteration starting from f. The base case is $c_0 = \widehat{\{a_{i0}\}_{i\in\mathbb{N}}} = \widehat{\{f_i\}_{i\in\mathbb{N}}} = f$. For the induction case:

$$\begin{aligned} c_{n+1} &= \widehat{\{a_{i(n+1)}\}_{i\in\mathbb{N}}} \\ &= \widehat{\{W_i(a_{in}, T_i(a_{in}), \lambda_n)\}_{i\in\mathbb{N}}} \\ &= W(\widehat{\{a_{in}\}_{i\in\mathbb{N}}}, \widehat{\{T_i(a_{in})\}_{i\in\mathbb{N}}}, \lambda_n) \\ &= W(c_n, T(\widehat{\{a_{in}\}}), \lambda_n) \\ &= W(c_n, T(c_n), \lambda_n). \end{aligned}$$

The proof is finished. □

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