Scientific report for Phase I of the research project "Proof Mining in Analysis and Optimization"

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This report details the scientific activity for Phase I (1 September – 31 December 2020) of the research project entitled *Proof Mining in Analysis and Optimization*, project code PN-III-P1-1.1-PD-2019-0396 (contract no. PD 56/2020).

The research team consists of: CS Dr. Andrei Sipoş (project director), CS II Dr. Liviu Păunescu (mentor).

1 Articles submitted for publication

- "Quantitative inconsistent feasibility for averaged mappings",
  arXiv:2001.01513 [math.OC], 2020.
- "Revisiting jointly firmly nonexpansive families of mappings",
  arXiv:2006.02167 [math.OC], 2020.
- "Construction of fixed points of asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces",
  arXiv:2008.03930 [math.MG], 2020.
- "Rates of metastability for iterations on the unit interval",
  arXiv:2008.03934 [math.CA], 2020.

2 Organized conferences

- Proof and Computation in Mathematics Minisymposium (together with Sam Sanders; part of the DMV Annual Meeting 2020; held online), September 2020.

3 Talks at conferences

- MFO Workshop no. 2046 (on 'Mathematical Logic: Proof Theory, Constructive Mathematics').
  Oberwolfach, Germany (held online), November 2020.
  Talk given: *Two recent results in proof mining.*

4 Research activity of the team members

Andrei Sipoş has obtained the following results, furthering the first objective of the project, "Quantitative analysis of optimization algorithms":

- *A rate of asymptotic regularity for the iteration of compositions of averaged operators.* If (x_n) is an iteration in a metric space associated to an operator T, then asymptotic regularity means that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

When T is a composition of projections on convex sets in a Hilbert space and (x_n) is its Picard iteration, the above was a conjecture which was solved in the positive by Bauschke in 2003. There is a longstanding theme in proof mining of analysing statements like that in order to extract rates of convergence, and Kohlenbach has analysed Bauschke's proof – which used highly non-constructive results like Minty's theorem – and managed to extract a polynomial rate of degree eight.

What we did was to analyse a recent result of Bauschke and Moursi which generalizes the above one to compositions of averaged operators, whose proof uses more natural techniques than the previous one, making crucial use of the concept of cocoercivity. We managed to obtain a quantitative version of the statement that cocoercive maps are rectangular and derive from that an upper bound for the norm of approximate fixed points of compositions. This immediately leads to a rate of asymptotic regularity.

- *Further study of jointly firmly nonexpansive families of mappings.* In 2017, in a paper written together with L. Leuştean and A. Nicolae, we introduced this notion in order to obtain an abstract version of the proximal point algorithm, a fundamental tool in convex optimization.

  What we did here was to continue to show how fruitful this notion is, specifically in the following ways:

  - We showed that if $(T_\gamma)_{\gamma>0}$ is a family of self-mappings in a CAT(0) space X, the following are equivalent:
    (a) (i) For all $\gamma > 0$, T_γ is nonexpansive.
    (ii) The so-called "resolvent identity" holds, i.e. for all $\gamma > 0$, $t \in [0,1]$ and $x \in X$,
    $$T_{(1-t)\gamma}((1-t)x + tT_\gamma x) = T_\gamma x.$$
    (b) $(T_\gamma)_{\gamma>0}$ is jointly firmly nonexpansive.
  - We showed how one can obtain a more powerful version of the "uniform case" (i.e. one can obtain a rate of convergence for the abstract proximal point algorithm in the case where the mappings are uniformly firmly nonexpansive using fewer additional hypotheses than in the 2017 paper), by means of a lemma of Kohlenbach and Powell.
  - We showed an abstract convergence theorem for approximating curves with minimal boundedness assumptions, i.e. we showed that if X is a complete CAT(0) space, $(T_\gamma)_{\gamma>0}$ is a jointly firmly nonexpansive family of self-mappings of X – with $F := \bigcap_{\gamma>0} Fix(T_\gamma)$ – $x \in X$, $b > 0$, and $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and for all n, $d(x, T_{\lambda_n} x) \leq b$, then we have that $F \neq \emptyset$ and the curve $(T_\gamma x)_{\gamma>0}$ converges to the unique point in F which is closest to x.

- *A construction of fixed points of asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces.* In 2010, Kohlenbach and Leuştean showed how any asymptotically nonexpansive mapping on a nonempty bounded UCW-hyperbolic space has a fixed point. We showed how to generalize a construction of Moloney in order to find an actual sequence which converges to such a fixed point.

- *Rates of metastability for iterations on the unit interval.* Frequently, a rate of convergence cannot be extracted using proof mining, not because of any limitation in the techniques, but simply because such a rate may be uncomputable. In this case, one analysed the (classically but not constructively) equivalent version of it, called "metastability" by Terence Tao (at the suggestion of Jennifer Chayes), expressed as
  $$\forall\varepsilon\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists N\ \forall i,j \in [N, N+g(N)]\ d(x_i, x_j) \leq \varepsilon$$
  and tries to extract a "rate of metastability", i.e. a bound on the N depending on ε, g and possibly some other parameters of the problem.

  What we did was to extract such rates of metastability for Picard, Mann and Ishikawa iterations of continuous functions on the unit interval $[0,1]$. Some work had already been done by Jaime Gaspar in his PhD thesis, where he analyzed a theorem due to Hillam, which states that if $f : [0,1] \to [0,1]$ is continuous, $x \in [0,1]$ and $\lim_{n\to\infty}(f^n x - f^{n+1}x) = 0$, then the sequence $(f^n x)$ converges. We first closed this circle of ideas, analysing more general versions of this due to Franks/Marzec and Rhoades, which do not depend on that asymptotic regularity assumption, thus extracting unconditional rates of metastability. Then, we analyzed another result due to Borwein/Borwein, which states that if $L > 0$, $f : [0,1] \to [0,1]$ is L-Lipschitz and (x_n), $(t_n) \subseteq [0,1]$ be such that for all n, $x_{n+1} = (1-t_n)x_n + t_n f(x_n)$, if there is a $\delta > 0$ such that for all n,

$$t_n \leq \frac{2-\delta}{L+1},$$

then the sequence (x_n) converges. The argument in the proof posed a significant challenge – it had not been so far analyzed using proof mining techniques. We managed to extract a rather complex rate of metastability for this kind of iteration, depending on ε, g and the δ above.

Liviu Păunescu, as the mentor of the project, suggested some avenues for proof mining in his own field of study, namely sofic group theory, and started a research seminar devoted to understanding some results in that field.

Director de proiect,
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