Scientific report for Phase II of the research project "Proof Mining in Analysis and Optimization"

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This report details the scientific activity for Phase II (1 January – 31 December 2021) of the research project entitled *Proof Mining in Analysis and Optimization*, project code PN-III-P1-1.1-PD-2019-0396 (contract no. PD 56/2020).

The research team consists of: CS Dr. Andrei Sipoş (project director), CS II Dr. Liviu Păunescu (mentor).

1 Phase summary: research and dissemination activity of the team members

Andrei Sipoş has obtained during this phase the following scientific results:

- within Objective 1 of the project, *Quantitative analysis of optimization algorithms* (which is now fully achieved):
  - *A quantitative multiparameter mean ergodic theorem.*

    One of the main goals of proof mining is to analyze concrete mathematical proofs in order to obtain quantitative information which may not be immediately apparent. Out of a convergence statement for a sequence (x_n) in a metric space with distance d, one might think of extracting a *rate of convergence*, but it is known that such a rate is most often neither computable nor uniform, so one then usually turns to the next best thing, which is called a *rate of metastability*, and which consists of a bound on the N in the following statement

    $$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists N \in \mathbb{N} \, \forall i, j \in [N, N + g(N)] \, \left(d(x_i, x_j) \leq \varepsilon \right),$$

    depending on the ε and g and possibly some other parameters of the problem at hand.

    Kohlenbach and Leuştean have extracted in 2008 such a rate for the convergence statement of the classical mean ergodic theorem from a proof by Birkhoff which works in uniformly convex Banach spaces. What we did is to obtain such a rate for a generalized 'multiparameter' mean ergodic theorem (with a proof essentially due to Riesz), which states the following: if X is a uniformly convex Banach space, $d \geq 1$ and $T_1, \ldots, T_d : X \to X$ are commuting linear operators such that for each l and for each $x \in X$, $\|T_l x\| \leq \|x\|$, then for any $x \in X$, the sequence (x_n), defined, for any n, by

    $$x_n := \frac{1}{(n+1)^d} \sum_{k_1=0}^{n} \cdots \sum_{k_d=0}^{n} T_1^{k_1} \ldots T_d^{k_d} x$$

    is convergent.

    We now detail a bit the challenges which we encountered. We first finitized the proof by noticing that the infimum of all the convex combinations of the iterates of x may be effectively replaced by that of just the arithmetic means of pairs of two given ergodic averages. Thus, we only needed to extend an quantitative arithmetical greatest lower bound principle of Kohlenbach and Leuştean to double sequences, which we did by means of the Cantor pairing function. Finally, we also had to use a combinatorial argument to deal with multiple dimensions.

This result has been written up in the paper "A quantitative multiparameter mean ergodic theorem" (already published in *Pacific Journal of Mathematics*, as listed below in the "Published papers" section.).

- *Abstract strongly convergent variants of the proximal point algorithm.*

  These results are a continuation of those contained in the author's previous papers "An abstract proximal point algorithm" (a 2018 paper written together with Laurenţiu Leuştean and Adriana Nicolae) and "Revisiting jointly firmly nonexpansive families of mappings" (reported in Phase I), which concern the application of our novel concept of *jointly firmly nonexpansive families of mappings* to the derivation of quite abstract convergence proofs for algorithms used in convex optimization, like the celebrated proximal point algorithm and also implicit algorithms involving approximating curves.

  Here, what we do is to extend those techniques in order to provide an abstract version in CAT(0) spaces of the (strongly convergent!) Halpern-type and Tikhonov-type proximal point algorithms. In the Halpern one, given a jointly firmly nonexpansive family of mappings (T_n), one fixes an 'anchor point' u and a sequence of 'weights' $(\alpha_n) \subseteq (0,1)$ and one considers a sequence (x_n) such that for every $n \in \mathbb{N}$,

  $$x_{n+1} := \alpha_n u + (1-\alpha_n)T_n x_n.$$

  We chose to base our approach to its abstraction on a 2017 argument of Aoyama and Toyoda – proof-mined by Kohlenbach in 2020 – because it yields strong convergence assuming quite general conditions on the weights and on the step-sizes. The main difficulty to overcome was that their original argument relied on a property of resolvents called *strong nonexpansiveness*, which is 'somewhat artificial' (Kohlenbach, 2016) to adapt to the metric context, where one usually has at most a property called uniform strong quasi-nonexpansiveness. We first showed that this latter property holds for our class of mappings, giving its corresponding 'SQNE-modulus', which turned out to be enough for the pure abstract strong convergence theorem. We needed, though, something more if we wanted to obtain a *rate of metastability*. What we did then was to mine the SQNE lemma in order to obtain a stronger property which we dubbed 'quantitative quasiness', and which made the whole argument go through.

  These results have been written up in the paper "Abstract strongly convergent variants of the proximal point algorithm" (submitted for publication) and have been presented both at the University of Bucharest LOS Seminar (see below) and at the international, invitation-based workshop on proof theory 'New Frontiers in Proofs and Computation' organized by IASM-BIRS, Hangzhou, China (held online because of the pandemic situation).

- within Objective 2 of the project, *Theoretical underpinnings of proof-theoretic techniques* (which is now halfway achieved):

  - *An analogue of Herbrand's theorem for systems having the proof-theoretic strength of first-order arithmetic.*

    This result can be considered to be part of the 'other half' of the proof mining program, aiming to produce logical explanations for results which have been obtained by more *ad hoc* methods. For example, the simplest example of a rate of metastability, the one for bounded monotone sequences, which has been featured by Georg Kreisel in 1952 and then redisovered by Terence Tao in 2008 – which is where the name 'metastability' originated, at the suggestion of Jennifer Chayes – takes the form of a Herbrand disjunction (as it might be expected from a proof using pure logic and not arithmetical principles), but of variable length (as observed by Kohlenbach around that same time). More precisely, denoting, for all $f : \mathbb{N} \to \mathbb{N}$ and for all $n \in \mathbb{N}$, by $\widetilde{f}(n)$ the quantity $n + f(n)$ and by $f^{(n)}$ the n-fold composition of f with itself, if we take $k \in \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$, and (a_n) a, say, nonincreasing sequence in the interval $[0,1]$, then there is an N such that for all i, $j \in [N, \widetilde{g}(N)]$, $|a_i - a_j| \leq 1/(k+1)$, and, moreover, N can be taken to be an element of the finite sequence 0,

$\widetilde{g}(0), \ldots, \widetilde{g}^{(k)}(0)$. Thus, the conclusion of the statement before can be written as

$$\bigvee_{i=0}^{k} \left(\left(\forall i,\, j \in [x, \widetilde{g}(x)]\, |a_i - a_j| \leq \frac{1}{k+1} \right) \left[x := \widetilde{g}^{(i)}(0) \right] \right),$$

so what we have here is a Herbrand disjunction which is **variable** in k, but does not depend on the g, which only shows up in the terms themselves.

What we did was to elucidate this fact through an extension to first-order arithmetic of the proof of (the ordinary) Herbrand's theorem due to Gerhardy and Kohlenbach which uses the Gödel's functional ('Dialectica') interpretation, in particular a variant inspired by that of Shoenfield, to construct witnesses that realize the interpreted formulas in a system similar to Gödel's T, but lacking recursors, and having case distinction functionals added in order to realize contraction. For a formula of the form $\exists x \varphi$, the extracted term was then β-reduced and it was shown that the resulting term has a sufficiently well-behaved form that one can read off it the classical Herbrand terms. Our extension of this proof to theories which are on the level of first-order arithmetic deals with the corresponding recursors (which one generally uses to interpret induction) by using Tait's infinite terms. This passage to the infinite allowed us to prove in this extended context the corresponding version of the well-behavedness property mentioned before. We also illustrated our result with a simplified variant of Tao's metastability statement.

The idea for this result originated during the first scientific visit of this year, to Ulrich Kohlenbach at Technische Universität Darmstadt.

This result has been written up in the paper "On extracting variable Herbrand disjunctions" (submitted for publication) and will be presented this month at the local logic seminar of IMAR and the University of Bucharest, as well as (most probably) at the relevant logic conferences of next year.

In addition, we make the following remarks:

- The papers reported for Phase I were all accepted for publication, and they are listed below in the "Published papers" and "Accepted papers" sections. In addition, some results of those papers were presented at online international seminars, including the logic seminar of the University of Bath.

- The second scientific visit of this year, to Genaro López-Acedo at Universidad de Sevilla, has yielded, in addition to the dissemination of this project's research via a talk in the local analysis seminar, the prospect of a collaboration group which would also involve Vittorio Colao (University of Calabria) and Adriana Nicolae (Babeş-Bolyai University, Cluj-Napoca).

- Andrei Sipoş has joined the Research Center for Logic, Optimization and Security (LOS) at the University of Bucharest, which also hosts two current collaborators, Laurenţiu Leuştean and Horaţiu Cheval, with whom some research endeavours involving the development of highly abstract logical metatheorems are already being worked on. Some other members of the group may also offer potential collaborations, especially Paul Irofti and Andrei Pătraşcu, whose work in convex optimization algorithms is adjacent to ours. Andrei Sipoş has given in the LOS working seminar an hour-long talk on his research involving abstract proximal point algorithms and jointly firmly nonexpansive families of mappings, including the result mentioned above.

Liviu Păunescu, the mentor of the project, has led a research seminar devoted to understanding some results in his own research (as we said in the report for Phase I), and recently we identified a research problem that we have started working on, namely the development of logical metatheorems suitable for algebraic-analytic structures from the field of geometric group theory.

2 Published papers

1. "Rates of metastability for iterations on the unit interval",
   Journal of Mathematical Analysis and Applications, Volume 502, Issue 1, 125235 [11 pages], 2021.

2. "Construction of fixed points of asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces",
   Numerical Functional Analysis and Optimization, Volume 42, Issue 6, 696–711, 2021.

3. "A quantitative multiparameter mean ergodic theorem",
   Pacific Journal of Mathematics, Volume 314, Number 1, 209–218, 2021.

3 Accepted papers

1. "Quantitative inconsistent feasibility for averaged mappings",
   arXiv:2001.01513 [math.OC], 2020. To appear in: *Optimization Letters.*

2. "Revisiting jointly firmly nonexpansive families of mappings",
   arXiv:2006.02167 [math.OC], 2020. To appear in: *Optimization.*

4 Submitted papers

1. "Abstract strongly convergent variants of the proximal point algorithm",
   arXiv:2108.13994 [math.OC], 2021.

2. "On extracting variable Herbrand disjunctions",
   arXiv:2111.12133 [math.LO], 2021.

5 Talks at conferences

1. IASM-BIRS Workshop 21w5156: New Frontiers in Proofs and Computation.
   Hangzhou, China (held online), 12-17 September 2021.
   Talk given: *On abstract proximal point algorithms and related concepts.*

6 Seminar talks

1. LOS Seminar (held online), University of Bucharest, 1 June 2021.
   Talk title: *On abstract proximal point algorithms and related concepts.*

2. Mathematical Foundations Group Seminar (held online), University of Bath, 20 July 2021.
   Talk title: *A proof mining case study on the unit interval.*

7 Scientific visits

1. 19-30 July 2021: Technische Universität Darmstadt, Germany (invited by Ulrich Kohlenbach).

2. 20-29 September 2021: Universidad de Sevilla, Spain (invited by Genaro López-Acedo).
   Invited talk: *A proof mining case study on the unit interval.*

Director de proiect,
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