Final scientific report for the research project "Proof Mining in Analysis and Optimization" (September 2020 – August 2022)

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This report details the scientific activity for the whole period (September 2020 – August 2022) of the post-doctoral research project of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI – UEFISCDI, within PNCDI III, entitled *Proof Mining in Analysis and Optimization*, having the project code PN-III-P1-1.1-PD-2019-0396 (contract no. PD 56/2020).

The project has been hosted by the Simion Stoilow Institute of Mathematics of the Romanian Academy. The research team consisted of: CS Dr. Andrei Sipoş (project director), CS II Dr. Liviu Păunescu (mentor).

The project web page is

https://cs.unibuc.ro/~asipos/pmao.html

1 Papers

1.1 Published papers

1. "Rates of metastability for iterations on the unit interval",
   Journal of Mathematical Analysis and Applications, Volume 502, Issue 1, 125235 [11 pages], 2021 ([31]).

2. "Construction of fixed points of asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces",
   Numerical Functional Analysis and Optimization, Volume 42, Issue 6, 696–711, 2021 ([32]).

3. "A quantitative multiparameter mean ergodic theorem",
   Pacific Journal of Mathematics, Volume 314, Number 1, 209–218, 2021 ([33]).

4. "Quantitative inconsistent feasibility for averaged mappings",
   Optimization Letters, Volume 16, Issue 6, 1915–1925, 2022 ([34]).

5. "On extracting variable Herbrand disjunctions",
   Studia Logica, Volume 110, Issue 4, 1115–1134, 2022 ([35]).

6. “Abstract strongly convergent variants of the proximal point algorithm”, *Computational Optimization and Applications*, Volume 83, Issue 1, 349–380, 2022 ([37]).

1.2 Papers accepted for publication

1. “Revisiting jointly firmly nonexpansive families of mappings”, to appear in *Optimization*. Online Ready, available at: https://doi.org/10.1080/02331934.2021.1915312 ([36]).

1.3 Submitted papers

1. “A proof-theoretic metatheorem for tracial von Neumann algebras” (with Liviu Păunescu), 2022 ([28]).

2. “An elementary lower bound for $L(1, \chi)$ of order $\sqrt{q} \log q$”, 2022 ([38]).

1.4 Preprints in preparation

1. “Logical metatheorems for abstract structures” (with Horaţiu Cheval and Laurenţiu Leuştean), 2022 ([6]).

2. “Effective strong uniqueness for rational best approximation”, 2022 ([39]).

2 Conference talks

1. MFO Workshop no. 2046 (on ‘Mathematical Logic: Proof Theory, Constructive Mathematics’).
Oberwolfach, Germany (held online), November 2020.
Talk given: *Two recent results in proof mining.*

2. IASM-BIRS Workshop 21w5156: New Frontiers in Proofs and Computation.
Hangzhou, China (held online), September 2021.
Talk given: *On abstract proximal point algorithms and related concepts.*

3. Scandinavian Logic Society Symposium – SLSS 2022.
Bergen, Norway, June 2022.
Talk given: *On extracting variable Herbrand disjunctions.*

4. Logic Colloquium 2022.
Reykjavík, Iceland, June 2022.
Talk given: *On extracting variable Herbrand disjunctions.*

5. International Conference on Applied Proof Theory 2022 (APT22).
Pescara, Italy, August 2022.
Talk given: *On extracting variable Herbrand disjunctions.*

3 Seminar talks

1. LOS Seminar (held online), University of Bucharest, 1 June 2021.
   Talk title: *On abstract proximal point algorithms and related concepts.*
2. Mathematical Foundations Group Seminar (held online), University of Bath, 20 July 2021.
   Talk title: *A proof mining case study on the unit interval.*

4 Participation in summer schools

1. Nordic Logic Summer School – NLS 2022 (Bergen, Norway, June 2022).

5 Organized workshops and seminars

1. Proof and Computation in Mathematics Minisymposium (together with Sam Sanders; part of the DMV Annual Meeting 2020; held online), September 2020.
2. LOS/IMAR Logic Seminar (together with Laurenţiu Leuştean; partially held online), University of Bucharest and Simion Stoilow Institute of Mathematics of the Romanian Academy (for the whole duration of the project).
3. Proof Theory Virtual Seminar (together with Lev Beklemishev, Yong Cheng, Anupam Das, Anton Freund, Thomas Powell, Sam Sanders, Monika Seisenberger and Henry Towsner), held online (2020-2021).
4. Proof Mining Seminar (together with Ulrich Kohlenbach and Laurenţiu Leuştean), held online (started in 2022).

6 Scientific visits

1. 19-30 July 2021: Technische Universität Darmstadt, Germany (invited by Ulrich Kohlenbach).
2. 20-29 September 2021: Universidad de Sevilla, Spain (invited by Genaro López-Acedo).
   Invited talk: *A proof mining case study on the unit interval.*

7 A discussion on the project objectives

This project concerned *proof mining*, an applied subfield of mathematical logic first suggested by the work of the logician Georg Kreisel in the 1950s (under the name 'unwinding of proofs') and then greatly developed by Ulrich Kohlenbach and his students and collaborators starting in the 1990s, that aims to analyze concrete proofs in mainstream mathematics from the point of view of proof theory. Its concrete goals are to find explicit and uniform witnesses or bounds and to remove superfluous premises from concrete mathematical statements by analyzing their proofs using proof-theoretic tools like proof interpretation whose adequacy to the goals is guaranteed by *general logical metatheorems*.

For the first objective of the project, *Quantitative analysis of optimization algorithms*, we have looked at hitherto unanalyzed strong convergence proofs in nonlinear analysis, convex optimization and ergodic theory in order to obtain new effective and uniform rates of metastability, a notion originally introduced by Terence Tao and highly sought out in proof mining. In convex optimization specifically, we have further developed our concept of jointly firmly nonexpansive families of mappings, in order to obtain abstract forms of strongly convergent proximal point algorithm, and also we have looked at novel inconsistent feasibility results in the literature in order to extract rates of asymptotic regularity.

For the second objective, *Theoretical underpinnings of proof-theoretic techniques*, which aims to go to the roots of the proof mining, developing new basic tools for it, we have obtained two results. The first one concerns one of the first rates of metastability, namely Tao's own finite convergence principle, which may appear as a Herbrand disjunction of a variable length. We explain logically this result by an appeal to an extension of the Gerhardy/Kohlenbach proof of Herbrand's theorem to first-order arithmetic. The second one, written up in a paper in collaboration with Horaţiu Cheval and Laurenţiu Leuştean, aims to give a highly abstract form of the fundamental tools of proof mining, the general logical metatheorems mentioned above, which covers in a uniform and most importantly syntactic way all classes of structures on which proof mining can be done.

In a separate paper, written in collaboration with the project mentor, Liviu Păunescu, we have applied this kind of ideas to the development of a metatheorem for a new class of structures, namely tracial von Neumann algebras. As our investigation can be seen to represent a pilot study in doing proof mining in sofic group theory (with a view towards possible applications in that area), it also connects to the third objective of the project, which focuses on doing proof mining in less-explored areas. In that vein, we have also obtained quantitative results in approximation theory – a conditional strong unicity result for best rational approximation – and in analytic number theory – an elementary proof of a lower bound for $L(1, \chi)$ of order $\sqrt{q} \log q$.

As it may be glimpsed from this short summary, we have obtained results in each objective and sub-objective of the project application. The seven papers which contain the results obtained in the first two phases of the project were all accepted for publication in notable journals, while the papers which were worked out during the last phase are either already submitted or in preparation. A detailed description of each of the project results is available in the final section of this report.

We now discuss the impact and future prospects of our results. The project director was invited to further major international workshops (at IASM-BIRS and in Pescara, Italy), in addition to the Oberwolfach workshop invitation mentioned in the project application. As a result, the project research was disseminated to the highest tier of researchers in proof theory in the world.

In addition, some results of those papers were presented in international seminars, including the logic seminar of the University of Bath and the analysis seminar of the University of Seville. In the latter location, the project director made a scientific visit, namely to Genaro López-Acedo, which has also yielded the prospect of a collaboration group which would also involve Vittorio Colao (University of Calabria) and Adriana Nicolae (Babeş-Bolyai University, Cluj-Napoca).

The other scientific visit was to Ulrich Kohlenbach at TU Darmstadt, a former mentor of the project director, which led to highly fruitful discussions on the project research, most importantly the seeds of the paper "On extracting variable Herbrand disjunctions", which may be considered one of the project's most important results, as we have chosen it for the topic of our contributed talks at the major logic conferences of 2022.

Last but not least, the project director has joined the Research Center for Logic, Optimization and Security (LOS) at the University of Bucharest, which also hosts two collaborators of one of the project papers, Laurenţiu Leuştean and Horaţiu Cheval. Some other members of the group may also offer in the future potential collaborations, especially Paul Irofti and Andrei Pătraşcu, whose work in convex optimization algorithms is adjacent to ours. Towards that end, we have given in the LOS working seminar an hour-long talk on our research on abstract proximal methods in convex optimization.

8 A detailed description of the project results and activity

We now detail each of the results obtained in this research project.

8.1 Abstract proximal methods in convex optimization

The proximal point algorithm is a fundamental tool of convex optimization. In its many variants, it usually operates by iterating on a starting point – in, say, a Hilbert space – a sequence of mappings dubbed 'resolvents', whose fixed points coincide with the solutions of the optimization problem that one is aiming at. Thus, if for any $\gamma > 0$ one denotes the resolvent of order γ corresponding to the given problem by J_γ, then one selects a sequence (γ_n) of 'step-sizes' and then forms the iterative sequence which bears the name 'proximal point algorithm' by putting, for any n, x_{n+1} to be equal to $J_{\gamma_n} x_n$.

Some years before the start of this research project, in the paper [25], we introduced, together with L. Leuştean and A. Nicolae, a unifying framework for studying the proximal point algorithm. Namely, we introduced the notion of a *jointly firmly nonexpansive family of mappings*, which abstracts from various kinds of resolvent families (J_γ) like in the above exactly the properties which are needed to prove e.g. convergence, not only in Hilbert spaces, but also in classes of geodesic spaces of non-positive curvature (CAT(0) spaces). The definition looks as follows.

Definition 8.1. *Let X be a CAT(0) space and $(T_\gamma)_{\gamma>0}$ be a family of self-mappings of X. We say that $(T_\gamma)_{\gamma>0}$ is* jointly firmly nonexpansive *if for all λ, $\mu > 0$, for all x, $y \in X$ and all α, $\beta \in [0,1]$ such that $(1-\alpha)\lambda = (1-\beta)\mu$, one has that*

$$d(T_\lambda x, T_\mu y) \leq d((1-\alpha)x + \alpha T_\lambda x, (1-\beta)y + \beta T_\mu y).$$

Thus, in that paper, it was shown that, indeed, examples of jointly firmly nonexpansive families of mappings are furnished by resolvent-type mappings used in convex optimization – specifically, by the following (where X is a CAT(0) space):

- the family $(J_{\gamma f})_{\gamma>0}$, where f is a proper convex lower semicontinous function on X and one denotes for any such function g its proximal mapping by J_g;
- the family $(R_{T,\gamma})_{\gamma>0}$, where T is a nonexpansive self-mapping of X and one denotes, for any $\gamma > 0$, its resolvent of order γ by $R_{T,\gamma}$;
- (if X is a Hilbert space) the family $(J_{\gamma A})_{\gamma>0}$, where A is a maximally monotone operator on X and one denotes for any such operator B its resolvent by J_B.

In the papers [36, 37], we continued this study of jointly firmly nonexpansive families of mappings by providing a conceptual characterization of them and by giving new applications.

The main results of the paper [36] are as follows:

- We showed that if $(T_\gamma)_{\gamma>0}$ is a family of self-mappings in a CAT(0) space X, the following are equivalent:
  - (a) (i) For all $\gamma > 0$, T_γ is nonexpansive.
    (ii) The 'resolvent identity' holds, i.e. for all $\gamma > 0$, $t \in [0, 1]$ and $x \in X$,
    $$T_{(1-t)\gamma}((1-t)x + tT_\gamma x) = T_\gamma x.$$
  - (b) $(T_\gamma)_{\gamma>0}$ is jointly firmly nonexpansive.
- We showed how one can obtain a more powerful quantitative version of the 'uniform case' (i.e. one can obtain a rate of convergence for the abstract proximal point algorithm in the case where the mappings are uniformly firmly nonexpansive using fewer additional hypotheses than in the paper [25]), by means of a lemma of Kohlenbach and Powell [21].
- We showed an abstract convergence theorem for approximating curves with minimal boundedness assumptions, i.e. we showed that if X is a complete CAT(0) space, $(T_\gamma)_{\gamma>0}$ is a jointly firmly nonexpansive family of self-mappings of X – with $F := \bigcap_{\gamma>0} Fix(T_\gamma)$ – $x \in X$, $b > 0$, and $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and for all n, $d(x, T_{\lambda_n} x) \le b$, then we have that $F \neq \emptyset$ and the (continuous) curve $(T_\gamma x)_{\gamma>0}$ converges to the unique point in F which is closest to x.

In the paper [37], we provided an abstract version in CAT(0) spaces of the strongly convergent Halpern-type and Tikhonov-type proximal point algorithms. In the Halpern one, given a jointly firmly nonexpansive family of mappings (T_n), one fixes an 'anchor point' u and a sequence of 'weights' $(\alpha_n) \subseteq (0, 1)$ and one considers a sequence (x_n) such that for every $n \in \mathbb{N}$,

$$x_{n+1} := \alpha_n u + (1-\alpha_n)T_n x_n.$$

We chose to base our approach to its abstraction on an argument of Aoyama and Toyoda [2], as it yields strong convergence assuming quite general conditions on the weights and on the step-sizes. The main difficulty to overcome was that their original argument relied on a property of resolvents called *strong nonexpansiveness*, which is 'somewhat artificial' [17, p. 229] to adapt to the metric context, where one usually has at most a property called uniform strong quasi-nonexpansiveness. We first showed that this latter property holds for our class of mappings, giving its corresponding 'SQNE-modulus', which turned out to be enough for the pure abstract strong convergence theorem. We needed, though, something more if we wanted to obtain a *rate of metastability* (for more details on this concept, see the next subsection). What we did then was to mine further the already partially quantitative SQNE lemma in order to obtain a stronger property which we dubbed 'quantitative quasiness', and which made the whole argument go through.

As befitting the applied nature of our convergence result, we also illustrated it with numerical experiments (simulating the iteration using an *Octave* script) in the simplest non-Hilbert example of a CAT(0) space, namely the Poincaré upper half-plane model.

These results have been published in leading optimization journals (that is, *Optimization* and *Computational Optimization and Applications*, respectively), and have been presented both at the University of Bucharest LOS Seminar and at the international, invitation-based workshop on proof theory 'New Frontiers in Proofs and Computation' organized by IASM-BIRS (held online due to the pandemic situation).

8.2 New rates of metastability and asymptotic regularity

One of the most traditional goals of proof mining is the extraction of rates of *metastability* and *asymptotic regularity*. Let us see what we mean by this. If we have a convergent sequence (x_n) in a metric space, what we would like to obtain using proof mining techniques would be a *rate of convergence*. Unfortunately, this is frequently precluded by computability-theoretic reasons, so one then usually has two options to turn to.

The first one is a finitary notion of convergence, introduced by Terence Tao in [41], usually called *metastability* (under a suggestion of Jennifer Chayes), which is formulated as follows, for a given sequence of reals (x_n):

$$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists N \in \mathbb{N} \, \forall i, j \in [N, N + g(N)] \; (|x_i - x_j| \leq \varepsilon),$$

and which is easily (but non-constructively) seen to be equivalent to (x_n) being Cauchy. Because of its reduced logical complexity, the proof mining metatheorems make it possible to extract a computable and uniform *rate of metastability* – a bound $\Theta(\varepsilon, g)$ on the N in the sentence above – from any proof that shows the convergence of a given class of sequences. (Such a rate of metastability was also obtained and mentioned in the previous subsection in the context of the abstract strongly convergent proximal point algorithm.)

The other option may be introduced as follows. Since the convergent sequence (x_n) usually approximates a fixed point of an operator T, a rate of convergence then states how close is an iteration x_n to such a point. Instead of that, one may measure how close it is to *being* a fixed point, i.e. to look at the statement

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

called the T-asymptotic regularity of the sequence (x_n), which is usually proved as the first step in proving the convergence of (x_n) and which, even though it is a convergence statement, may have a proof of reduced logical complexity which can allow the extraction of a rate of its convergence, which is then called a *rate of asymptotic regularity* for the original sequence.

The main challenge in the extraction of such rates is to deal with new kinds of arguments that have not been analyzed before using proof-theoretic tools. We now list the results of this form that we obtained in our project (in addition to the one in the previous subsection):

1. *A rate of asymptotic regularity for the iteration of compositions of averaged operators.* When T is a composition of projections on convex sets in a Hilbert space and (x_n) is its Picard iteration, the T-asymptotic regularity of (x_n) was a conjecture which was solved in the positive by Bauschke in 2003. Kohlenbach has then analysed in [18] (which continued the work in [17]) Bauschke's proof – which used highly non-constructive results like Minty's theorem – and managed to extract a rate of asymptotic regularity which is polynomial of degree eight.

   In the paper [34] (recently published in *Optimization Letters*), we analysed a recent result of Bauschke and Moursi [4] which generalizes the above one to compositions of averaged operators, whose proof uses more natural techniques than the previous one, making crucial use of the concept of cocoercivity. We did manage to extract a rate of asymptotic regularity, and the crucial step was to obtain the following quantitative version of the statement that cocoercive maps are rectangular.

   Proposition 8.2. *Put, for all β, L_1, L_2, $L_3 > 0$,*

   $$\Psi(\beta, L_1, L_2, L_3) := \frac{L_1 + L_2 + 2\beta L_3 + \sqrt{L_1^2 + L_2^2 + 2L_1L_2 + 8\beta L_1L_3 + 4\beta L_2L_3}}{2\beta}$$

   and

   $$\Theta(\beta, L_1, L_2, L_3) := (L_1 + L_2)\left(L_3 + \Psi(\beta, L_1, L_2, L_3)\right).$$

   Let X be a Hilbert space. Let β, L_1, L_2, $L_3 > 0$ and $A : X \to X$ be β-cocoercive. Let b, $c \in X$ with $\|b\| \leq L_1$, $\|c\| \leq L_2$ and $\|Ab\| \leq L_3$. Then for all $a \in X$,

   $$\langle a - c, Ab - Aa\rangle \leq \Theta(\beta, L_1, L_2, L_3).$$

2. *Rates of metastability for iterations on the unit interval.* In the paper [31] (published in *Journal of Mathematical Analysis and Applications*), we extracted rates of metastability for Picard, Mann and Ishikawa iterations of continuous functions on the unit interval $[0, 1]$. Some work had already been done by Gaspar [8], who analyzed a theorem due to Hillam, which states that if $f : [0, 1] \to [0, 1]$

is continuous, $x \in [0,1]$ and $\lim_{n\to\infty}(f^n x - f^{n+1}x) = 0$, then the sequence $(f^n x)$ converges. We first closed this circle of ideas, analysing more general versions of this due to Franks/Marzec and Rhoades, which do not depend on that asymptotic regularity assumption, thus extracting unconditional rates of metastability. Then, we analyzed another result due to Borwein and Borwein [5], which states that if $L > 0$, $f : [0,1] \to [0,1]$ is L-Lipschitz and (x_n), $(t_n) \subseteq [0,1]$ be such that for all n, $x_{n+1} = (1-t_n)x_n + t_n f(x_n)$, if there is a $\delta > 0$ such that for all n,

$$t_n \leq \frac{2-\delta}{L+1},$$

then the sequence (x_n) converges. The argument in this latter proof had not been so far analyzed using proof mining techniques. We managed to extract a rather complex rate of metastability for this kind of iteration, depending on ε, g and the δ above.

3. *A quantitative multiparameter mean ergodic theorem.* Kohlenbach and Leuştean have extracted [19] a rate of metastability for the convergence statement of the classical mean ergodic theorem from a proof by Birkhoff which works in uniformly convex Banach spaces. In the paper [33] (published in the *Pacific Journal of Mathematics*), we extracted such a rate for a generalized 'multiparameter' mean ergodic theorem (with a proof essentially due to Riesz), which states the following: if X is a uniformly convex Banach space, $d \geq 1$ and $T_1, \ldots, T_d : X \to X$ are commuting linear operators such that for each l and for each $x \in X$, $\|T_l x\| \leq \|x\|$, then for any $x \in X$, the sequence (x_n), defined, for any n, by

$$x_n := \frac{1}{(n+1)^d} \sum_{k_1=0}^{n} \ldots \sum_{k_d=0}^{n} T_1^{k_1} \ldots T_d^{k_d} x$$

is convergent.

We now detail a bit the challenges which we encountered. We first finitized the proof by noticing that the infimum of all the convex combinations of the iterates of x may be effectively replaced by that of just the arithmetic means of pairs of two given ergodic averages. Thus, we only needed to extend an quantitative arithmetical greatest lower bound principle of Kohlenbach and Leuştean to double sequences, which we did by means of the Cantor pairing function. Finally, we also had to use a combinatorial argument to deal with multiple dimensions.

This and the previous rate of metastability were presented at the the international, invitation-based workshop on proof theory 'Mathematical Logic: Proof Theory, Constructive Mathematics' organized by the Mathematical Research Institute of Oberwolfach (held online due to the pandemic situation), as well as in invited talks in regularly-held seminars of the universities of Bath and Seville.

8.3 Construction of fixed points in hyperbolic spaces

This line of research is a spillover from proof mining into pure, non-quantitative nonlinear analysis. The setting we work in is the one of UCW-hyperbolic spaces, a class of hyperbolic geodesic spaces, introduced by Leuştean in [23, 24] as a non-linear generalization of uniformly convex Banach spaces; he also proved the adaptation of the classical Browder-Göhde-Kirk fixed point theorem to this setting. Later, in [20], Kohlenbach and Leuştean also proved the corresponding Goebel-Kirk extension, which generalizes the Browder-Göhde-Kirk result to the class of *asymptotically nonexpansive mappings*, which are mappings T having the property that there is a $(k_n) \subseteq [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that for any x, y in the domain of T and any $n \in \mathbb{N}$,

$$d(T^n x, T^n y) \leq (1 + k_n) d(x, y).$$

In the paper [32] (published in *Numerical Functional Analysis and Optimization*), we show that one can adapt a construction due to Moloney [26] to UCW-hyperbolic spaces, i.e. we explicitly construct, for any asymptotically nonexpansive self-mapping of a bounded complete nonempty UCW-hyperbolic space, a sequence that converges strongly to one of its fixed points.

8.4 The extraction of variable Herbrand disjunctions

We now move to the second objective of the project, which aims to explore the underpinnings of the techniques used in proof mining. In particular, practical extractions of terms that were obtained by *ad hoc* methods fall sometimes outside the scope of existing metatheorems, and new theoretical results are needed in order to explain why said extractions were possible.

The logical result that is the main precursor to ours is the proof of Herbrand's theorem due to Gerhardy and Kohlenbach [10] which uses the *Dialectica* interpretation to construct witnesses that realize the interpreted formulas in a system similar to Gödel's T, but lacking recursors, and having case distinction functionals added in order to realize contraction. For a formula of the form $\exists x \varphi$ as in the hypothesis of Herbrand's theorem, the extracted term is then β-reduced and it is shown that the resulting term has a sufficiently well-behaved form that one can read off it the classical Herbrand terms which form the expression that is usually called a *Herbrand disjunction*.

The empirical result which, in our opinion, needed a logical explanation was the fact that the simplest example of a rate of metastability, the one for bounded monotone sequences, explained by Terence Tao in 2008 [41], takes the form of a Herbrand disjunction (as it might be expected from a proof using pure logic and not arithmetical principles), but 'of variable length' [15, p. 34]. More precisely, denoting, for all $f : \mathbb{N} \to \mathbb{N}$ and for all $n \in \mathbb{N}$, by $\widetilde{f}(n)$ the quantity $n + f(n)$ and by $f^{(n)}$ the n-fold composition of f with itself, if we take $k \in \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$, and (a_n) a nonincreasing sequence in the interval $[0, 1]$, then there is an N such that for all i, $j \in [N, \widetilde{g}(N)]$, $|a_i - a_j| \leq 1/(k + 1)$, and, moreover, N can be taken to be an element of the finite sequence 0, $\widetilde{g}(0), \ldots, \widetilde{g}^{(k)}(0)$. Thus, the conclusion of the statement before can be written as

$$\bigvee_{i=0}^{k} \left(\left(\forall i,\, j \in [x, \widetilde{g}(x)]\, |a_i - a_j| \leq \frac{1}{k+1} \right) \left[x := \widetilde{g}^{(i)}(0) \right] \right),$$

so what we have here is a Herbrand disjunction which is **variable** in k, but does not depend on the g, which only shows up in the terms themselves.

In the paper [35] (recently published in *Studia Logica*), we extended the proof of Gerhardy and Kohlenbach to theories which are on the level of first-order arithmetic, dealing with the corresponding recursors (which one generally uses to interpret induction) by using Tait's infinite terms [40]. This passage to the infinite allowed us to prove in this extended context the corresponding version of the well-behavedness property mentioned before, which forms the technical, combinatorial core of our argument. We also illustrated our result with the above metastability example.

The idea behind this result originated during the first scientific visit of 2021, to Ulrich Kohlenbach at Technische Universität Darmstadt. The result has been presented in the form of contributed talks at the Scandinavian Logic Society Symposium and Logic Colloquium conferences (both held in June 2022), as well as in the invitation-based International Conference on Applied Proof Theory 2022.

8.5 Metatheorems for abstract and tracial structures

In the paper [6], we give, together with H. Cheval and L. Leuştean, a highly abstract form of the usual logical metatheorems of proof mining. Since that paper is still in preparation, we shall try to give here as many details as possible of its motivation and results.

First of all, we note that our presentation of the metatheorems has the following advantages:

- by its abstractness, we have that the bulk of the proofs of the metatheorems may be given uniformly for all known structures (e.g. metric spaces, normed/Banach spaces, hyperbolic spaces, spaces which are formalizable in positive-bounded logic);
- most of the proof (including the majorization step, which is a generalization of Kohlenbach's monotone functional interpretation) is done syntactically;
- one allows for arbitrary Δ-sentences (see below for the definition) in the analyzed systems;
- we use systems of proof-theoretic strength of the order of first-order arithmetic, which do not require in their interpretation the use of bar-recursive functionals, and thus allow us to use classical set-theoretic models instead of strongly majorizable ones, and, in addition, we can dispense with strong majorization altogether in favour of Howard's original majorization notion.

The framework in which the abstract metatheorem is expressed is parametrized by the concept of a *higher-order signature*, which is a pair $\Sigma = (\mathcal{X}, \mathcal{F})$, where $\mathcal{X}$ is a set whose elements are called the *abstract base types* and which generate (together with 0, the type of natural numbers) the free type algebra $\mathbb{T}^{\mathcal{X}}$, while $\mathcal{F}$ is a (disjoint) family of sets indexed by $\mathbb{T}^{\mathcal{X}}$, the element of each set representing the language constants of the corresponding type.

One then defines, for each signature Σ, two 'higher-order' logical systems $\mathcal{T}_i^\omega[\Sigma]$ and $\mathcal{T}_c^\omega[\Sigma]$ (to be considered, in a sense, 'intuitionistic' and 'classical', respectively), and a third denoted by $\mathcal{T}_m^\omega[\Sigma]$, all of them with associated formulas, sentences, proof systems and set-theoretic models. Among the sentences of these systems, one distiguishes the class of Δ-sentences, which are of the form

$$\forall \underline{a}^{\underline{\delta}} \exists \underline{b}^{\underline{\sigma}} \preceq_{\underline{\sigma}} \underline{r}\underline{a} \forall \underline{c}^{\underline{\gamma}} B(\underline{a}, \underline{b}, \underline{c}),$$

where B is quantifier-free and all its free variables are among the tuples $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{r}$ is a tuple of closed terms of type-tuple $\underline{\delta} \to \underline{\sigma}$ of $\mathcal{T}_i^\omega[\Sigma]$.

Our most general semantical result is then formulated as follows.

Theorem 8.3. *Let $\Sigma = (\mathcal{X}, \mathcal{F})$ be a higher-order signature and Γ be a set of Δ-sentences over Σ. Let A be a quantifier-free Σ-formula having at most the free variable z of type 0. Assume that*

$$\mathcal{T}_c^\omega[\Sigma] + \Gamma \vdash \exists z A.$$

Then one can extract from the proof a closed term t of $\mathcal{T}_m^\omega[\Sigma]$ of type 0, which is hereditarily of types not containing the abstract base types in $\mathcal{X}$, such that for every set-theoretic model $\mathcal{S}$ of $\mathcal{T}_m^\omega[\Sigma] + \Gamma$, we have that

$$\mathcal{S} \models \exists z (z \leq t \wedge A).$$

As one can see, the term t above represents a 'uniform bound' that is extractable by the algorithm implicit in the proof. The metatheorem is then instantiated to obtain the familiar cases of metric spaces, normed spaces etc.

In addition, in the paper [28] (submitted for publication), we have introduced, together with L. Păunescu, a brand new concrete instantiation of this metatheorem. This result lies at the intersection of the second objective and the third objective – the latter of those referring to finding applications of proof mining in the less-explored areas of approximation theory, analytic number theory and sofic group theory (the results in the remaining subsections will deal solely with this objective) – as the class of structures to which this metatheorem is directed is the one of *tracial von Neumann algebras*, which are of great importance in the fields of operator algebras and geometric group theory (the latter including the theory of sofic groups listed before). For this result, the expertise of L. Păunescu, as the mentor of this project, proved crucial. In fact, the idea behind it originated in the seminars organized by L. Păunescu featuring tracial von Neumann algebras, the first one devoted to presenting his own results and the second one to the recent claimed negative solution to the Connes Embedding Problem via the complexity-theoretic statement of $\mathsf{MIP}^* = \mathsf{RE}$ [12], to which the model-theoretic axiomatization of tracial von Neumann algebras, due to Farah, Hart and Sherman [7], on which we based our work, is connected, namely model theory can provide a 'shortcut' [11, Section 7] in showing that $\mathsf{MIP}^* = \mathsf{RE}$ implies the failure of the CEP.

Again, in order to be able to 'do' proof mining on tracial von Neumann algebras, we introduce three 'higher-order' logical systems $\mathcal{T}_{\mathrm{vN},i}$, $\mathcal{T}_{\mathrm{vN},c}$, and $\mathcal{T}_{\mathrm{vN},m}$, which codify into the abstract systems the axiomatization that we just mentioned in the form of Δ-sentences (actually, purely universal sentences). We also associate to each von Neumann algebra a specific set-theoretic model of these systems. This follows a long line of research in extending metatheorems to various analytic structures, see e.g. our previous paper [29]. The main difficulty that we had to deal with arose from the fact that complex numbers had never been considered before in proof mining, and unlike the norm or distance functions which only take non-negative values, the (real and imaginary) components of the trace function may take arbitrary real values, and thus a new representation function $(\cdot)_+$ had to be devised.

Our main metatheorem for tracial von Neumann algebras is expressed as follows.

Theorem 8.4. *Let Γ be a set of Δ-sentences. Let B be a quantifier-free formula having at most the free variables y of an arbitrary type τ, and z of type 0. Assume that*

$$\mathcal{T}_{\mathrm{vN,c}} + \Gamma \vdash \forall y \exists z B.$$

Then one can extract from the proof a higher-order primitive recursive functional $\Phi : \mathcal{S}_{\widehat{\tau}} \to \mathbb{N}$ such that for every tracial von Neumann algebra A with $A \models \Gamma$, letting $\mathcal{S}$ be the model associated to A, and for every $b \in \mathcal{S}_\tau$ and $b' \in \mathcal{S}_{\widehat{\tau}}$ such that we have that $\mathcal{S} \models b \lesssim_\tau b'$, we have that

$$A \models \exists z (z \leq \Phi(b') \wedge B[y := b]).$$

This investigation represents a pilot study in doing proof mining in sofic group theory, as we have in mind some future possible applications from that area: for example, from the result in [3], which states that every finite list of permutations that 'almost' commute is 'near' another such list of actually commuting permutations and which is proven non-constructively using tracial von Neumann algebras, one could extract moduli quantifying exactly what is meant by 'almost' and 'near'.

8.6 Strong uniqueness for rational best approximation

In the paper [39], we further the third objective of the research project by obtaining more quantitative results in approximation theory. Since, again, that paper is still in preparation, we shall try to detail it here as much as possible.

A typical kind of result which comes up in approximation theory is the uniqueness of the best approximation of a function taken from a generally large class towards a reasonably well-behaved object such as a polynomial or a piecewise linear function. In that vein, one may cite the classical uniqueness theorem for uniform Chebyshev approximation. We shall denote in the sequel the supremum norm by $\|\cdot\|$ and – for any $n \in \mathbb{N}$ – the class of real polynomials of degree at most n by P_n. Then, the theorem states that for any continuous $f : [0,1] \to \mathbb{R}$ and any $n \in \mathbb{N}$, there is a unique $p \in P_n$ such that

$$\|f - p\| = \min_{q \in P_n} \|f - q\|.$$

In his 1990 PhD thesis [13], Ulrich Kohlenbach carried out a program of applying proof mining techniques to proofs of theorems such as the above in order to compute explicit so-called 'moduli of uniqueness': functions Ψ such that for any f and n as in the above, any $\varepsilon > 0$ and any p_1, $p_2 \in P_n$ that have the corresponding 'approximation errors' $\|f - p_1\|$ and $\|f - p_2\|$ within $\Psi(f, n, \varepsilon)$ of the desired minimum, one can then be sure that $\|p_1 - p_2\| \leq \varepsilon$. Such a modulus would help, for example, in calibrating the number of steps an algorithm should be run in order to obtain a polynomial as close as desired to the optimal one. If the modulus is linear in ε (i.e., of the form $\gamma \cdot \varepsilon$), one recovers a well-established property dubbed 'strong unicity' in approximation theory.

Two years before the start of this research project, we carried out in [30] the extraction of a linear modulus of uniqueness for the best Chebyshev approximation by polynomials of bounded degree with some constraints on the coefficients. The main novelty in that proof was the application of Schur polynomials to obtain useful explicit formulas for the interpolation results which were needed in the original, non-quantitative proof.

What we did now in [39] was to look at a best rational approximation result due to Akhiezer [1] and extract a modulus of uniqueness which is linear if we impose a novel bounding condition. The framework of that theorem is known to lack strong unicity in the classical sense, so our result is new even in a qualitative way.

The original uniqueness result is expressed as follows. For all m, $n \in \mathbb{N}$, we set

$$R_{m,n} := \{p/q \mid p \in P_m,\ q \in P_n,\ \text{such that for all } x \in [0,1],\ q(x) > 0\}.$$

Then for any continuous $f : [0,1] \to \mathbb{R}$ and any m, $n \in \mathbb{N}$, there is a unique $\psi \in R_{m,n}$ such that

$$\|f - \psi\| = \min_{\chi \in R_{m,n}} \|f - \chi\|.$$

We have obtained the following result.

Theorem 8.5. *For any continuous $f : [0,1] \to \mathbb{R}$ and any m, $n \in \mathbb{N}$, denoting by $\psi \in R_{m,n}$ the minimizing rational function as in the above, and putting $E := \|f - \psi\|$ and $\psi = p/q$ with $p \in P_m$, $q \in P_n$, p/q irreducible and $\|q\| = 1$, then for any $l > 0$ with $E \geq l$ and any $b > 0$ such that for all $x \in [0,1]$ one has $q(x) \geq b$, there is a $\gamma > 0$ such that for all $\varepsilon > 0$ and all $\chi \in R_{m,n}$, putting $\chi = p_1/q_1$ with $p_1 \in P_m$, $q_1 \in P_n$, p_1/q_1 irreducible, $\|q_1\| = 1$ and for all $x \in [0,1]$ one has $q_1(x) \geq b$, we have that if $\|f - \chi\| \leq E + \gamma \cdot \varepsilon$, then $\|\psi - \chi\| \leq \varepsilon$.*

Finally, we note that our analysis resembles a bit the one of Kohlenbach in [14] of a second proof of the classical Chebyshev uniqueness result, in the sense that one does not need to further analyze the alternation theorem which is used in the proof, as it can be treated like a black box.

8.7 Elementary lower bounds in analytic number theory

In the paper [38] (submitted for publication), we obtain some quantitative results in analytic number theory, the last area of interest mentioned in the project application. The focus is on L-functions associated to Dirichlet characters, and specifically on the property that for any non-principal character χ, one has that $L(1, \chi) \neq 0$, a classical result used in the proof of Dirichlet's theorem on the infinitude of primes in arithmetic progressions.

The natural question to ask is whether one can obtain an effective lower bound on $|L(1, \chi)|$ in terms of the modulus of χ (a topic also suggested by Kreisel [22, p. 139]), in particular in the case where χ is real since that is where the difficulty of the proof of the non-vanishing largely lies.

The classical complex-analytic result then states that there is an effective constant c such that for all q and all real χ of modulus q, $L(1,\chi) \geq c/\sqrt{q}$. Without using complex analysis, Gel'fond has obtained in [9] a lower bound of $1/(33\sqrt{q}(\log q)^2)$. A simplification of this argument in a non-quantitative variant is due to Monsky [27]. It has been suggested (see the introduction of our paper for more historical details) that various refinements of this class of arguments may be used to obtain a bound of order $\sqrt{q}\log q$.

What we do is to take the Gel'fond-Monsky argument and make it even more elementary by removing the real integral used in the first estimate. Using that integral, the parameter we denote by N could have been an arbitrary real x; therefore, we must take special care to find a specific natural number to instantiate N that has the same asymptotics as the x chosen by Gel'fond. After we overcome that step, we are still able to produce a final bound of $1/(512\sqrt{q}\log q)$.

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