Some notes on sheaf-theoretic model theory

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0 Preamble

This is an exposition of a particular intersection of sheaf theory and model theory initiated by D. Ellerman's work from his 1971 PhD thesis [1] and his subsequent 1974 article [2]. We do not claim originality about what is said here, except for the fact that some constructions and results are reformulated using sheaf-theoretic machinery that has since become standard.

Section 1 contains a survey of the various concepts and tools needed, like the equivalence between preordered sets and Alexandrov spaces, the complete basic theory of Heyting algebras and their spectra or some category-theoretic and model-theoretic notions. Section 2 describes in detail the notion of a structured space as a generalization of the structured set which is the central object of study in ordinary first-order model theory. We focus on sheaves on spaces that are obtained by canonical constructions, like the ones from Section 1 – we therefore use the full machinery of indexed categories to easily derive the internal structure of such spaces. For example, we show that the classical ultraproduct construction can be obtained by such a sheaf-theoretic procedure. Finally, Section 3 introduces the way first-order logic can be adapted to express properties of such generalized structures, culminating in a massive generalization of the classical Łoś theorem.

We apologise for any errors that have slipped through the editing process.

1 Fundamentals

1.1 Spaces

By "space" we shall mean a topological space. Generally, if X is a space, then X will also be the name of its underlying set (by abuse of language) and $\mathcal{O}(X)$ will be its lattice of open sets. The category of all spaces is denoted by **Top**.

Definition 1.1. For any space X any any subset A of it, we will denote by $int(A)$ the interior of A and by $ic(A)$ the interior of the closure of A. An open set D of X is called **regular** if $ic(D) = D$. The set of the regular open sets of X is denoted by $RO(X)$.

Lemma 1.2. *If X is a space, $A \subseteq X$ and $B \in \mathcal{O}(X)$, we have that:*

$$int(A) \cap B = int(A \cap B)$$

Proof. The left hand side is equal to $(\bigcup_{V \in \mathcal{O}(X) \cap \mathcal{P}(A)} V) \cap B = \bigcup_{V \in \mathcal{O}(X) \cap \mathcal{P}(A)} (V \cap B)$, while the right hand side us equal to $\bigcup_{V \in \mathcal{O}(X) \cap \mathcal{P}(A \cap B)} W$. We prove that the same sets participate in both unions: i.e. that for any open $W \subseteq A \cap B$ there is an open $V \subseteq A$ with $W = V \cap B$. But then we can simply take $V := W$. $\square$

Definition 1.3. A space is called **discrete** if any subset of it is open. We denote the full subcategory of **Top** consisting of discrete spaces by **DisTop**.

Remark 1.4. *The functor* **Set** $\to$ **DisTop** *taking a set I into the unique discrete space with I as its underlying set is an isomorphism of categories.*

Definition 1.5. An **Alexandrov space** is a space such that the intersection of an arbitrary family of open sets of it is open.

Remark 1.6. *Any discrete space is Alexandrov.*

Theorem 1.7. *Let $(I, \leq)$ be a preordered set. Denote by $\mathcal{O}(I)$ the set of all subsets A of I which are* **order-closed***, i.e. which have the following property: for every $i, j \in I$ s.t. $i \in A$ and $i \leq j$ we have that $j \in A$. Then $\mathcal{O}(I)$ is a topology on I and $(I, \leq)^a := (I, \mathcal{O}(I))$ is an Alexandrov space.*

Proof. The sets $\emptyset$ and I are immediately seen to be order-closed.

Let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a family of order-closed subsets and $i, j \in I$ s.t. $i \leq j$. Suppose i is in the union (intersection) of the family $\{A_\alpha\}$. Then i is in some (all) A_α and since each A_α is order-closed, we conclude that j is also in some (all) A_α and so in the union (intersection) of the family. $\square$

Remark 1.8. *A basis for the above topology can be given by the sets $E_x = \{y \in I \mid x \leq y\}$.*

Proof. Let D be an order-closed set. Then $D = \bigcup_{x \in D} E_x$. $\square$

Remark 1.9. *On any space X we can define a relation $\leq_X$ as follows: for any $x, y \in X$ we have that $x \leq_X y$ if and only if x is in the closure of the set $\{y\}$. By the laws of the closure operation, it follows that $\leq_X$ is a preorder. It is called the* **specialization preorder** *of X.*

Remark 1.10. *Any space X can be endowed with the Alexandrov topology induced by the specialization preorder. This topology is necessarily finer than the original one.*

Proof. Let D be an open set in X. We must show that it is order-closed. Let $x, y \in X$ such that x is in the closure of the set $\{y\}$. Suppose $x \in D$ and $y \notin D$. Then y is in the complement of D, which is closed, and hence the closure of $\{y\}$ (including x) is contained within the complement of D, which contradicts the fact that $x \in D$. $\square$

Theorem 1.11. *Let X be a space. TFAE:*

1) any set which is order-closed under the specialization preorder of X is open;
2) there exists a preorder $\leq$ on the set X such that $X = (X, \leq)^a$ (and if so, the preorder is equal to the specialization preorder and hence it is unique);
3) X is Alexandrov.

Proof. ($1 \Rightarrow 2$) It is clear that the specialization preorder does the job. Suppose we have another preorder $\leq$ in that situation, i.e. that the open sets are exactly the order-closed sets under $\leq$. We need to show that for every $x, y \in X$, $x \leq y$ iff x is contained in the closure of $\{y\}$.

Suppose first that x is not contained in the closure of $\{y\}$. Then the complement of the closure of $\{y\}$ is an set that contains x, does not contain y and is open, hence order-closed for $\leq$. So we cannot have $x \leq y$.

Suppose now that x is not smaller or equal to y. Then y is not contained into the order-closed set of all elements greater or equal to x, which is an open set. Then the closure of $\{y\}$ is a subset of the complement of that open set, and so it does not contain x.

($2 \Rightarrow 3$) This has been proven above.

($3 \Rightarrow 1$) Let D be order-closed under the specialization preorder. We need to show that it is open. The space X being Alexandrov, it is sufficient to show that D is the intersection of all open sets that contain D, i.e. that for every $y \notin D$ there is a D' open that is a superset of D and does not contain y. Let y be outside of D. We take D' to be the complement of the closure of $\{y\}$. Clearly it does not contain y. Suppose that it is not a superset of D, i.e. that there is an $x \in D$ that is not in D'. Then x is in the closure of $\{y\}$, so by the order-closedness of D we obtain that y is in D, contradicting our choice of y. $\square$

Thus we may use a preorder and an Alexandrov topology on a specific set interchangeably. In particular, we have:

Corollary 1.12. *There exists a bijection between the class of preordered sets and the class of Alexandrov spaces.*

Remark 1.13. *In the above correspondence, discrete spaces correspond to preordered sets where the only elements of the preorder are those of the form (x, x).*

Theorem 1.14. *Let I, J be sets endowed with preorders and $f : I \to J$ a function. Then f is nondecreasing iff f is continuous relative to the Alexandrov topology.*

Proof. ($\Rightarrow$) Let E_y be an element of the basis of J (as defined above), where $y \in J$. We must show that $f^{-1}(E_y)$ is open, i.e. that it is order-closed. Suppose $x \in f^{-1}(E_y)$ and $x \leq z$. Then $f(x) \in E_y$, so $y \leq f(x)$. Given that f is nondecreasing, $f(x) \leq f(z)$, so $y \leq f(z)$ and z is in $f^{-1}(D_y)$.

($\Leftarrow$) The preorders are the specialization preorders, so we can take x, y to be such that x is in the closure of $\{y\}$ and we must show that $f(x)$ is in the closure of $\{f(y)\}$. Suppose, on the contrary, that there is an open set D such that $f(x)$ is in D and $f(y)$ is not in D. Then, as f is continuous, $f^{-1}(D)$ is an open set that contains x and does not contain y, contradicting the fact that x is in the closure of $\{y\}$. $\square$

Corollary 1.15. *The category of preordered sets and the one of Alexandrov spaces are isomorphic.*

Note that because we have proved the bijection between structures on a set-by-set basis, we are able to use the "evil" concept of isomorphism instead of the usual equivalence of categories. Another related observation is that the isomorphism is "concrete", i.e. it commutes with the canonical "underlying set" functors.

1.2 Heyting algebras

We presume all basic facts regarding general lattices, Boolean algebras and filters on Boolean algebras to be known.

Theorem–Definition 1.16. *Let H be a lattice and $a, b, c \in H$. TFAE:*

- *c is the greatest element of H s.t. $c \wedge a \leq b$;*
- *for all $x \in H$, $x \wedge a \leq b$ iff $x \leq c$.*

In this case, c is called the **relative pseudo-complement** *of the pair (a, b) and we write this as $c = a \to b$.*

Definition 1.17. A **Heyting algebra** is a bounded lattice such that any pair of elements admits a pseudo-complement. In the sequel, the least and greatest element of any Heyting algebra will be denoted by 0 and 1, respectively.

Definition 1.18. If H is a Heyting algebra and $x \in H$, the **pseudo-complement** of H is the element $\neg x = x \to 0$.

Theorem 1.19. *Let H be a Heyting algebra. The following hold:*

1. *H is a distributive lattice.*
2. *for every $x \in H$, $x \wedge \neg x = 0$.*
3. *for every $x, y \in H$, $\neg x \wedge \neg y = \neg(x \vee y)$.*
4. *for every $x \in H$, $x \leq \neg\neg x$.*
5. *for every $x, y \in H$ s.t. $x \leq y$, we have $\neg y \leq \neg x$ (and so $\neg\neg x \leq \neg\neg y$ and $\neg\neg\neg x = \neg x$).*
6. *$\neg 1 = 0$, $\neg 0 = 1$.*
7. *for every $x, y \in H$, $y \to \neg x = \neg(x \wedge y)$.*
8. *for every $a, b, c \in H$, $(a \to b) \wedge (b \to c) \leq a \to c$.*
9. *for every $x, y \in H$, $\neg\neg x \wedge \neg\neg y \leq \neg\neg(x \wedge y)$.*

Proof.

1. By the laws of lattices, we only need to prove $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. In fact, we will show that $(\bigvee_{i \in I} x_i) \wedge x = \bigvee_{i \in I}(x_i \wedge x)$, for an arbitrary family $\{x_i\}_{i \in I} \subseteq H$. We have that, for every $u \in H$:

$$\begin{aligned} \bigvee_{i \in I}(x_i \wedge x) \leq u &\Leftrightarrow x_j \wedge x \leq u, \forall j \in I \\ &\Leftrightarrow x_j \leq x \to u, \forall j \in I \\ &\Leftrightarrow \bigvee_{i \in I} x_i \leq x \to u \\ &\Leftrightarrow (\bigvee_{i \in I} x_i) \wedge x \leq u, \end{aligned}$$

   and so the statement follows "by Yoneda".
2. Let y be a common lower bound of x and $\neg x$. Then $y \leq x \to 0$, hence $y \wedge x \leq 0$. Since $y \leq x$, we have that $y \wedge x = y$ and so $y = 0$.

3. We have that for every $p \in H$:

$$\begin{aligned} p \leq \neg x \wedge \neg y &\Leftrightarrow p \leq \neg x \text{ and } p \leq \neg y \\ &\Leftrightarrow p \wedge x = 0 \text{ and } p \wedge y = 0 \\ &\Leftrightarrow (p \wedge x) \vee (p \wedge y) = 0 \\ &\Leftrightarrow p \wedge (x \vee y) = 0 \\ &\Leftrightarrow p \leq \neg(x \vee y). \end{aligned}$$

4. We have that $\neg x \leq \neg x = x \to 0$, so $\neg x \wedge x \leq 0$. We derive that $x \leq \neg x \to 0 = \neg\neg x$.
5. From $x \leq y$ and the previous inequality, we have $x \leq \neg\neg y$ and so $x \leq \neg y \to 0$, which is equivalent to $x \wedge \neg y = 0$ or $\neg y \leq \neg x$.
6. Since for all x, $x \wedge 1 = x$, 0 is the only element x such that $x \wedge 1 \leq 0$. So $1 \to 0$ ($=\neg 1$) is 0. Now, $1 \wedge 0 = 0 \leq 0$, so $1 \leq 0 \to 0 = \neg 0$ and $1 = \neg 0$.
7. For every $p \in H$:

$$\begin{aligned} p \leq y \to x &\Leftrightarrow p \wedge y \leq x \\ &\Leftrightarrow p \wedge y \leq x \to 0 \\ &\Leftrightarrow p \wedge y \wedge x = 0 \\ &\Leftrightarrow p \wedge (x \wedge y) = 0 \\ &\Leftrightarrow p \leq \neg(x \wedge y). \end{aligned}$$

8. Since $a \wedge (a \to b) \leq b$, $a \wedge (a \to b) \wedge (b \to c) \leq b \wedge (b \to c) \leq c$ and so $(a \to b) \wedge (b \to c) \leq a \to c$.
9. In the previous inequality, by taking $a = y$, $b = \neg x$ and $c = 0$, we obtain:

$$(y \to \neg x) \wedge \neg\neg x) \leq \neg y$$

and so

$$\neg(x \wedge y) \wedge \neg\neg x \wedge \neg\neg y \leq \neg y \wedge \neg\neg y = \neg y \wedge (\neg y \to 0) \leq 0$$

and

$$\neg\neg x \wedge \neg\neg y \leq \neg\neg(x \wedge y).$$

□

Remark 1.20. *Any Boolean algebra is a Heyting algebra, with the usual complementation operation, i.e.* $a \to b = \neg a \vee b$.

Proof. For any a, b, x, we have (using the Boolean property "$c \leq d$ iff $\neg c \vee d = 1$"):

$$\begin{aligned} x \wedge a \leq b &\Leftrightarrow \neg(x \wedge a) \vee b = 1 \\ &\Leftrightarrow \neg x \vee \neg a \vee b = 1 \\ &\Leftrightarrow x \leq \neg a \vee b \\ &\Leftrightarrow x \leq a \to b. \end{aligned}$$

□

Remark 1.21. *Conversely, if a Heyting algebra satisfies the law "$x \vee \neg x = 1$", it is a Boolean algebra.*

Proof. The Heyting algebra becomes a complemented distributive lattice (and the Boolean complement is exactly the $\neg$). □

Theorem–Definition 1.22. *Let H be a Heyting algebra and $x \in H$. TFAE:*

- $\neg\neg x = x$;

- *there exists $y \in H$ s.t. $x = \neg y$.*

Such an x is called **regular***. The set of them is denoted by $Reg(H)$.*

Proof. The forward direction is clear (take $y = \neg x$). Now, if $x = \neg y$, then $\neg\neg x = \neg\neg\neg y = \neg y = x$. $\square$

Theorem 1.23. *$Reg(H)$ is a Boolean algebra.*

Proof. We first show that it is a bounded lattice. It is bounded because, as we have shown above, 1 and 0 are regular.

Suppose now that a and b are regular and we write it as $a = \neg c$ and $b = \neg d$. Then $a \wedge b = \neg c \wedge \neg d = \neg(c \vee d)$. So $a \wedge b$ is also regular and hence it is a greatest lower bound in $Reg(H)$ also. If r is a regular element that is an upper bound of a and b, then $a \vee b \leq r$ and so $\neg\neg(a \vee b) \leq \neg\neg r = r$. Since $\neg\neg(a \vee b)$ is greater than $a \vee b$ and regular, it is the lowest upper bound of a and b in $Reg(H)$.

We have shown that it is a bounded lattice. Now we will prove that it is a Heyting algebra. It suffices to show that if a, b are regular, then $a \to b$ is regular. We have that $a \wedge (a \to b) \leq b)$, so $\neg\neg(a \wedge (a \to b)) \leq \neg\neg b$, from which we have $\neg\neg a \wedge \neg\neg(a \to b) \leq \neg\neg b$ and since a and b are regular, we deduce $a \wedge \neg\neg(a \to b) \leq b$. Hence $\neg\neg(a \to b) \leq a \to b$. Clearly, $a \to b \leq \neg\neg(a \to b)$, so $a \to b = \neg\neg(a \to b)$, which is regular. In particular $\neg a$ is the same, whether in H or in $Reg(H)$.

To show that it is a Boolean algebra, it remains to show that the lowest regular upper bound of a and $\neg a$, for a regular, is 1. Let y be a regular upper bound of them. Since $a \leq y$, $\neg y \leq \neg a$ and so $\neg y \leq y \leq \neg\neg y = \neg y \to 0$. Then $\neg y \wedge \neg y \leq 0$ and so $\neg y \leq 0$. Then $1 = \neg 0 \leq \neg\neg y = y$. So y is 1. $\square$

Note that in $Reg(H)$, $\wedge$ and $\to$ (and so also $\neg$) are the same as in H, but the $\vee$ is potentially different.

Definition 1.24. A **complete Heyting algebra** is a complete lattice that is a Heyting algebra.

Example 1.25. *If X is a space, $\mathcal{O}(X)$ is a complete Heyting algebra.*

Proof. It is clearly a complete bounded lattice (the greatest lower bound of a family of open sets can be achieved by taking the interior of the intersection).

Now, for any two open sets A and B we must find a largest open set X s.t. $X \cap A \subseteq B$. But that is equivalent to $X \subseteq C(A) \cup B$, where by C we have denoted the set-theoretic complement. Clearly, then, we may take X to be $int(C(A) \cup B)$. In particular $\neg A$ will be $int(C(A))$. $\square$

Remark 1.26. *The regular elements of $\mathcal{O}(X)$ are precisely the regular open sets.*

Proof. We show that the operation ic is the $\neg\neg$ of $\mathcal{O}(X)$. First, note that the interior of the complement of a set is the largest open contained in the complement of the set, so it is the complement of the smallest closed that contains the set. So $C(int(C(A)))$ is $C(C(closure(A)))$ – that is, $closure(A)$. Then $\neg\neg(A) = int(C(int(C(A)))) = int(closure(A)) = ic(A)$. $\square$

Remark 1.27. *If H is a complete Heyting algebra, then $Reg(H)$ is a complete Boolean algebra.*

Proof. Let $\{a_i\}_{i \in I}$ be a family of elements of H. We show that $\neg\neg \bigvee_{i \in I} a_i$ $(\neg\neg \bigwedge_{i \in I} a_i)$ is the supremum (infimum) of the family in $Reg(H)$. Clearly, since $\neg\neg$ is compatible with the order relation, it is an upper bound (lower bound). Suppose now that b is another upper bound (lower bound) that is also regular. Then, given the completeness of H, we have $\bigvee_{i \in I} a_i \leq b$ (or $\bigwedge_{i \in I} a_i \geq b$). Then we apply $\neg\neg$ to the inequality and we obtain $\neg\neg \bigvee_{i \in I} a_i \leq \neg\neg b = b$ (or $\neg\neg \bigwedge_{i \in I} a_i \geq \neg\neg b = b$). $\square$

Corollary 1.28. *$RO(X)$, defined as $Reg(\mathcal{O}(X))$, is a complete Boolean algebra.*

Definition 1.29. A **filter** in a Heyting algebra H is a non-empty set $F \subseteq H$ which is closed under $\wedge$ and s.t. for every $a \in F$ and $b \in H$ with $a \leq b$, we have $b \in F$. It is called a **proper filter** if $0 \notin F$.

Remark 1.30. *These notions coincide with the Boolean ones when H is a Boolean algebra.*

Remark 1.31. *If F is a filter in a Heyting algebra H and $a, b \in H$ s.t. $a \to b \notin F$, then the smallest filter which contains both F and a does not contain b.*

Proof. Clearly the set of all elements x s.t. there is some $f \in F$ with $f \wedge a \leq x$ is a filter which contains both F and a. Suppose this filter contains b. Then there is some $f \in F$ s.t. $f \wedge a \leq b$, so $f \leq a \to b$. Then $a \to b \in F$, contradicting the hypothesis. □

Corollary 1.32. *If F is a filter in a Heyting algebra H and $a \in H$ s.t. $\neg a \notin F$, then the smallest filter which contains both F and a is proper.*

Proof. Take $b = 0$ in the previous remark. □

Definition 1.33. A proper filter is called **prime** if whenever $a \vee b \in F$, we have $a \in F$ or $b \in F$.

Remark 1.34. *If F is a filter in a Heyting algebra H and $a \in H \setminus F$, then F is contained in a prime filter that does not contain a.*

Proof. By a Zorn's lemma argument, let F' be a filter which contains F and does not contain a, maximal among those with these properties. Let us show that this F' is prime. Suppose there are x, y not in F' s.t. $x \vee y \in F'$. Take F'' to be the smallest filter which contains both F' and x. Then, by the maximality of F', we must have $a \in F'$, so there is some $f \in F'$ s.t. $f \wedge x \leq a$, so $f \leq x \to a$. Then $x \to a \in F'$. Similarly $y \to a \in F'$ and so $(x \to a) \wedge (y \to a) \in F'$. Now, for every $b \in H$, we have:

$$\begin{aligned} b \leq (x \vee y) \to a &\Leftrightarrow b \wedge (x \vee y) \leq a \\ &\Leftrightarrow (b \wedge x) \vee (b \wedge y) \leq a \\ &\Leftrightarrow b \wedge x \leq a \text{ and } b \wedge y \leq a \\ &\Leftrightarrow b \leq x \to a \text{ and } b \leq y \to a \\ &\Leftrightarrow b \leq (x \to a) \wedge (y \to a). \end{aligned}$$

So $(x \vee y) \to a = (x \to a) \wedge (y \to a) \in F'$. But then, since $(x \vee y) \wedge ((x \vee y) \to a) \leq a$ and $x \vee y \in F'$, we must have $a \in F'$, which contradicts our choice of F'. □

Corollary 1.35. *Every maximal filter (that is, maximal in the poset of proper filters) is prime.*

Proof. By the previous remark, taking F to be a maximal filter and a to be 0, the constructed prime filter must, by maximality, be equal to F. □

Corollary 1.36. *Any proper filter of a Heyting algebra is contained in some maximal filter. In particular any nonzero element belongs to some maximal filter, by taking the filter containing all elements greater than it.*

Proof. Taking F to be the proper filter in the statement and a to be 0, we obtain the required maximal filter. □

Remark 1.37. *If a is an element of a Heyting algebra H and M is a maximal filter in H, then either $a \in M$ or $\neg a \in M$ (but not both!).*

Proof. Suppose $\neg a \notin M$. Take F to be the smallest filter containing M and a. By an earlier remark, F is proper and since M is maximal, $M = F$. Then a is in M. They cannot be both in M since $a \wedge \neg a = 0$ and M is proper. □

Corollary 1.38. *If a is an element of a Heyting algebra H and M is a maximal filter in H, then $a \in M$ iff $\neg\neg a \in M$.*

Proof. If $a \in M$, then, since $a \leq \neg\neg a$, we have that $\neg\neg a \in M$. Say $a \notin M$. Then, by the previous remark, $\neg a \in M$. Applying the previous remark again, $\neg\neg a \notin M$. $\square$

Remark 1.39. *In a Boolean algebra, any prime filter is maximal (also called an* **ultrafilter***).*

Theorem–Definition 1.40. *Let H be a Heyting algebra. For any $a \in H$, denote by D_a the set of all prime filters that contain a. This family of sets is a basis for a topology on the set of prime filters in H, which becomes a space called the* **prime spectrum** *of H, denoted by $Spec(H)$.*

Proof. Clearly, D_1 is the set of all prime filters and $D_a \cap D_b = D_{a \wedge b}$ for any $a, b \in H$. This suffices for the family to be a topological basis. $\square$

Remark 1.41. *In addition to the compatibility property from the above proof, the basis sets of $Spec(H)$ also satisfy $D_a \cup D_b = D_{a \vee b}$ and $\neg D_a = D_{\neg a}$ (where the first $\neg$ is performed in the Heyting algebra $\mathcal{O}(Spec(H))$).*

Proof. The first one is immediate from the definition of the prime filter. Now, since no proper filter contains both a and $\neg a$, $D_{\neg a}$ is included in the complement of D_a, but since $D_{\neg a}$ is open, it is included in the interior of the complement of D_a – that is, in $\neg D_a$. For the converse, suppose F is a prime in $\neg D_a$. Then, there is a basic open D_b s.t. $F \in D_b$ and $D_a \cap D_b = \emptyset$. But since $D_a \cap D_b = D_{a \wedge b}$ and any nonzero element of H is contained in some prime filter, we must have $a \wedge b = 0$, so $b \leq \neg a$ and so $\neg a$ is also in F. But that means that F is in $D_{\neg a}$. $\square$

Remark 1.42. *If H is a complete Heyting algebra that also satisfies the infinite distributive law:*

$$x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i)$$

for any $x \in H$ and any family $\{y_i\}_{i \in I}$ of elements of H (this happens, for example, if $H = \mathcal{O}(X)$ for some space X) then, for any family $\{y_i\}_{i \in I}$ of elements of H we have that:

$$D_{\bigvee_{i \in I} y_i} \subseteq \neg\neg \bigcup_{i \in I} D_{y_i}$$

Proof. Since $D_{\bigvee_{i \in I} y_i}$ is already open, we only need to show that it is contained in the closure of $\bigcup_{i \in I} D_{y_i}$. So we must prove that for any prime filter F that contains $\bigvee_{i \in I} y_i$ and for any basic open D_x that contains F – that is, for any $x \in F$ – there is a prime filter F' containing x and an $i \in I$ s.t. $y_i \in F'$. Briefly, what is to be shown is that for any $x \in F$ there is an $i \in I$ s.t. $x \wedge y_i \neq 0$. Suppose, on the contrary, that there is an $x \in F$ s.t. for any $i \in I$, $x \wedge y_i = 0$, so $\bigvee_{i \in I}(x \wedge y_i) = 0$. Applying the infinite distributive law, we get that $x \wedge (\bigvee_{i \in I} y_i) = 0$. But that is a contradiction, since both x and $\bigvee_{i \in I} y_i$ are in F. $\square$

Remark 1.43. *Let H be a Heyting algebra. The specialization preorder on $Spec(H)$ coincides with the inclusion relation between filters.*

Proof. Let F and F' be two prime filters. We have that F' is in the closure of $\{F\}$ iff any basic open set D_a (with $a \in H$) that contains F' also contains F iff any $a \in F'$ is also in F iff $F' \subseteq F$. $\square$

Theorem 1.44. *For any Heyting algebra H, the space $Spec(Reg(H))$ embeds canonically into $Spec(H)$ as the subspace of maximal filters in H.*

Proof. Define, for any U in $Spec(Reg(H))$, $f(U) = \{u \in H \mid \neg u \notin U\}$, and for any M a maximal filter in H, $g(M) = \{\neg\neg x \mid x \in M\}$.

Step 1. For any U, $f(U)$ is a maximal filter in H.

Since $\neg 1 = 0 \notin U$, we have that 1 is in $f(U)$, so $f(U)$ is nonempty.

Let $x \in f(U)$ and $x \leq y$. Then $\neg x \notin U$ and $\neg y \leq \neg x$, so $\neg y \notin U$ and $y \in f(U)$.

Let $x, y \in f(U)$. Then $\neg x, \neg y \notin U$ and because U is an ultrafilter, $\neg\neg x$ and $\neg\neg y$ belong to U. Then $\neg\neg x \wedge \neg\neg y \in U$. We have proved earlier that $\neg\neg x \wedge \neg\neg y \leq \neg\neg(x \wedge y)$, so $\neg\neg(x \wedge y)$ is in U. Therefore $\neg(x \wedge y)$ is not in U, so $x \wedge y$ is in $f(U)$.

So $f(U)$ is a filter. To see that it is proper, suppose 0 is in $f(U)$ – then $\neg 0 = 1 \notin U$, clearly false. To see that it is maximal, suppose it is contained in another proper filter G that contains at least an additional element y. Then, since y is not in $f(U)$, $\neg y$ must be in U. Then $\neg\neg y$ is not in U, so $\neg y$ is in $f(U)$ and also in G. But then G contains both y and $\neg y$, which is absurd.

Step 2. For any M, $g(M)$ is an ultrafilter in $Reg(H)$.

Since $1 = \neg 0 = \neg\neg 1$, we have that 1 is in $g(M)$, so $g(M)$ is nonempty.

Let $x \in g(M)$ and $y \in Reg(H)$ s.t. $x \leq y$. Then $x = \neg\neg z$ for some $z \in M$ and $y = \neg\neg y$, so $\neg\neg z \leq \neg\neg y$. Suppose $\neg y$ is in M. Then $z \wedge \neg y \in M$, but $z \leq \neg\neg z \leq \neg\neg y$, so $z \wedge \neg y \leq \neg\neg y \wedge \neg y = 0$, contradicting the fact that M is proper. Thus $\neg y$ is not in M, therefore y is in M and $\neg\neg y = y \in g(M)$.

Let now be $x, y \in g(M)$, so $x = \neg\neg z$ and $y = \neg\neg t$ with $z, t \in M$. So $x \wedge y = \neg\neg z \wedge \neg\neg t$ which is greater than $\neg\neg(z \wedge t)$ by the property of the infimum. Since $\neg\neg(z \wedge t)$ is in M, $x \wedge y$ is in $g(M)$. So $g(M)$ is a filter.

To see that it is proper, suppose 0 is of the form $\neg\neg x$ with $x \in M$. But since $x \leq \neg\neg x$, we conclude that $0 \in M$, contradiction. To see that $g(M)$ is an ultrafilter, take $x \in Reg(H)$. Either $x \in M$ or $\neg x \in M$, so either $\neg\neg x$ or $\neg\neg\neg x$ is in $g(M)$, but since $x = \neg\neg x$, we have that either $x \in g(M)$ or $\neg x \in g(M)$, a defining property of the ultrafilter in a Boolean algebra.

Step 3. We will show that $f(g(M)) = M$ for any M.

It boils down to showing that $\{u \in H \mid \nexists x \in M \text{ s.t. } \neg u = \neg\neg x\} = M$.

Let u be such that for any $x \in M$, $\neg u \neq \neg\neg x$. Suppose $u \notin M$. Then $\neg u \in M$, so by taking $x = \neg u$, we have $\neg u = \neg\neg x$, contradicting the hypothesis.

Now take $u \in M$ and suppose there is some $x \in M$ s.t. $\neg u = \neg\neg x$. Then $\neg u \notin M$, so $\neg\neg x \notin M$, $\neg x \in M$ and $x \notin M$, contradicting the choice of x.

Step 4. We will show that $g(f(U)) = U$ for any U.

We have:

$$\begin{aligned} g(f(U)) &= g(\{u \in H \mid \neg u \notin U\}) \\ &= \{\neg\neg u \mid \neg u \notin U\} \\ &= \{\neg\neg u \mid \neg\neg u \in U\} \\ &= U. \end{aligned}$$

For the last two steps, we will use D_a for an arbitrary basic open set of $Spec(H)$, where $a \in H$, and D_b^{reg} for an arbitrary basic open set of $Spec(Reg(H))$, where $b = \neg\neg b$.

Step 5. We show that f is continuous.

We have:

$$\begin{aligned} U \in f^{-1}(D_a) &\Leftrightarrow f(U) \in D_a \\ &\Leftrightarrow a \in f(U) \\ &\Leftrightarrow \neg a \notin U \\ &\Leftrightarrow \neg\neg a \in U \\ &\Leftrightarrow U \in D^{reg}_{\neg\neg a}. \end{aligned}$$

Step 6. We show that g is continuous.

We have:

$$\begin{aligned} M \in g^{-1}(D^{reg}_b) &\Leftrightarrow g(M) \in D^{reg}_b \\ &\Leftrightarrow b \in g(M) \\ &\Leftrightarrow \exists x \in M \text{ s.t. } \neg\neg x = b \\ &\Leftrightarrow b \in M \\ &\Leftrightarrow M \in D_b, \end{aligned}$$

where at the fourth equivalence we used that $x \leq \neg\neg x = b$ and that $b = \neg\neg b$.

Since f and g are continuous and inverse to each other, the two spaces considered are homeomorphic. □

Definition 1.45. We define the **spectrum** of a space X to be $Spec(X) := Spec(\mathcal{O}(X))$ and its **Stone spectrum** to be $St(X) := Spec(RO(X))$ (which is homeomorphic, as per above, to the space of maximal filters in $\mathcal{O}(X)$).

Theorem–Definition 1.46. *Spec can be "extended" to an endofunctor of* **Top***, by defining, for any continuous $f : X \to Y$ and any $F \in Spec(X)$, $Spec(f)(F)$ to be the set of all opens D of Y s.t. $f^{-1}(D) \in F$.*

Proof. For any space X and any filter F in $\mathcal{O}(X)$, we have that $Spec(id_X)(F)$ is the set of all opens D of X such that $id_X^{-1}(D) = D \in F$. But that set is simply $F = id_{Spec(X)}(F)$.

To prove that the association is compositional, one must only remark that $(f \circ g)^{-1}$ equals $g^{-1} \circ f^{-1}$ and the conclusion follows. □

Theorem 1.47. *For each x in a space X, the set of all opens that contain x is a prime filter. Moreover, the map $\eta_X : X \to Spec(X)$ sending each point to the prime filter containing it is continuous, and $\eta = \{\eta_X\}_{X \in \mathbf{Top}}$ is a natural transformation between the identity functor on* **Top** *and the Spec functor.*

Proof. The prime filter conditions are easily checked. Let us now consider a basic open D_U of $Spec(X)$, where $U \in \mathcal{O}(X)$. The preimage of D_U is the set of all x such that U contains X, which is U, so open. Hence η_X is continuous.

To prove that η is a natural transformation, we must check that for any continuous $f : X \to Y$, we have that $\eta_Y \circ f = Spec(f) \circ \eta_X$. But $\eta_Y(f(x)) = \{U \mid f(x) \in U\} = \{U \mid x \in f^{-1}(U)\} = \{U \mid f^{-1}(U) \in \eta_X(x)\} = Spec(f)(\eta_X(x))$, for an arbitrary $x \in X$, so the equality holds. □

1.3 First-order structures

We shall consider only relational structures (excepting occasional use of constants in an unproblematic way). If A is a set and $n \in \mathbb{N}$ then A^n is the cartesian product of A taken n times (A^0 will be the singleton $\{\emptyset\}$). Elements of A^n will be sometimes denoted by vector-letters like $\vec{a} = (a_1, ..., a_n)$. If $f : A \to B$ is a function and $n \in \mathbb{N}$, then we define the function $f^{(n)} : A^n \to B^n$ by $f^{(n)}(a_1, ..., a_n) = (f(a_1), ..., f(a_n))$.

Definition 1.48. If A is a set and $n \in \mathbb{N}$, an **n-ary relation** on A is a subset of A^n.

Definition 1.49. An n-ary relation R on A is called a **function** if for every $(a_1, ..., a_{n-1}) \in A^{n-1}$ there exists an unique $a \in A$ s.t. $(a_1, ..., a_{n-1}, a) \in R$.

Definition 1.50. A (first-order, relational) **signature** is a pair (γ, μ) where γ is an ordinal and $\mu : \gamma \to \mathbb{N}$ is a function.

Definition 1.51. Let $\Sigma = (\gamma, \mu)$ be a signature. A **Σ-structure** M is a pair $(M, \{R_i\}_{i\in\gamma})$, where M is a set (denoted by the same letter as the structure, by abuse of language) and for every $i \in \gamma$, R_i is a $\mu(i)$-ary relation on M, and we say that its **index** is i. Those relations will be called the **relations of** M. (This terminology will make possible for us to largely forget about the γ and the μ, as we shall see shortly.)

From now on, we shall fix a signature Σ and use it throughout the rest of these notes (and frequently drop the explicit reference to it).

Definition 1.52. A Σ-structure is called **algebraic** if all its relations are functions.

Definition 1.53. Let M and N be two Σ-structures. A **homomorphism** between M and N is a function $f : M \to N$ having the property that for any two n-ary relations R and R' of M and N, respectively, of the same index, and for any $\vec{a} \in R$ we have that $f^{(n)}(\vec{a}) \in R'$.

Remark 1.54. *The Σ-structures together with the homomorphisms between them form a category that will be denoted by* Σ**-Str**.

Definition 1.55. A poset is called **directed** if any two elements have a common upper bound.

Example 1.56. *For any space X and any $x \in X$, $(\eta_X(x))^{op}$ is a directed poset.*

Proof. For any two opens that contain x, their intersection also contains x. □

Theorem 1.57. *The category* Σ**-Str** *admits colimits of diagrams indexed by directed posets. Moreover, such a directed colimit of algebraic structures is necessarily algebraic.*

Proof. Let $(I, \leq)$ be a poset and $D : I \to \Sigma$**-Str** be a functor. Consider the relation $\sim$ on the disjoint union of all the $D(i)$'s (where $i \in I$; it consists of all pairs (i, m) with $i \in I$ and $m \in D(i)$), defined by: $(i, m) \sim (j, n)$ iff there is some $k \in I$ s.t. $k \geq i, j$ and $D(i \leq k)(m) = D(j \leq k)(n)$. The relation $\sim$ is an equivalence relation (the poset I needs to be directed for it to be transitive). We denote by D_∞ the resulting quotient set.

To make D_∞ into a colimit candidate we must endow it with a family of relations as prescribed by Σ and a family of homomorphisms $\{\psi_i : D(i) \to D_\infty\}_{i\in I}$. Define, for each i and each $m \in D(i)$, $\psi_i(m)$ to be the class in D_∞ of (i, m). Clearly, these maps commute with the $D(i \leq j)$'s (we take in the definition of $\sim$ the k to be j). We will define the relations on D_∞ such that all the ψ_i's are homomorphisms. Let $\{R_i\}_{i\in I}$ be s.t. for any i, R_i is a relation of $D(i)$ and all the relations are of the same index, hence there is an n such that all are n-ary. Let R be the corresponding relation that we seek to define on D_∞. Put $\vec{a} \in D_\infty^n$ in R iff there is an i and an $\vec{a'} \in R_i$ such that $\psi_i^{(n)}(\vec{a'}) = \vec{a}$.

To show universality, let $(F, \{\chi_i\}_{i\in I})$ be another cocone over our diagram. We seek to show that there is an unique homomorphism $\chi : D_\infty \to F$ such that, for all i, we have $\chi_i = \chi \circ \psi_i$. To show uniqueness, it is sufficient to remark that each element of D_∞ is the class of an (i, m), so it is a $\psi_i(m)$. But $\chi(\psi_i(m)) = \chi_i(m)$, which is already fixed.

It remains to be seen whether this assignment really defines a morphism. To show that it is a well-defined function, suppose there is an element that can be written as both $\psi_i(m)$ and $\psi_j(n)$. But then $(i, m) \sim (j, n)$, so there is a $k \leq i, j$ s.t. $D(i \leq k)(m) = D(j \leq k)(n)$. So $\chi_i(m) = \chi_k(D(i \leq k)(m)) = \chi_k(D(j \leq k)(n)) = \chi_i(n)$, so the map is well-defined.

To show that the map is a morphism, let, as before, $\{R_i\}_{i\in I}$ be s.t. for any i, R_i is a relation of $D(i)$ and all the relations are of the same index (and n-ary), R the corresponding relation of D_∞ and R' the one of F. Start with $\vec{a} \in R$. We need to prove that $\chi^{(n)}(\vec{a}) \in R'$. If $\vec{a} \in R$, then there is some k and some $\vec{a'} \in R_k$ such that $\psi_k^{(n)}(\vec{a'}) = \vec{a}$. But then $\chi^{(n)}(\vec{a}) = \chi(\psi_k^{(n)}(\vec{a'})) = \chi_k^{(n)}(\vec{a'})$, the last of which being in R', by virtue of χ_k being a homomorphism. The colimit's existence has been shown.

Let us turn now our attention to the last claim. We use the notations from the previous paragraph and we suppose that all R_i's are functions. Let $(a_1, ..., a_{n-1})$ be in D_∞^{n-1}. Then for all $p \in \overline{1, n-1}$ there is an $i_p \in I$ and some $a'_p \in D(i_p)$ s.t. $\psi_{i_p}(a'_p) = a_p$. Since I is directed, there is $k \in I$ greater than all i_p's. Define a''_p as $D(i_p \leq k)(a'_p) \in D(k)$. Then, for all p, $\psi_k(a''_p) = a_p$. Because R_k is a function, there is $b'' \in D(k)$ such that $(a''_1, ..., a''_{n-1}, b'') \in R_k$, so, by taking b to be $\psi_k(b'')$ we have that $(a_1, ..., a_{n-1}, b)$ is in R. To show uniqueness, let b_1 be another element of D_∞ s.t. $(a_1, ..., a_{n-1}, b_1)$ is in R. Then there is some k_1 and $a'''_1, ..., a'''_{n-1}, b'''_1 \in D(k_1)$ such that for any $p \in \overline{1, n-1}$, $\psi_{k_1}(a'''_p) = a_p$ and $\psi_{k_1}(b'''_1) = b_1$. Now, for any p we have that $a_p = \psi_k(a''_p) = \psi_{k_1}(a'''_p)$ so there is some $l_p \geq k, k_1$ s.t. $D(k \leq l_p)(a''_p) = D(k_1 \leq l_p)(a'''_p)$. Let l be greater than all l_p's. Then, by applying the fact that R_l is a function, we get that $D(k \leq l)(b'') = D(k_1 \leq l)(b'''_1)$ and so that $b = b_1$. Thus R is also a function. $\square$

Definition 1.58. If $(I, \leq)$ is a preordered set, a **Kripke Σ-structure** on I is a functor $K : (I, \leq) \to \Sigma$**-Str**.

Kripke structures come naturally equipped with the following forcing relation.

Definition 1.59. If K is a Kripke structure on an $(I, \leq)$, we inductively define, for any $i \in I$ and any ϕ in the language of Σ augmented by constants corresponding to the elements of $K(i)$ (so that we can refer, for all $i' \geq i$, to some elements of $K(i')$, evaluating a c_m to $K(i \leq i')(m)$), the relation $i \Vdash \phi$ (read "i **forces** ϕ"):

- if ϕ is atomic, then $i \Vdash \phi$ iff $K(i) \models \phi$;
- $i \Vdash \neg\psi$ iff for any $i' \geq i$ we have $i' \nVdash \psi$;
- $i \Vdash \psi \wedge \chi$ iff $i \Vdash \psi$ and $i \Vdash \chi$;
- $i \Vdash \psi \vee \chi$ iff $i \Vdash \psi$ or $i \Vdash \chi$;
- $i \Vdash \exists x \psi(x)$ iff there is some $a \in K(i)$ s.t. $i \Vdash \psi(a)$.

Note that if I is discrete (and so $i \leq i'$ iff $i = i'$), then Kripke forcing coincides with usual satisfaction, i.e. $i \vdash \phi \Leftrightarrow K(i) \models \phi$.

1.4 Indexed categories

Definition 1.60. An **indexed category** is a **Cat**-valued (quasi-)functor, where **Cat** is the (quasi-)category of all categories.

Definition 1.61. If $L : \mathcal{C} \to \mathbf{Cat}$ is an indexed category, we define its **Grothendieck flattening** to be the category $\widetilde{L}$, given by the following data:

- objects are pairs (i, A), where i is an object of $\mathcal{C}$ and A is an object of $L(i)$;
- morphisms between two such objects (i, A) and (j, B) are pairs (f, f'), where $f : i \to j$ is a morphism of $\mathcal{C}$ and f' is a morphism in $L(j)$ between B and $L(f)(A)$;
- if $\tilde{f} = (f, f') : (i, A) \to (j, B)$ and $\tilde{g} = (g, g') : (j, B) \to (k, C)$ are two morphisms in $\widetilde{L}$ we define their composition $\tilde{g} \circ \tilde{f}$ to be the morphism $(g \circ f, L(g)(f') \circ g')$.

The category axioms can be easily checked.

Theorem 1.62. *If $L : \mathcal{C} \to \mathbf{Cat}$ is an indexed category, $F : \mathcal{C} \to \mathcal{C}$ is a functor and $\eta : id_{\mathcal{C}} \Rightarrow F$ is a natural transformation, we can define a functor $\widetilde{F_\eta} : \widetilde{L} \to \widetilde{L}$, by setting:*

- *for any object (i, A) of $\widetilde{L}$, $\widetilde{F_\eta}(i, A)$ to be the pair $(F(i), L(\eta_i)(A))$;*

- *for any morphism* $\tilde{f} = (f, f') : (i, A) \to (j, B)$ *of* $\widetilde{L}$, $\widetilde{F}_\eta(\tilde{f})$ *to be* $(F(f), L(\eta_j)(f'))$.

Proof. Easy check. $\square$

2 Structured spaces

2.1 Basic notions and constructions

Definition 2.1. If X is a space, a **Σ-presheaf** on X is a functor $P : \mathcal{O}(X)^{op} \to \Sigma$-**Str**.

Remark 2.2. *The Σ-presheaves on a space X, together with morphisms between them (i.e., natural transformations), form a category, denoted by* $\mathbf{Psh}_\Sigma(X)$.

Remark 2.3. *If X is a space, P is a presheaf on X, $U \subseteq V$ two opens of X and $f \in P(V)$, then the element $P(U \subseteq V)(f) \in P(U)$ will be denoted simply by $f_{|U}$. Also, we denote $P(U \subseteq V)^{(n)}(\vec{f})$ by $\vec{f}_{|U}$, for any $n \in \mathbb{N}$.*

Definition 2.4. A presheaf is called **algebraic** if its image consists only of algebraic structures.

Definition 2.5. Let X be a space, $x \in X$ and P a presheaf on X. The colimit of the restriction of P to $(\eta_X(x))^{op}$ (which, as we have seen, is directed) is called the **stalk of P at x**. It is denoted by P_x.

Remark 2.6. *As above, when $x \in U$ and $f \in P(V)$ we denote the image of f in the colimit P_x by f_x.*

Remark 2.7. *The stalks of an algebraic presheaf are algebraic structures.*

Proof. It follows from the last claim of the theorem on directed colimits. $\square$

Theorem 2.8. *If P and Q are two presheaves on a space X and $\phi : P \Rightarrow Q$, then for any $x \in X$ we have a canonical morphism $\phi_x : P_x \to Q_x$, taking a $h_x \in P_x$, where h is in an $P(U)$, to $(\phi_U(h))_x$.*

Proof. For any U in $\eta_X(x)$, consider the map $\chi_U = \psi_U \circ \phi_U$ (where the ψ's are the injections into Q_x). We prove that $(Q_x, \{\chi_U\}_{U \in \eta_X(x)})$ is a cocone over the diagram $P_{|\eta_X(x)}$. We must show that for any $U \subseteq V$, we have that $\chi_U \circ P(U \subseteq V) = \chi_V$. This follows because of the ϕ being natural and because the ψ's form a cocone over the corresponding restriction of Q. By the universality of the colimit P_x, we get the desired map. $\square$

Remark 2.9. *If P is a presheaf on a space X, and $x, y \in X$ such that x is in the closure of $\{y\}$, there is a natural morphism $P_{x,y} : P_x \to P_y$ such that for any $U \ni x$ (so y is also in U) and any $f \in P(U)$ we have that $P_{x,y}(f_x) = f_y$.*

Proof. Since $\eta_X(x) \subseteq \eta_X(y)$, the map is naturally induced by the universal property of the colimit. $\square$

Definition 2.10. A presheaf P on a space X is called a **sheaf** if for any $U \in \mathcal{O}(X)$ and any open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of U, we have that:

1. for any family $\{h_\alpha\}_{\alpha \in \Lambda}$ s.t. for any α we have that $h_\alpha \in P(U_\alpha)$ and for any α, β we have that $h_{\alpha|U_\alpha \cap U_\beta} = h_{\beta|U_\alpha \cap U_\beta}$, there exists an unique $f \in P(U)$ s.t. for any α we have that $f_{|U_\alpha} = f_\alpha$;
2. if we denote by $\{R_\alpha\}_\alpha$ a family of relations s.t. for any α, R_α is a relation of $P(U_\alpha)$, and all the relations are of the same index, and their common arity is n, and if R is the corresponding relation of $P(U)$, then for any $\vec{f} \in (P(U))^n$ s.t. for any α we have that $\vec{f}_{|U_\alpha} \in R_\alpha$, we have that $\vec{f} \in R$.

Definition 2.11. The full subcategory of $\mathbf{Psh}_\Sigma(X)$ consisting of sheaves on X will be denoted by $\mathbf{Sh}_\Sigma(X)$.

Example 2.12. *Let X be a space and A be a structure. Then we define the "A-valued"* **constant sheaf** $\underline{A}$ *on X by postulating $\underline{A}(U)$ to be, for any $U \in \mathcal{O}(X)$, the set of all $f : U \in A$ that are locally constant (equivalently, they are continuous when A is given the discrete topology) and the restriction maps to be the ordinary restriction operations on functions. This construction can easily be seen to form a sheaf on X.*

Remark 2.13. *An algebraic presheaf P that satisfies condition (1) above is a sheaf.*

Proof. Let $\{R_\alpha\}_\alpha$, R, $\vec{f}$ be as in the antecedent of condition (2). Since $(f_1, ..., f_{n-1}) \in (P(U))^n$ and R is a function, there exists an unique $g \in P(U)$ s.t. $(f_1, ..., f_{n-1}, g) \in R$. We need to prove that $g = f_n$. Since the "restrictions" are homomorphisms, we have that for any α, $(f_{1|U_\alpha}, ..., f_{n-1|U_\alpha}, g_{|U_\alpha}) \in R_\alpha$. But since $(f_{1|U_\alpha}, ..., f_{n-1|U_\alpha}, f_{n|U_\alpha}) \in R_\alpha$ and R_α is a function, $g_{|U_\alpha} = f_{n|U_\alpha}$. But then, applying condition (1), $g = f_n$. □

Since sheaf theory is usually done for purely algebraic structures (sets, rings, abelian groups etc.), in the majority of books on sheaf theory, only condition (1) is given in the definition of a sheaf.

Remark 2.14. *If P is a sheaf on a space X, then $P(\emptyset)$ is a one-element set.*

Proof. Apply the first property of the sheaf for $\Lambda = \emptyset$ (so U is also $\emptyset$). □

Definition 2.15. If $f : X \to Y$ is a continuous map and P is a sheaf on X, we define the **direct image of P along f** to be the sheaf f_*P, defined by $f_*(P)(U) = P(f^{-1}(U))$ and $f_*P(U \subseteq V) = P(f^{-1}(U) \subseteq f^{-1}(V))$. It can easily be seen to satisfy all the necessary conditions, since taking the preimage of a set commutes with all the usual set-theoretic operations.

Remark 2.16. *If $f : X \to Y$ is a continuous map, P a sheaf on X and $x \in X$, there is a natural map $(f_*P)_{f(x)} \to P_x$, taking an $h_{f(x)}$, where h is in an $(f_*P)(U) = P(f^{-1}(U))$, to h_x.*

Proof. We establish P_x as a cocone over $(\eta_Y(f(x)))^{op}$ and by applying the universal property of the colimit, the conclusion will follow. If $U \in \eta_Y(f(x))$, then $f(x) \in U$ and so $x \in f^{-1}(U)$. So we have a natural map from $(f_*P)(U) = P(f^{-1}(U))$ to P_x. These maps together form the desired cocone. □

Remark 2.17. *The map f_* can be "extended" to a functor, by defining, for any space X, $P, Q \in \mathbf{Sh}_\Sigma(X)$ and $\phi : P \Rightarrow Q$, $(f_*\phi)_U$ to be $\phi_{f^{-1}(U)}$.*

Theorem–Definition 2.18. *The map $\mathbf{Sh}_\Sigma$ can be considered to be an indexed category by setting $\mathbf{Sh}_\Sigma(f) = f_*$. The category $\widetilde{\mathbf{Sh}_\Sigma}$ is called the category of* **(Σ-)structured spaces** *– that is, a structured space is a pair (X, P) where X is a space and P is a sheaf on X.*

Remark 2.19. *If $\tilde{f} = (f, f') : (X, P) \to (Y, Q)$ is a map of structured spaces and $x \in X$, we have a natural map $\tilde{f}_x : Q_{f(x)} \to P_x$.*

Proof. By the definition of the Grothendieck flattening, f' is a map between Q and f_*P, so it induces (by Theorem 2.8) a map $Q_{f(x)} \to (f_*P)_{f(x)}$. We get the desired one by composing this one with the natural map $(f_*P)_{f(x)} \to P_x$, obtained in Remark 2.16. □

Theorem–Definition 2.20. *If (X, P) is a structured space, then, denoting by $\leq_X$ the specialization preorder on X, we can define its* **associated Kripke structure** $P^k : (X, \leq_X) \to \mathbf{\Sigma\text{-}Str}$ *by the following rules:*

- *if $x \in X$ then $P^k(x)$ is P_x;*
- *if $x \leq_X y$ then $P^k(x \leq_X y)$ is $P_{x,y}$ (constructed in Remark 2.9).*

Proof. We only need to prove that the functor law holds. If $x \leq_X y \leq_X z$ then for any $U \ni x$ and any $f \in P(U)$,

$$(P_{y,z} \circ P_{x,y})(f_x) = P_{y,z}(P_{x,y}(f_x)) = P_{y,z}(f_y) = f_z = P_{x,z}(f_x)$$

and we are done. □

Moreover, if I is an Alexandrov space and $\leq$ its associated preorder, we can characterize the sheaves on I. Note first that if $x \in I$ then every open set containing x must include E_x, which is therefore a terminal object in $(\eta_I(x))^{op}$, so the stalk of any sheaf P at x is isomorphic to $P(E_x)$. If P is a sheaf on I, we can easily define a Kripke structure K on $(I, \leq)$ by setting $K(x) = P(E_x)$ for any $x \in I$, and $K(x \leq y) = P(E_y \subseteq E_x)$. This K is isomorphic to the P^k constructed in the above theorem. Conversely, if K is a Kripke structure on $(I, \leq)$, we can define a sheaf P on I by setting $P(U)$, for any $U \in \mathcal{O}(I)$, to be the set of all $h \in \prod_{i \in U} K(i)$ s.t. for any $i \leq j$, $K(i \leq j)(h_i) = h_j$ (the relations are the natural ones). This can easily be seen to define a correspondence between sheaves on I and Kripke structures on $(I, \leq)$.

We can also note that if, in particular, I is discrete, the above correspondence reduces to one between sheaves on I and families of structures $\{A_i\}_{i \in I}$.

2.2 Sheaves on spectra

By applying Theorem 1.62 to the case where $\mathcal{C}$ is **Top**, L is $\mathbf{Sh}_\Sigma$, F is *Spec* and η is the one from Theorem 1.47, we obtain an endofunctor:

$$\widetilde{Spec}_\eta : \widetilde{\mathbf{Sh}_\Sigma} \to \widetilde{\mathbf{Sh}_\Sigma},$$

where for any object (X, P) of $\widetilde{\mathbf{Sh}_\Sigma}$, $\widetilde{Spec}_\eta(X, P)$ will be $(Spec(X), (\eta_X)_* P)$. Our goal, from now on, will be mainly to study the (model-theoretic) properties of the stalks of this structured space.

Let us see what happens when X is a discrete space. Firstly, note that in this case $\mathcal{O}(X)$ is equal to $\mathcal{P}(X)$, the powerset of X, which is a Boolean algebra, so $Spec(X)$ is the Stone space of the ultrafilters on X. Also, by the discussion in the previous subsection, a sheaf P on X is "induced" by a family of structures $\{A_x\}_{x \in X}$, in the sense that for any $B \subseteq X$, $P(B)$ is isomorphic to $\prod_{x \in B} A_x$ (and the restriction maps are the standard ones). Let us assume that each A_x is nonempty and choose for any x an element $c_x \in A_x$.

If we denote by D_B the basic open set of all ultrafilters that contain a given $B \in \mathcal{P}(X)$, then we have seen earlier that $\eta_X^{-1}(D_B) = B$, so $((\eta_X)_* P)(D_B) = P(B) = \prod_{x \in B} A_x$. Let U be an ultrafilter on X and let us compute the stalk of $(\eta_X)_* P$ at U. This is equal to:

$$\varinjlim_{U \in D} ((\eta_X)_* P)(D),$$

but the same result can be achieved by considering the colimit only over the values of $(\eta_X)_* P$ on basic open sets containing U, which form a cofinal family. So the stalk becomes:

$$\varinjlim_{B \in U} ((\eta_X)_* P)(D_B),$$

but this is equal, from the earlier discussion, to:

$$\varinjlim_{B \in U} \prod_{x \in B} A_x.$$

By the general existence theorem (Theorem 1.57), we can identify the elements of that colimit with pairs (B, m), where $B \in U$ and $m \in \prod_{x \in B} A_x$, factored by the relation $\sim$, where $(B, m) \sim (C, n)$ iff there is $D \in U$ with $D \subseteq B \cap C$ s.t. for any $x \in D$, $m(x) = n(x)$. Let (B, m) be such a pair and remark that if we define $n \in \prod_{x \in X} A_i$ s.t. $n(x)$ is $m(x)$ when $x \in B$ and c_x otherwise, then (by choosing $D = B$) $(B, m) \sim (I, n)$. So we can restrict ourselves to considering pairs of the form (I, m), and even drop the I. Two such elements m and n are equivalent iff there is a $D \in U$ s.t. $D \subseteq \{x \mid m(x) = n(x)\}$. But that only happens when $\{x \mid m(x) = n(x)\}$ is itself in U – in other words, if and only if m and n are equivalent with respect to the ultrafilter U! So the stalk of $(\eta_X)_* P$ at U is equal to the ultraproduct $\prod_U A_x$.

Remember that, in order for this to work, we have assumed each A_x to be nonempty. There is strong evidence that this happens only because we use the "wrong" definition of the ultraproduct. Let us see why this is so. If ϕ is the sentence $(\exists x)(x = x)$, then a structure satisfies ϕ if and only if it is nonempty. If $\{A_n\}$ is a family of structures indexed by $\mathbb{N}$ s.t. A_n is nonempty iff $n = 1$, and if U is the ultrafilter on $\mathbb{N}$ generated by the singleton $\{1\}$, then, since $\prod_n A_n$ is empty, the ultraproduct $\prod_U A_n$ will also be empty and so it will not satisfy ϕ. We have then that, by Łoś's theorem, the set of all n such that $A_n \models \phi$ is not in U. But that set is $\{1\}$, which is in U, by the definition of U. We have obtained a contradiction. That is why, in common treatments of Łoś's theorem, the condition that each model is nonempty is outright stated. But this can be avoided by replacing the definition of the ultraproduct of a family $\{A_x\}_{x\in X}$, with respect to an ultrafilter U on X, with the colimit above, namely:

$$\varinjlim_{B\in U} \prod_{x\in B} A_x,$$

which makes Łoś's theorem work even in edge conditions.

Because of this situation that happens when X is discrete, the functor $\widetilde{Spec}_\eta$ can be considered to be the rightful generalization of the ultraproduct construction.

3 Logic

In order to study the properties of the stalks of the structured space that is obtained via the $\widetilde{Spec}_\eta$ functor, we first need to define a notion of *forcing*.

Definition 3.1. If (X, P) is a structured space and $U \in \mathcal{O}(X)$, we define, for any sentence ϕ in the language of Σ augmented by constants corresponding to elements of $P(U)$ (so that we can refer inside ϕ to any element of a $P(V)$ with $V \in \mathcal{O}(U)$, given that a constant c_m would evaluate to $m_{|V}$, similarly to the way Kripke forcing is defined; also, we will refer to the constant as m or $m_{|V}$ instead of c_m in the sequel), its **(strong) forcing value**, which is an open subset of U and is denoted by $\|\phi\|^s_U$, by the following inductive rules:

- if ϕ is atomic, then:

$$\|\phi\|^s_U := \bigcup_{\substack{V\in\mathcal{O}(U)\\ P(V)\models\phi}} V$$

  (or, equivalently, by using the second defining property of the sheaf, it is the largest $V \in \mathcal{O}(U)$ such that $P(V) \models \phi$);
- $\|\neg\psi\|^s_U := \neg\|\psi\|^s_U$ (i.e. the interior of the complement relative to U of $\|\psi\|^s_U$, since the operation $\neg$ is performed in $\mathcal{O}(U)$);
- $\|\psi \wedge \chi\|^s_U := \|\psi\|^s_U \cap \|\chi\|^s_U$;
- $\|\psi \vee \chi\|^s_U := \|\psi\|^s_U \cup \|\chi\|^s_U$;
- $\|\exists x\psi(x)\|^s_U := \bigcup_{V\in\mathcal{O}(U)} \bigcup_{a\in P(V)} \|\psi(a)\|^s_V$.

Strong forcing satisfies the following natural property.

Lemma 3.2. *Let (X, P) be a structured space, $U, U' \in \mathcal{O}(X)$ with $U' \subseteq U$, $n \in \mathbb{N}$, $\vec{x} = (x_{k_1}, ..., x_{k_n})$, where $\{x_{k_i} \mid i \in \overline{1, n}\}$ is a subset of the set of variables of the language (and we may have $x_{k_i} = x_{k_j}$ for $i \neq j$), $\vec{f} \in \mathcal{P}(U)^n$ and $\phi(\vec{x})$ a formula in the language of Σ augmented by constants corresponding to elements of $P(U)$ (as in the definition of strong forcing). Then:*

$$\|\phi(\vec{f}_{|U'})\|^s_{U'} = \|\phi(\vec{f})\|^s_U \cap U'$$

In particular, $\|\phi(\vec{f}_{|U'})\|^s_{U'} \subseteq \|\phi(\vec{f})\|^s_U$.

Proof. We proceed by structural induction.

If $\phi(\vec{x})$ is atomic, then:

$$\|\phi(\vec{f}_{|U'})\|^s_{U'} = \bigcup_{\substack{V \in \mathcal{O}(U') \\ P(V) \models \phi(\vec{f}_{|V})}} V$$

and:

$$\|\phi(\vec{f})\|^s_U \cap U' = \left(\bigcup_{\substack{W \in \mathcal{O}(U) \\ P(W) \models \phi(\vec{f}_{|W})}} W \right) \cap U' = \bigcup_{\substack{W \in \mathcal{O}(U) \\ P(W) \models \phi(\vec{f}_{|W})}} (W \cap U')$$

We will show that the same sets participate in both unions. We first note that all the sets considered are opens in U', so we only have to show that for any $V \in \mathcal{O}(U')$, we have that $P(V) \models \phi(\vec{f}_{|V})$ iff there is a $W \in \mathcal{O}(U)$ with $V = W \cap U'$ and $P(W) \models \phi(\vec{f}_{|W})$. From left to right, we simply take $W := V$, while from right to left we use the fact that $P(V \subseteq W)$ is a homomorphism and therefore preserves atomic formulas.

Now, if $\phi(\vec{x})$ is of the form $\neg\psi(\vec{x})$, we have that:

$$\begin{aligned}
\|\phi(\vec{f}_{|U'})\|^s_{U'} &= \|\neg\psi(\vec{f}_{|U'})\|^s_{U'} \\
&= int(U' \setminus \|\psi(\vec{f}_{|U'})\|^s_{U'}) \\
&= int(U' \setminus (\|\psi(\vec{f})\|^s_U \cap U')) && \text{(by the induction hypothesis)} \\
&= int(U' \setminus \|\psi(\vec{f})\|^s_U) \\
&= int(U \setminus \|\psi(\vec{f})\|^s_U) \cap U' && \text{(by Lemma 1.2, with } A := U \setminus \|\psi(\vec{f})\|^s_U \text{ and } B := U') \\
&= \|\neg\psi(\vec{f})\|^s_U \cap U' \\
&= \|\phi(\vec{f})\|^s_U \cap U'
\end{aligned}$$

If $\phi(\vec{x})$ is of one of the forms $\psi(\vec{x}) \wedge \chi(\vec{x})$, $\psi(\vec{x}) \vee \chi(\vec{x})$, the induction proceeds straightforwardly.

Finally, if $\phi(\vec{x})$ is of the form $\exists y \psi(\vec{x}, y)$, we have that:

$$\|\phi(\vec{f}_{|U'})\|^s_{U'} = \|\exists y \psi(\vec{f}_{|U'}, y)\|^s_{U'} = \bigcup_{V \in \mathcal{O}(U')} \bigcup_{a \in P(V)} \|\psi(\vec{f}_{|V}, a)\|^s_V$$

and that:

$$\begin{aligned}
\|\phi(\vec{f})\|^s_U \cap U' &= \|\exists y \psi(\vec{f}, y)\|^s_U \cap U' \\
&= \left(\bigcup_{W \in \mathcal{O}(U)} \bigcup_{a \in P(W)} \|\psi(\vec{f}_{|W}, a)\|^s_W \right) \cap U' \\
&= \bigcup_{W \in \mathcal{O}(U)} \bigcup_{a \in P(W)} (\|\psi(\vec{f}_{|W}, a)\|^s_W \cap U') \\
&= \bigcup_{W \in \mathcal{O}(U)} \bigcup_{a \in P(W)} \|\psi(\vec{f}_{|W \cap U'}, a_{|W \cap U'})\|^s_{W \cap U'} && \text{(by the induction hypothesis)}
\end{aligned}$$

We show the two quantities such obtained to be equal, i.e. extensionally equal – that is, we have to prove that for any $x \in U$:

$$\exists V \in \mathcal{O}(U'), a \in P(V) \text{ s.t. } x \in \|\psi(\vec{f}_{|V}, a)\|^s_V \Leftrightarrow \exists W \in \mathcal{O}(U), a \in P(W) \text{ s.t. } x \in \|\psi(\vec{f}_{|W \cap U'}, a_{|W \cap U'})\|^s_{W \cap U'}$$

But then, from left to right we may simply take $W := V$ and $a := a$. From right to left we take $V := W \cap U'$ and $a := a_{|W \cap U'}$ and we are done. $\square$

Definition 3.3. If (X, P), U and ϕ are as in the definition of strong forcing, we define the **weak forcing value** of ϕ by $\|\phi\|^w_U := \|\neg\neg\phi\|^s_U$ (which is also equal, by the second rule above to $\neg\neg\|\phi\|^s_U$, and, by the Heyting algebra structure of $\mathcal{O}(U)$, to the interior of the closure of $\|\phi\|^s_U$).

Not surprisingly, this notion of forcing coincides with the Kripke one (defined earlier), in the following sense.

Theorem 3.4. *If $(I, \leq)$ is a preordered set and K is a Kripke structure on I (or, equivalently, as we have seen, a sheaf on the Alexandrov space corresponding to I), then, for any $i \in I$, any $U \ni i$ and ϕ in the language of $K(U)$:*

$$i \in \|\phi\|^s_U \Leftrightarrow i \Vdash \phi$$

Proof. The only interesting case is the one where ϕ is $\neg\psi$ for some ψ. Then:

$$\begin{aligned} i \Vdash \neg\psi &\Leftrightarrow \forall i' \geq i, i' \nVdash \psi \\ &\Leftrightarrow \forall i' \geq i, i' \notin \|\psi\|^s_U \\ &\Leftrightarrow E_i \cap \|\psi\|^s_U = \emptyset \\ &\Leftrightarrow E_i \subseteq U \setminus \|\psi\|^s_U \\ &\Leftrightarrow E_i \subseteq int(U \setminus \|\psi\|^s_U) \\ &\Leftrightarrow E_i \subseteq \|\neg\psi\|^s_U \\ &\Leftrightarrow i \in \|\neg\psi\|^s_U, \end{aligned}$$

by using the structural induction hypothesis and the properties of the Alexandrov topology of I. □

Corollary 3.5. *If (I, P) is a discrete structured space and so it corresponds to a family $\{A_i\}_{i \in I}$ of structures, then the following equivalence holds:*

$$\|\phi\|^s_U = \{i \in I \mid A_i \models \phi\}$$

Proof. We have seen before that in this case $i \Vdash \phi$ iff $A_i \models \phi$. □

Theorem 3.6. *(Prime Stalk Lemma for Strong Forcing) If (X,P) is a structured space, then for any $U \in \mathcal{O}(X)$ and any ϕ in the language of $P(U) = ((\eta_X)_* P)(D_U)$:*

$$\|\phi\|^s_{D_U} \subseteq D_{\|\phi\|^s_U} \subseteq \neg\neg\|\phi\|^s_{D_U}$$

Proof. We proceed by structural induction.

- if ϕ is an atomic sentence, we will prove a stronger statement, namely that:

  $$\|\phi\|^s_{D_U} = D_{\|\phi\|^s_U}$$

  For any $F \in Spec(X)$, we have that:

  $$\begin{aligned} F \in \|\phi\|^s_{D_U} &\Leftrightarrow \exists V \ni F, ((\eta_X)_* P)(V) \models \phi \\ &\Leftrightarrow \exists W \in \mathcal{O}(X) \text{ with } F \in D_W, ((\eta_X)_* P)(D_W) \models \phi \\ &\Leftrightarrow \exists W \in \mathcal{O}(X) \text{ with } F \in D_W, P(W) \models \phi \\ &\Leftrightarrow \exists W \in F, P(W) \models \phi \\ &\Leftrightarrow \exists W \in F, W \subseteq \|\phi\|^s_U \\ &\Leftrightarrow \|\phi\|^s_U \in F. \end{aligned}$$

  The equality is proved.

- if ϕ is $\neg\psi$, we first derive:

$$\|\phi\|^s_{D_U} = \|\neg\psi\|^s_{D_U} = \neg\|\psi\|^s_{D_U} = \neg\neg\neg\|\psi\|^s_{D_U}$$

  but, since by the induction hypothesis $D_{\|\psi\|^s_U} \subseteq \neg\neg\|\psi\|^s_{D_U}$, we can write further:

$$\neg\neg\neg\|\psi\|^s_{D_U} \subseteq \neg D_{\|\psi\|^s_U} = D_{\|\neg\psi\|^s_U} = D_{\|\phi\|^s_U}$$

  For the second part, consider:

$$D_{\|\phi\|^s_U} = D_{\|\neg\psi\|^s_U} = \neg D_{\|\psi\|^s_U}$$

  and since we have that $\|\psi\|^s_{D_U} \subseteq D_{\|\psi\|^s_U}$, we can write:

$$\neg D_{\|\psi\|^s_U} \subseteq \neg\|\psi\|^s_{D_U} = \neg\neg\neg\|\psi\|^s_{D_U} = \neg\neg\|\neg\psi\|^s_{D_U} = \neg\neg\|\phi\|^s_{D_U}$$

  and we are done.

- if ϕ is $\psi \wedge \chi$, then:

$$\|\phi\|^s_{D_U} = \|\psi \wedge \chi\|^s_{D_U} = \|\psi\|^s_{D_U} \cap \|\chi\|^s_{D_U} \subseteq D_{\|\psi\|^s_U} \cap D_{\|\chi\|^s_U} = D_{\|\psi\wedge\chi\|^s_U} = D_{\|\phi\|^s_U}$$

  and

$$D_{\|\phi\|^s_U} = D_{\|\psi\wedge\chi\|^s_U} = D_{\|\psi\|^s_U} \cap D_{\|\chi\|^s_U} \subseteq \neg\neg\|\psi\|^s_{D_U} \cap \neg\neg\|\chi\|^s_{D_U} \subseteq \neg\neg(\|\psi\|^s_{D_U} \cap \|\chi\|^s_{D_U}) =$$
$$= \neg\neg\|\psi \wedge \chi\|^s_{D_U} = \neg\neg\|\phi\|^s_{D_U},$$

  where at the second inclusion we used Theorem 1.19, (9).

- if ϕ is $\psi \vee \chi$, then:

$$\|\phi\|^s_{D_U} = \|\psi \vee \chi\|^s_{D_U} = \|\psi\|^s_{D_U} \cup \|\chi\|^s_{D_U} \subseteq D_{\|\psi\|^s_U} \cup D_{\|\chi\|^s_U} = D_{\|\psi\vee\chi\|^s_U} = D_{\|\phi\|^s_U}$$

  and

$$D_{\|\phi\|^s_U} = D_{\|\psi\vee\chi\|^s_U} = D_{\|\psi\|^s_U} \cup D_{\|\chi\|^s_U} \subseteq \neg\neg\|\psi\|^s_{D_U} \cup \neg\neg\|\chi\|^s_{D_U} \subseteq$$
$$\subseteq \neg\neg\|\psi \vee \chi\|^s_{D_U} \cup \neg\neg\|\psi \vee \chi\|^s_{D_U} = \neg\neg\|\psi \vee \chi\|^s_{D_U} = \neg\neg\|\phi\|^s_{D_U}$$

- if ϕ is $\exists x\psi(x)$, then:

$$\begin{aligned}
\|\phi\|^s_{D_U} &= \|\exists x\psi(x)\|^s_{D_U} \\
&= \bigcup_{V\in\mathcal{O}(D_U)} \bigcup_{a\in((\eta_X)_*P)(V)} \|\psi(a)\|^s_V \\
&= \bigcup_{W\in\mathcal{O}(U)} \bigcup_{a\in P(W)} \|\psi(a)\|^s_{D_W} \\
&\subseteq \bigcup_{W\in\mathcal{O}(U)} \bigcup_{a\in P(W)} D_{\|\psi(a)\|^s_W} \\
&\subseteq D_{\bigcup_{W\in\mathcal{O}(U)} \bigcup_{a\in P(W)} \|\psi(a)\|^s_W} \\
&= D_{\|\exists x\psi(x)\|^s_U} \\
&= D_{\|\phi\|^s_U}.
\end{aligned}$$

and

$$
\begin{aligned}
D_{\|\phi\|^s_U} =& D_{\|\exists x\psi(x)\|^s_U} \\
=& D_{\bigcup_{W\in\mathcal{O}(U)}\bigcup_{a\in P(W)}\|\psi(a)\|^s_W} \\
\subseteq& \neg\neg \bigcup_{W\in\mathcal{O}(U)} \bigcup_{a\in P(W)} D_{\|\psi(a)\|^s_W} \\
\subseteq& \neg\neg \bigcup_{W\in\mathcal{O}(U)} \bigcup_{a\in P(W)} \neg\neg\|\psi(a)\|^s_{D_W} \\
\subseteq& \neg\neg\neg\neg \bigcup_{W\in\mathcal{O}(U)} \bigcup_{a\in P(W)} \|\psi(a)\|^s_{D_W} \\
=& \neg\neg \bigcup_{W\in\mathcal{O}(U)} \bigcup_{a\in P(W)} \|\psi(a)\|^s_{D_W} \\
=& \neg\neg \bigcup_{V\in\mathcal{O}(D_U)} \bigcup_{a\in((\eta_X)_*P)(V)} \|\psi(a)\|^s_V \\
=& \neg\neg\|\exists x\psi(x)\|^s_{D_U} \\
=& \neg\neg\|\phi\|^s_{D_U},
\end{aligned}
$$

where at the first (crucial!) inclusion we used Remark 1.42, at the second one we used the induction hypothesis and at the third one we used the same trick we used in the case $\phi = \psi \vee \chi$.

The proof is finished. □

Corollary 3.7. *(Prime Stalk Lemma for Weak Forcing) If (X,P) is a structured space, then for any $U \in \mathcal{O}(X)$ and any ϕ in the language of $P(U) = ((\eta_X)_*P)(D_U)$:*

$$\|\phi\|^w_{D_U} = D_{\|\phi\|^w_U}$$

Proof. We have that:

$$\|\phi\|^w_{D_U} = \|\neg\neg\phi\|^s_{D_U} \subseteq D_{\|\neg\neg\phi\|^s_U} = D_{\|\phi\|^w_U}$$

and that:

$$D_{\|\phi\|^w_U} = D_{\|\neg\neg\phi\|^s_U} \subseteq \neg\neg\|\neg\neg\phi\|^s_{D_U} = \neg\neg\neg\neg\|\phi\|^s_U = \neg\neg\|\phi\|^s_U = \|\neg\neg\phi\|^s_{D_U} = \|\phi\|^w_{D_U}$$

□

From now on, "Prime Stalk Lemma" will refer simply to the above corollary.

Theorem 3.8. *(Genericity of maximal stalks) If (X,P) is a structured space, then for any maximal filter M in $\mathcal{O}(X)$, any $U \in M$ and any ϕ in the language of $P(U) = ((\eta_X)_*P)(D_U)$:*

$$((\eta_X)_*P)_M \models \phi \Leftrightarrow M \in \|\phi\|^w_{D_U}$$

Proof. We proceed by structural induction.

If ϕ is atomic, then:

$$\begin{aligned}
((\eta_X)_*P)_M \models \phi &\Leftrightarrow \exists V \in M, ((\eta_X)_*P)(D_V) \models \phi \\
&\Leftrightarrow \exists V \in M, P(V) \models \phi \\
&\Leftrightarrow \exists V \in M, V \subseteq \{x \in U \mid P_x \models \phi\} \\
&\Leftrightarrow \{x \in U \mid P_x \models \phi\} \in M \\
&\Leftrightarrow \|\phi\|_U^s \in M \\
&\Leftrightarrow \|\phi\|_U^w \in M && \text{(by Corollary 1.38)} \\
&\Leftrightarrow M \in D_{\|\phi\|_U^w} \\
&\Leftrightarrow M \in \|\phi\|_{D_U}^w && \text{(by the Prime Stalk Lemma)}
\end{aligned}$$

Now, if ϕ is $\neg\psi$, then:

$$\begin{aligned}
((\eta_X)_*P)_M \models \phi &\Leftrightarrow ((\eta_X)_*P)_M \models \neg\psi \\
&\Leftrightarrow ((\eta_X)_*P)_M \not\models \psi \\
&\Leftrightarrow M \notin \|\psi\|_{D_U}^w && \text{(by the induction hypothesis)} \\
&\Leftrightarrow M \notin D_{\|\psi\|_U^w} && \text{(by the Prime Stalk Lemma)} \\
&\Leftrightarrow \|\psi\|_U^w \notin M \\
&\Leftrightarrow \|\psi\|_U^s \notin M && \text{(by Corollary 1.38)} \\
&\Leftrightarrow \|\neg\psi\|_U^s \in M && \text{(by Remark 1.37)} \\
&\Leftrightarrow \|\neg\psi\|_U^w \in M && \text{(by Corollary 1.38)} \\
&\Leftrightarrow M \in D_{\|\neg\psi\|_U^w} \\
&\Leftrightarrow M \in \|\neg\psi\|_{D_U}^w && \text{(by the Prime Stalk Lemma)} \\
&\Leftrightarrow M \in \|\phi\|_{D_U}^w.
\end{aligned}$$

If ϕ is $\psi \wedge \chi$ or $\psi \vee \chi$, the proof proceeds roughly among the same lines as for $\neg\psi$.

Finally, if ϕ is $\exists x\psi(x)$, letting M_U be $M \cap \mathcal{O}(U)$:

$$\begin{aligned}
((\eta_X)_*P)_M \models \phi &\Leftrightarrow ((\eta_X)_*P)_M \models \exists x\psi(x) \\
&\Leftrightarrow \exists f \in ((\eta_X)_*P)_M, ((\eta_X)_*P)_M \models \psi(f) \\
&\Leftrightarrow \exists V \ni M, \exists f \in ((\eta_X)_*P)(V), ((\eta_X)_*P)_M \models \psi(f) \\
&\Leftrightarrow \exists U_3 \in M, \exists f \in ((\eta_X)_*P)(D_{U_3}), ((\eta_X)_*P)_M \models \psi(f) \\
&\Leftrightarrow \exists U_2 \in M_U, \exists f \in ((\eta_X)_*P)(D_{U_2}), ((\eta_X)_*P)_M \models \psi(f) && \text{(take } U_2 := U_3 \cap U\text{)} \\
&\Leftrightarrow \exists U_2 \in M_U, \exists f \in P(U_2), ((\eta_X)_*P)_M \models \psi(f) \\
&\Leftrightarrow \exists U_2 \in M_U, \exists f \in P(U_2), M \in \|\psi(f)\|_{D_{U_2}}^w && \text{(by the induction hypothesis)} \\
&\Leftrightarrow \exists U_2 \in M_U, \exists f \in P(U_2), \|\psi(f)\|_{U_2}^w \in M \\
&\Leftrightarrow \exists U_2 \in M_U, \exists f \in P(U_2), \|\psi(f)\|_{U_2}^s \in M && \text{(by Corollary 1.38)} \\
&\Leftrightarrow \bigcup_{V \in \mathcal{O}(U)} \bigcup_{a \in P(V)} \|\psi(a)\|_V^s \in M && \text{(tricky, see below)} \\
&\Leftrightarrow \|\exists x\psi(x)\|_U^s \in M \\
&\Leftrightarrow \|\exists x\psi(x)\|_U^w \in M && \text{(by Corollary 1.38)} \\
&\Leftrightarrow M \in D_{\|\exists x\psi(x)\|_U^w} \\
&\Leftrightarrow M \in \|\exists x\psi(x)\|_{D_U}^w && \text{(by the Prime Stalk Lemma)} \\
&\Leftrightarrow M \in \|\phi\|_{D_U}^w.
\end{aligned}$$

It remains to prove the "tricky" equivalence signalled above. Left to right is easy (take $V := U_2$ and $a := f$), so we will focus on right to left. Suppose that

$$\bigcup_{V \in \mathcal{O}(U)} \bigcup_{a \in P(V)} \|\psi(a)\|_V^s \in M$$

and denote the above union of opens by U_1. Take $\mathcal{A}$ to be the subset of $\mathcal{P}(\mathcal{O}(U_1))$ containing all X such that any two opens in X are disjoint and for any $W \in X$ there is some $V \in \mathcal{O}(U)$ and $a \in P(V)$ such that $W \subseteq \|\psi(a)\|_V^s$. Now, $\mathcal{A}$ is non-empty and closed under unions of chains, so it is well-ordered by inclusion. By Zorn's lemma, it contains a maximal element B. Take U_2 to be the union of all opens in B, and since all of them are in $\mathcal{O}(U_1)$, we have that $U_2 \subseteq U_1$. We will now prove that U_2 is a dense subset of U_1. Suppose that, on the contrary, there is some $x \in U_1$ and $Z \in \mathcal{O}(U_1)$ containing x which is disjoint from U_2. Then, for every $W \in B$, $Z \cap W = \emptyset$. Remember, though, that $x \in U_1$, so there is some suitable V and a such that $x \in \|\psi(a)\|_V^s$. Taking Z' to be the set $Z \cap \|\psi(a)\|_V^s$, we have that $B \cup \{Z'\} \in \mathcal{A}$, contradicting the maximality of B. So $U_1 \subseteq \neg\neg U_2$ and by applying Corollary 1.38, $U_2 \in M$. Also, since $U_1 \subseteq U$ and $U_2 \subseteq U_1$, $U_2 \in \mathcal{O}(U)$, so $U_2 \in M_U$.

Choose now for any $W \in B$ an open $V \in \mathcal{O}(U)$ and an $a \in P(U)$ such that $W \subseteq \|\psi(a)\|_V^s$. Denote $a_{|W}$ by a_W. Since the opens in B form a covering of U_2, applying the first property of the sheaf (since the intersections are empty, the restrictions coincide by Remark 2.14) we have that there is a $f \in P(U_2)$ such that for any $W \in B$, $f_{|W} = a_W$.

Let now x be an element of U_2. Then there is a $W \in B$ s.t. $x \in W$. Then $x \in \|\psi(a_W)\|_W^s$. We apply Lemma 3.2 with $U := U_2$, $U' := W$, $n := 1$, $\vec{f} := f$, $\phi(\vec{x}) := \psi(x)$ (and therefore $\vec{f}_{|U'}$ will be a_W), and we obtain that $\|\psi(a_W)\|_W^s \subseteq \|\psi(f)\|_{U_2}^s$, so $x \in \|\psi(f)\|_{U_2}^s$. Since x was chosen arbitrarily, we have that $U_2 \subseteq \|\psi(f)\|_{U_2}^s \in M$. The proof is now finished. □

Corollary 3.9. *(Maximal Stalk Theorem) If (X,P) is a structured space, then for any $U \in \mathcal{O}(X)$, any ϕ in the language of $P(U) = ((\eta_X)_* P)(D_U)$ and any maximal filter $M \in D_U$:*

$$((\eta_X)_* P)_M \models \phi \Leftrightarrow \|\phi\|_U^w \in M \Leftrightarrow \|\phi\|_U^s \in M$$

Proof. The first equivalence follows from Theorem 3.8 and the Prime Stalk Lemma. The last equivalence follows from applying Corollary 1.38 for $\|\phi\|_U^s$ and M. □

Corollary 3.10. *(Łoś's Fundamental Theorem of Ultraproducts) If $\{A_i\}_{i \in I}$ is a family of structures, F is an ultrafilter in $\mathcal{P}(I)$ and ϕ is a sentence of Σ (i.e., without constants), then:*

$$\prod_F A_i \models \phi \Leftrightarrow \{i \in I \mid A_i \models \phi\} \in F$$

Proof. It follows from Corollary 3.5, the Maximal Stalk Theorem (Corollary 3.9) and the discussion at the end of the previous section. □

4 References

[1] D. Ellerman. Sheaves of relational structures and ultraproducts. Boston University Math. Dept. Research Report 71-19, 1971.

[2] D. Ellerman. Sheaves of structures and generalized ultraproducts. *Annals of Math. Logic*, 7:163–195, 1974.