Products of hyperbolic spaces

Andrei Sipoș
(joint work with Pedro Pinto)

Universitatea din București
Institute of Mathematics of the Romanian Academy
Institute for Logic and Data Science

October 9, 2024
Colloquium Logicum 2024
Wien, Österreich

The p-norms

L^p spaces are a representative example of Banach spaces which are both uniformly convex and uniformly smooth. In particular, for every $p > 1$ and $n \in \mathbb{N}^*$, one can define the corresponding p-norm on the n-dimensional Euclidean space, $\| \cdot \|_p : \mathbb{R}^n \to \mathbb{R}_+$, making it into a normed space which we shall denote by $\mathbb{R}^n_p$, by putting, for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\|(x_1, \ldots, x_n)\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

In particular, we shall denote $\| \cdot \|_2$ simply by $\| \cdot \|$.

Motivation: the work of Pinto

Recently, Pedro Pinto introduced the class of uniformly smooth hyperbolic spaces as a nonlinear generalization of uniformly smooth Banach spaces, similarly to how Leuștean (2007, 2010) had nonlinearly generalized uniformly convex Banach spaces in the form of UCW-hyperbolic spaces.

Theorem (Pinto 2024)

If X is a bounded complete uniformly smooth UCW-hyperbolic space, then Reich's theorem holds.

Question: Is there any example of such a space which is neither a CAT(0) space, nor a convex subset of a normed space?
This is what I will answer today.

Hyperbolic spaces

Let's recap the following definition, due to Kohlenbach (2005).

Definition

A W-**hyperbolic space** is a triple (X, d, W) where (X, d) is a metric space and $W : X^2 \times [0, 1] \to X$ such that, for all x, y, z, $w \in X$ and λ, $\mu \in [0, 1]$, we have that

1. $d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)$;
2. $d(W(x, y, \lambda), W(x, y, \mu)) = |\lambda - \mu| d(x, y)$;
3. $W(x, y, \lambda) = W(y, x, 1 - \lambda)$;
4. $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)$.

Clearly, any normed space may be made into a W-hyperbolic space in a canonical way.

One generally denotes $W(x, y, \lambda)$ by $(1 - \lambda)x + \lambda y$.

CAT(0) spaces

CAT(0) spaces, which are the nonlinear generalization of Hilbert spaces, are a particular case of W-hyperbolic spaces. If X is a CAT(0) space, then, for any x, y, $z \in X$ and $\lambda \in [0,1]$ we have that

$$d^2(z,(1-\lambda)x+\lambda y) \leq (1-\lambda)d^2(z,x)+\lambda d^2(z,y)-\lambda(1-\lambda)d^2(x,y).$$

A representative (nonlinear, as we shall see) example of a (complete) CAT(0) space is the *(hyperbolic) Poincaré upper half-plane model*, having as the underlying set

$$\mathbb{H} := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 > 0\},$$

where, given the function $\operatorname{arcosh} : [1,\infty) \to [0,\infty)$, where for every $t \in [1,\infty)$, $\operatorname{arcosh} t = \ln(t+\sqrt{t^2-1})$, the distance function is defined as follows: for any $x=(x_1,x_2)$, $y=(y_1,y_2) \in \mathbb{H}$, one sets

$$d(x,y) := \operatorname{arcosh}\left(1+\frac{(y_1-x_1)^2+(y_2-x_2)^2}{2x_2y_2}\right).$$

The Poincaré half-plane continued

One may prove that geodesic lines of this space are of two types: for every $a \in \mathbb{R}$ and $r > 0$, one has the semicircle

$$\mathcal{C}_{a,r} = \{(x_1, x_2) \in \mathbb{H} \mid (x_1 - a)^2 + x_2^2 = r^2\},$$

while, for every $a \in \mathbb{R}$, one has the ray

$$\mathcal{R}_a = \{(x_1, x_2) \in \mathbb{H} \mid x_1 = a\}.$$

It may be then easily shown that for every two points there is exactly one geodesic segment that joins them. The general formula for the convex combination of two points is somewhat involved and we shall omit it, instead giving only the specialized formula for the midpoint, which shall be used later. For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{H}$, we have that

$$W\left(x, y, \frac{1}{2}\right) = \left(\frac{x_1 y_2 + x_2 y_1}{x_2 + y_2}, \frac{\sqrt{x_2 y_2} \cdot \sqrt{(x_2 + y_2)^2 + (x_1 - y_1)^2}}{x_2 + y_2}\right).$$

Smoothness in Banach spaces

We will now explore the class of uniformly smooth Banach spaces, in order to better understand the way it was generalized by Pinto to uniformly smooth hyperbolic spaces.

A Banach space X is called *smooth* if, for any $x \in X$ with $\|x\| = 1$, we have that, for any $y \in X$ with $\|y\| = 1$, the limit

$$\lim_{h \to 0} \frac{\|x + hy\| - \|x\|}{h}$$

exists, and *uniformly smooth* if the limit is attained uniformly in x and y.

Duality mappings and smoothness

Definition

Let X be a Banach space. We define the **normalized duality mapping of** X to be the map $J : X \to 2^{X^*}$, defined, for all $x \in X$, by

$$J(x) := \{x^* \in X^* \mid x^*(x) = \|x\|^2,\ \|x^*\| = \|x\|\}.$$

The condition of (not necessarily uniform) smoothness has been proven to be equivalent to the fact that the normalized duality mapping of the space, $J : X \to 2^{X^*}$, is single-valued – and we shall denote its unique section by $j : X \to X^*$.

Hilbert spaces are smooth, and in that case $j(x)(y)$ is then simply $\langle y, x\rangle$, for any x, y in the space. Because of this, we may consider the j to be a generalized variant of the inner product, sharing some of its nice properties.

Hyperbolic smoothness

This view of j as a generalization of the inner product led Pinto to the following definition.

Definition (Pinto 2024)

A **smooth hyperbolic space** is a quadruple (X, d, W, π), where (X, d, W) is a W-hyperbolic space and $\pi : X^2 \times X^2 \to \mathbb{R}$, such that, for any x, y, u, $v \in X$ (where an ordered pair of points $(a, b) \in X^2$ is denoted by $\overrightarrow{ab}$):

1. $\pi(\overrightarrow{xy}, \overrightarrow{xy}) = d^2(x, y)$;
2. $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = -\pi(\overrightarrow{yx}, \overrightarrow{uv}) = -\pi(\overrightarrow{xy}, \overrightarrow{vu})$;
3. $\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{yz}, \overrightarrow{uv}) = \pi(\overrightarrow{xz}, \overrightarrow{uv})$;
4. $\pi(\overrightarrow{xy}, \overrightarrow{uv}) \leq d(x, y)d(u, v)$;
5. $d^2(W(x, y, \lambda), z) \leq (1 - \lambda)^2 d^2(x, z) + 2\lambda\pi(\overrightarrow{yz}, \overrightarrow{W(x, y, \lambda)z})$.

Hyperbolic uniform smoothness

It is a classical result that, in uniformly smooth Banach spaces, the duality mapping is norm-to-norm uniformly continuous on bounded subsets. Bénilan proved that the norm-to-norm uniform continuity on bounded subsets of an arbitrary duality selection mapping is in fact equivalent to uniform smoothness.

Motivated by this fact, Pinto defined a *uniformly smooth hyperbolic space* to be a smooth hyperbolic space (X, d, W, π) such that there is an $\omega : (0, \infty) \times (0, \infty) \to (0, \infty)$, called a *modulus of uniform continuity for* π, having the property that, for any r, $\varepsilon > 0$ and a, u, v, x, $y \in X$ with $d(u, a) \leq r$, $d(v, a) \leq r$ and $d(u, v) \leq \omega(r, \varepsilon)$, one has that

$$|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})| \leq \varepsilon \cdot d(x, y).$$

Hyperbolic uniform convexity

As I said, the other relevant class of Banach spaces is that of uniformly convex Banach spaces. We shall now turn to exploring uniform convexity directly in the hyperbolic setting, as first introduced by Leuștean.

Definition

If (X, d, W) is a W-hyperbolic space, then a **modulus of uniform convexity** for (X, d, W) is a function $\eta : (0, \infty) \times (0, \infty) \to (0, 1]$ such that, for any r, $\varepsilon > 0$ and any a, x, $y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$, $d(x, y) \geq \varepsilon r$, we have that

$$d\left(\frac{x+y}{2}, a\right) \leq (1 - \eta(r, \varepsilon))r.$$

We call the modulus **monotone** if, for any r, s, $\varepsilon > 0$ with $s \leq r$, we have $\eta(r, \varepsilon) \leq \eta(s, \varepsilon)$.
A UCW-**hyperbolic space** is a W-hyperbolic space that admits a monotone modulus of uniform convexity.

Property (G)

The crucial property of uniform convexity that was used by Kohlenbach and the speaker was the following, which we now reify for the first time.

Definition

Let $\psi : (0, \infty) \times (0, \infty) \to (0, \infty)$. We say that a W-hyperbolic space (X, d, W) **has property** (G) **with modulus** ψ if, for any r, $\varepsilon > 0$ and any a, x, $y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$, $d(x, y) \geq \varepsilon$, we have that

$$d^2\left(\frac{x+y}{2}, a\right) \leq \frac{1}{2}d^2(x, a) + \frac{1}{2}d^2(y, a) - \psi(r, \varepsilon).$$

We have shown that uniformly convex Banach spaces have property (G) with a modulus which is easily computable in terms of the modulus of uniform convexity (the qualitative result essentially goes back to Zălinescu). Also, CAT(0) may be easily shown to have this property.

Property (M)

We do not know (and we leave it as an open problem) whether, generally, UCW-hyperbolic spaces have property (G); fortunately, Pinto identified the following weaker property of them (which, again, we now reify for the first time) as being enough for the proof to go through.

Definition

Let $\psi : (0, \infty) \times (0, \infty) \to (0, \infty)$. We say that a W-hyperbolic space (X, d, W) **has property** (M) **with modulus** ψ if, for any r, $\varepsilon > 0$ and any a, x, $y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$, $d(x, y) \geq \varepsilon$, we have that

$$d^2\left(\frac{x+y}{2}, a\right) \leq \max(d^2(x, a), d^2(y, a)) - \psi(r, \varepsilon).$$

The main results of Pinto may be expressed by saying that Reich's theorem and its consequences hold for uniformly smooth hyperbolic spaces having property (M).

Products of hyperbolic spaces

Let $n \in \mathbb{N}^*$ and fix n metric spaces $(X_1, d_1), \ldots, (X_n, d_n)$. Put $X := \prod_{i=1}^n X_i$ and define $d : X \times X \to \mathbb{R}$, by putting, for any $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$,

$$d(x, y) := \left(\sum_{i=1}^n d_i^2(x_i, y_i) \right)^{\frac{1}{2}}.$$

It is a classical result that (X, d) is also a metric space.

Now fix $W_1, \ldots, W_n$ such that, for each i, (X_i, d_i, W_i) is a W-hyperbolic space. Define $W : X^2 \times [0, 1] \to X$ by putting, for any $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$ and $\lambda \in [0, 1]$, and for any i,

$$W(x, y, \lambda)_i := W_i(x_i, y_i, \lambda).$$

Proposition

(X, d, W) is a W-hyperbolic space.

Products and property (G)

Proposition

Assume that, for each i, (X_i, d_i, W_i) has property (G). Then (X, d, W) has property (G).

Proof

For each i, let ψ_i be a modulus for the property (G) of (X_i, d_i, W_i). Define $\psi : (0, \infty) \times (0, \infty) \to (0, \infty)$ by putting, for any r, $\varepsilon > 0$, $\psi(r, \varepsilon) := \min_i \ \psi_i\left(r, \frac{\varepsilon}{\sqrt{n}}\right)$. We will show that (X, d, W) has property (G) with modulus ψ.
Let r, $\varepsilon > 0$ and a, x, $y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$, $d(x, y) \geq \varepsilon$. Assume that, for every i, $d_i(x_i, y_i) < \varepsilon/\sqrt{n}$. Then

$$\varepsilon^2 \leq d^2(x, y) = \sum_{i=1}^{n} d_i^2(x_i, y_i) < n \cdot \frac{\varepsilon^2}{n} = \varepsilon^2,$$

a contradiction. Thus, there is a $j \in \{1, \ldots, n\}$ such that $d_j(x_j, y_j) \geq \varepsilon/\sqrt{n}$.

Products and property (G)

Proof (cont'd)

We have, using a basic property for the first inequality, that

$$\begin{aligned}
&\frac{1}{2}d^2(x,a) + \frac{1}{2}d^2(y,a) - d^2\left(\frac{x+y}{2}, a\right) \\
&\quad = \sum_{i=1}^{n}\left(\frac{d_i^2(x_i,a_i) + d_i^2(y_i,a_i)}{2} - d_i^2\left(\frac{x_i+y_i}{2}, a_i\right)\right) \\
&\quad \geq \frac{d_j^2(x_j,a_j) + d_j^2(y_j,a_j)}{2} - d_j^2\left(\frac{x_j+y_j}{2}, a_j\right) \\
&\quad \geq \psi_j\left(r, \frac{\varepsilon}{\sqrt{n}}\right) \\
&\quad \geq \psi(r,\varepsilon).
\end{aligned}$$

Products and convexity

Proposition

Assume that, for each i, (X_i, d_i, W_i) is a UCW-hyperbolic space. Then (X, d, W) is a UCW-hyperbolic space.

Proof

For each i, let η_i be a monotone modulus of uniform convexity for (X_i, d_i, W_i). Define η, $\check{\eta} : (0, \infty) \times (0, 2] \to (0, \infty)$ by putting, for any $r > 0$ and any $\varepsilon \in (0, 2]$,

$$\eta(r, \varepsilon) := \min \left(\{\eta_i(r, \varepsilon) \mid i \in \{1, \ldots, n\}\} \cup \left\{ \frac{1}{2} \right\} \right)$$

$$\check{\eta}(r, \varepsilon) := \min \left(\frac{\varepsilon^4}{4608 n^4} \cdot \eta^2 \left(r, \frac{\varepsilon}{\sqrt{n}} \right), \frac{\varepsilon^2}{16n} \cdot \eta \left(r, \frac{\varepsilon}{\sqrt{n}} \right) \right).$$

We showed that $\check{\eta}$ is a monotone modulus of uniform convexity for (X, d, W).

Products and property (M)

Proposition

Assume that, for each i, (X_i, d_i, W_i) has property (M). Then (X, d, W) has property (M).

Proof

For each i, let ψ_i be a modulus for the property (M) of (X_i, d_i, W_i). Define ψ, $\check{\psi} : (0,\infty) \times (0,\infty) \to (0,\infty)$ by putting, for any r, $\varepsilon > 0$,

$$\psi(r,\varepsilon) := \min_i \ \psi_i(r,\varepsilon)$$

and

$$\check{\psi}(r,\varepsilon) := \min\left(\frac{\psi^2(r,\varepsilon/\sqrt{n})}{64n^2r^2}, \frac{\psi(r,\varepsilon/\sqrt{n})}{2}\right).$$

We showed that (X, d, W) has property (M) with modulus $\check{\psi}$.

Products and smoothness

Now fix $\pi_1, \ldots, \pi_n$ such that, for each i, (X_i, d_i, W_i, π_i) is a smooth hyperbolic space. Define $\pi : X^2 \times X^2 \to \mathbb{R}$ by putting, for any $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in X$,

$$\pi(\overrightarrow{xy}, \overrightarrow{uv}) := \sum_{i=1}^{n} \pi_i(\overrightarrow{x_i y_i}, \overrightarrow{u_i v_i}).$$

Proposition

(X, d, W, π) is a smooth hyperbolic space.

Products and uniform smoothness

Proposition

Assume that, for each i, (X_i, d_i, W_i, π_i) is a uniformly smooth hyperbolic space. Then (X, d, W, π) is a uniformly smooth hyperbolic space.

Proof

For each i, let ω_i be a modulus of uniform continuity for π_i. Define $\omega : (0, \infty) \times (0, \infty) \to (0, \infty)$ by putting, for any r, $\varepsilon > 0$, $\omega(r, \varepsilon) := \min_i \ \omega_i \left(r, \frac{\varepsilon}{\sqrt{n}}\right)$. We will show that ω is a modulus of uniform continuity for π.

Let r, $\varepsilon > 0$ and a, u, v, x, $y \in X$ with $d(u, a) \leq r$, $d(v, a) \leq r$ and $d(u, v) \leq \omega(r, \varepsilon)$. We want to show that

$$|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})| \leq \varepsilon \cdot d(x, y).$$

It is immediate that, for each i, $d_i(u_i, v_i) \leq \omega(r, \varepsilon) \leq \omega_i \left(r, \frac{\varepsilon}{\sqrt{n}}\right)$, and, thus, $|\pi_i(\overrightarrow{x_i y_i}, \overrightarrow{u_i a_i}) - \pi_i(\overrightarrow{x_i y_i}, \overrightarrow{v_i a_i})| \leq \frac{\varepsilon}{\sqrt{n}} \cdot d_i(x_i, y_i)$, so

Products and uniform smoothness

Proof (cont'd)

$$
\begin{aligned}
&|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})| \\
&\quad = \left| \sum_{i=1}^{n} \left(\pi_i(\overrightarrow{x_i y_i}, \overrightarrow{u_i a_i}) - \pi_i(\overrightarrow{x_i y_i}, \overrightarrow{v_i a_i})\right) \right| \\
&\quad \leq \sum_{i=1}^{n} \left|\pi_i(\overrightarrow{x_i y_i}, \overrightarrow{u_i a_i}) - \pi_i(\overrightarrow{x_i y_i}, \overrightarrow{v_i a_i})\right| \\
&\quad \leq \frac{\varepsilon}{\sqrt{n}} \sum_{i=1}^{n} d_i(x_i, y_i) \\
&\quad = \frac{\varepsilon}{\sqrt{n}} \cdot \langle (1, \ldots, 1), (d_1(x_1, y_1), \ldots, d_n(x_n, y_n)) \rangle \\
&\quad \leq \frac{\varepsilon}{\sqrt{n}} \cdot \|(1, \ldots, 1)\| \cdot \|(d_1(x_1, y_1), \ldots, d_n(x_n, y_n))\| \\
&\quad = \frac{\varepsilon}{\sqrt{n}} \cdot \sqrt{n} \cdot \left(\sum_{i=1}^{n} d_i^2(x_i, y_i) \right)^{\frac{1}{2}} = \varepsilon \cdot d(x, y).
\end{aligned}
$$

The example

From the above results, it follows that $\mathbb{H} \times \mathbb{R}^2_3$ is a uniformly smooth *UCW*-hyperbolic space and, thus, Reich's theorem and its consequences hold for it. This space:

- is not a CAT(0) space
  - because $\mathbb{R}^2_3$ can be seen as a convex subset of $\mathbb{H} \times \mathbb{R}^2_3$ and it is not a CAT(0) space
    - because a Banach space is a CAT(0) space iff it is Hilbert
- is not a convex subset of a normed space (with the canonical convexity structure)
  - because $\mathbb{H}$ can be seen as a convex subset of $\mathbb{H} \times \mathbb{R}^2_3$ and it is not a convex subset of a normed space
    - this is what remains to be shown! – on the next slide

The last proof

Proof

If it were a convex subset of a normed space, for any x, y, $z \in X$ and any $\lambda \in [0,1]$, $d(W(x,z,\lambda), W(y,z,\lambda)) = (1-\lambda)d(x,y)$. We will now exhibit x, y, $z \in \mathbb{H}$ and $\lambda \in [0,1]$ such that $d(W(x,z,\lambda), W(y,z,\lambda)) < (1-\lambda)d(x,y)$. Take $x := (0,1)$, $y := (1,1)$, $z := (0,2)$, $\lambda := 1/2$. Then $W(x,z,\lambda) = (0,\sqrt{2})$ and $W(y,z,\lambda) = (2/3, 2\sqrt{5}/3)$, so

$$d(x,y) = \text{arcosh}(3/2) = \ln((3+\sqrt{5})/2)$$

and

$$d(W(x,z,\lambda), W(y,z,\lambda)) = \text{arcosh}(7/(2\sqrt{10})) = \ln(\sqrt{10}/2).$$

We have to show that $\ln(\sqrt{10}/2) < \ln((3+\sqrt{5})/2)/2$, but this follows from the immediate strict inequality

$$(\sqrt{10}/2)^2 < (3+\sqrt{5})/2.$$

These results may all be found in:

P. Pinto, A. Sipoș, Products of hyperbolic spaces.
arXiv:2408.14093 [math.MG], 2024.

Thank you for your attention.