The computational content of super strongly nonexpansive mappings

(and uniformly monotone operators)

Andrei Sipoș

Research Center for Logic, Optimization and Security, University of Bucharest
Institute of Mathematics of the Romanian Academy
Institute for Logic and Data Science

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Fixed points

Let X be a Hilbert space. A standard problem in nonlinear analysis is determining ways to find fixed points of mappings $T : X \to X$.

For example, if T is a contraction – i.e. there is a $k \in (0, 1)$ such that, for all x, $y \in X$, $\|Tx - Ty\| \leq k\|x - y\|$ – the classical theorem of Banach from 1922 states that, for any $x \in X$, the **Picard iteration** $(T^n x)_{n\in\mathbb{N}}$ converges to a fixed point of T (the unique such one, in fact).

For more general classes of mappings, like **nonexpansive mappings** – for all x, $y \in X$, $\|Tx - Ty\| \leq \|x - y\|$ – this result does not hold anymore. In some cases, we may hope for weaker variants of convergence, like *weak convergence* itself...

Convex sets

If C is a convex, closed, nonempty subset of X, we denote by $P_C : X \to X$ the projection onto C (which is nonexpansive), and we see that C is the set of fixed points of P_C.

Assume we have n such sets $C_1, \ldots, C_n$ with nonempty intersection in which we want to find a point: this is called a *(consistent) convex feasibility problem*. Let $T := P_{C_n} \circ \ldots \circ P_{C_1}$. Then, a result of Bregman (1965) states that $(T^n x)_{n \in \mathbb{N}}$ converges weakly to such a point.

What if the intersection is empty (*inconsistent feasibility*), so we don't have anything to converge to? Then we have to go to something even weaker, called *asymptotic regularity*...

Asymptotic regularity

The property of T being asymptotically regular (also used in proofs of convergence) was defined by Browder and Petryshyn (1966), and says that, for all $x \in X$, $\lim_{n\to\infty} \|T^n x - TT^n x\| = 0$ (intuition: close to a fixed point vs. close to *being* a fixed point). It is clear that such a mapping has the **approximate fixed point (afp) property**, i.e. for all $\delta > 0$ there is an x such that $\|x - Tx\| \leq \delta$.

Bauschke, Borwein and Lewis conjectured in 1995 that asymptotic regularity still holds when the intersection is empty. This was proven by Bauschke (Proc. AMS, 2003).

Bauschke/Martín-Márquez/Moffat/Wang later generalized this result from projections to **firmly nonexpansive mappings** (assuming they have the afp property). So, how are these kinds of results proven?

Strong nonexpansiveness

The proofs are quite involved, so let us first focus on the simplest part, which uses the concept of strong nonexpansiveness. A **strongly nonexpansive (SNE)** mapping is a nonexpansive map U such that for all sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ such that $(x_n - y_n)_{n\in\mathbb{N}}$ is bounded and $\|x_n - y_n\| - \|Ux_n - Uy_n\| \to 0$, we have that $(x_n - y_n) - (Ux_n - Uy_n) \to 0$.

SNE mappings are closed under composition and include firmly nonexpansive mappings, so the T in the theorem is SNE.

Bruck and Reich have shown in 1977 (in the same paper where they introduced SNE mappings) that a SNE mapping which is afp is asymptotically regular, so afp is all one has to show. We now have afp both in the hypothesis and in the conclusion, so one might think an induction approach may work, but it doesn't, since firmly nonexpansive mappings are not closed under composition. We hold on to the idea, though.

Proof mining

This is, however, a talk on *proof mining*[1], so we may ask what sort of extra information we could want out of these asymptotic regularity proofs. The answer is a *rate of asymptotic regularity*, which is the Φ in the following:

$$\forall \varepsilon > 0 \, \exists N \leq \Phi(\varepsilon) \, \forall n \geq N \, \|T^n x - T^{n+1} x\| \leq \varepsilon.$$

Since, by the nonexpansiveness of T, the sequence $(\|T^n x - T^{n+1} x\|)_{n \in \mathbb{N}}$ is nonincreasing, the above is equivalent to

$$\forall \varepsilon > 0 \, \exists N \leq \Phi(\varepsilon) \, \|T^N x - T^{N+1} x\| \leq \varepsilon,$$

which arises from a sentence which is (in a sense) Π_2, and, thus, the goal to extract such a Φ is tractable by the general logical metatheorems of proof mining.

[1]The research program, originating with G. Kreisel in the 1950s and given maturity by the school of U. Kohlenbach in the 1990s, which aims to use proof-theoretical tools in order to obtain additional (usually quantitative) information out of proofs in mainstream mathematics.

SNE moduli

A proof mining analysis of SNE mappings was first given by Kohlenbach (Israel J. Math., 2016). He showed that a mapping U is SNE iff there is an $\omega : (0, \infty)^2 \to (0, \infty)$ (called an **SNE-modulus**), such that for any b, $\varepsilon > 0$ and all x, $y \in X$ with $\|x - y\| \leq b$ and $\|x - y\| - \|Ux - Uy\| < \omega(b, \varepsilon)$, one has that $\|(x - y) - (Ux - Uy)\| < \varepsilon$.

Notice that we don't have an absolute value in the ω condition, which allows us to not assume nonexpansiveness for U (Kohlenbach's trick).

Kohlenbach has obtained such moduli for firmly nonexpansive mappings, and for compositions of SNE mappings for which moduli are known.

Moduli for afp and the extraction

In order to have a quantitative version of the Bruck/Reich result (afp and SNE imply asymptotic regularity), one needs only to define a modulus for the afp property: an α such that for all $\delta > 0$ there is an x with $\|x\| \leq \alpha(\delta)$ such that $\|x - Tx\| \leq \delta$. Afterwards, one only needs to obtain this α for the T in the theorem(s).

This was done by Kohlenbach (FoCM, 2019). We will not go into the details (because those will be different in our case), but we mention that in the projection case the final rate of asymptotic regularity is polynomial of degree eight (an instance of the *proof-theoretic tameness* phenomenon).

Averaged mappings

A further generalization is given by the class of **averaged mappings**. For an $\alpha \in (0,1)$, a mapping U is α-averaged if U is of the form $(1-\alpha)\mathrm{id}_X + \alpha T$, where T is nonexpansive.

Averaged mappings are still SNE, and they are themselves closed under composition. Moreover, $\frac{1}{2}$-averaged mappings are exactly the firmly nonexpansive ones, so the mapping $U \mapsto 2U - \mathrm{id}_X$ from firmly nonexpansive mappings to plainly nonexpansive ones is bijective.

These two facts suggest that, for this case, an induction-based, more conceptual proof might be possible. Such a proof was indeed given by Bauschke and Moursi (FoCM, 2020). Let us see what its ingredients are.

Monotone and cocoercive operators

A set-valued operator $A \subseteq X \times X$ is **monotone** if, for any (a, b), $(c, d) \in A$, $\langle a - c, b - d \rangle \geq 0$. It is **maximally monotone** if it is maximal among monotone operators as ordered by inclusion.

For a maximally monotone operator A, one may define its **resolvent** $J_A := (\mathrm{id}_X + A)^{-1} : X \to X$ which is a (single-valued) firmly nonexpansive mapping, and this association is bijective. Composing it with the previous bijection, we obtain the **reflected resolvent** $R_A := 2J_A - \mathrm{id}_X$.

Furthermore, for a $\beta > 0$, an $A \subseteq X \times X$ is called β-**cocoercive** if, for any (a, b), $(c, d) \in A$, $\langle a - c, b - d \rangle \geq \beta \|b - d\|^2$. For such an A, one can see (immediately) that it is single-valued and (nontrivially) that it has full domain, so one can think of it as a mapping $A : X \to X$ and rewrite the condition as: for any x, $y \in X$, $\langle x - y, Ax - Ay \rangle \geq \beta \|Ax - Ay\|^2$.

Rectangularity

An $A \subseteq X \times X$ is called 3^*-**monotone**[2] or **rectangular**[3] if, for any c in the domain of A and any b' in the range of A, $\sup_{(a,a')\in A} \langle a - c, b' - a' \rangle < \infty$. For single-valued full domain mappings, the condition may be written as: for any b, $c \in X$, $\sup_{a\in X} \langle a - c, Ab - Aa \rangle < \infty$.

It is known that:

- if A is maximally monotone and $\beta > 0$, then A is β-cocoercive iff R_A is $\frac{1}{1+\beta}$-averaged;
- β-cocoercive operators are rectangular.

All these concepts are, thus, used in the Bauschke/Moursi proof, and we managed to extract (Optim. Letters, 2022) a rate of asymptotic regularity for this case of averaged mappings.

[2]Notion introduced by Brézis and Haraux in 1976.

[3]Named as such by Simons in 2006.

Uniform rectangularity

A relevant feature is that here (and also in Kohlenbach's 2019 paper) one has to first extract a 'rate' for rectangularity, namely a Θ such that for any β, L_1, L_2, $L_3 > 0$, any β-cocoercive A, and any a, b, $c \in X$ with $\|b\| \leq L_1$, $\|c\| \leq L_2$ and $\|Ab\| \leq L_3$,

$$\langle a - c, Ab - Aa \rangle \leq \Theta(\beta, L_1, L_2, L_3).$$

This was later codified by Kohlenbach and Pischke (Phil. Trans. R. Soc. A, 2023) under the name of **modulus of uniform rectangularity**.

Uniform monotonicity

How may one further generalize this line of results?

Well, cocoercive operators have the following generalization: an A is called **inverse uniformly monotone** with a (classical) modulus $\varphi : [0, \infty) \to [0, \infty]$ (nondecreasing and vanishing only at 0) if, for any (a, b), $(c, d) \in A$, $\langle a - c, b - d \rangle \geq \varphi(\|b - d\|)$. Again, one can prove that such an A is single-valued and has full domain, so the condition is rewritten as: for any x, $y \in X$, $\langle x - y, Ax - Ay \rangle \geq \varphi(\|Ax - Ay\|)$.

If cocoercive operators correspond (via the reflected resolvent) to averaged mappings, then what do inverse uniformly monotone operators correspond to?

Super strong nonexpansiveness

The answer was given by Liu/Moursi/Vanderwerff (arXiv, 2022) in the form of **super strongly nonexpansive (SSNE)** mappings. Such a mapping is a nonexpansive map U such that for all sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ such that $\|x_n - y_n\|^2 - \|Ux_n - Uy_n\|^2 \to 0$, we have that $(x_n - y_n) - (Ux_n - Uy_n) \to 0$.

We have shown (using a similar trick to the one of Kohlenbach) that this is equivalent to the existence of an **SSNE-modulus** χ.

Examples of SSNE mappings include (obviously) averaged mappings, as well as a class of mappings dubbed *contractions for large distances*.

We need to make the correspondence between SSNE mappings and inverse uniformly monotone operators to be quantitative.

Abstract moduli

For that, we first need to 'prepare' the modulus φ, and thus we reified an old idea of Kohlenbach ('make the sentence monotone instead of the function') in the following highly abstract way.

Proposition

Let M be a set, $A : M \to [0, \infty)$ and $B : M \to \mathbb{R}$. We define a function $\varphi_{(M,A,B)} : [0, \infty) \to [-\infty, \infty]$ (definition omitted here). TFAE, and, in this case, we say that (M, A, B) is **adequate***:*

- $\varphi_{(M,A,B)}(0) = 0$ *and, for all* $\varepsilon > 0$, $\varphi_{(M,A,B)}(\varepsilon) > 0$;
- *there is a function* $\varphi : [0, \infty) \to [0, \infty]$*, nondecreasing and vanishing only at* 0*, such that, for all* $x \in M$, $\varphi(A(x)) \leq B(x)$;
- *there is a function* $\psi : (0, \infty) \to (0, \infty)$ *such that for all* $x \in M$ *and all* $\varepsilon > 0$ *with* $A(x) \geq \varepsilon$*, we have that* $\psi(\varepsilon) \leq B(x)$.

We shall call a function φ as above a **classical modulus** *for it and a function ψ as above simply a* **modulus** *for it.*

Part of the correspondence

The condition becomes: $A : X \to X$ is inverse uniformly monotone with a modulus ψ iff, for all $\varepsilon > 0$ and x, $y \in X$ with $\|Ax - Ay\| \leq \varepsilon$, $\langle x - y, Ax - Ay \rangle \geq \psi(\varepsilon)$.

Now that the moduli have been suitably defined, the quantitative correspondence result gets the following nice form.

Proposition

For any $\chi : (0, \infty) \to (0, \infty)$ *and any* $\varepsilon > 0$, *put*

$$\psi_\chi(\varepsilon) := \frac{\chi(2\varepsilon)}{4}.$$

Let $\chi : (0, \infty) \to (0, \infty)$ *and* A *be a maximally monotone operator on* X *such that* R_A *is super strongly nonexpansive with modulus* χ. *Then* A *is inverse uniformly monotone with modulus* ψ_χ.

Towards asymptotic regularity

One notices that Liu/Moursi/Vanderwerff do not have any asymptotic regularity result in their paper. Maybe this can be done, since part of the promise of proof mining is that one should get new insights into the proofs one is analyzing.

We seek to extend the Bauschke/Moursi proof. However, one does not know that arbitrary inverse uniformly monotone operators are rectangular. This is known only for the 'supercoercive' case, which is defined, in terms of the classical modulus φ, as the condition that

$$\lim_{s \to \infty} \frac{\varphi(s)}{s} = \infty.$$

In order to work with this sort of condition, we need a similar 'preparation' as before.

Abstract moduli, the supercoercive case

Proposition

Let (M, A, B) be adequate in the previous sense. Let $\eta : (0, \infty) \to (0, \infty)$. TFAE, and, in this case, we say that η is a **supercoercivity modulus** *for (M, A, B):*

- *for all $N > 0$ and all $s \geq \eta(N)$, $\varphi_{(M,A,B)}(s) \geq Ns$;*
- *there exists a classical modulus φ for (M, A, B) such that for all $N > 0$ and all $s \geq \eta(N)$, $\varphi(s) \geq Ns$;*
- *there exists a modulus ψ for (M, A, B) such that for all $N > 0$ and all $s \geq \eta(N)$ such that there is an $x \in M$ with $A(x) \geq s$, $\psi(s) \geq Ns$;*
- *for all $N > 0$, all $s \geq \eta(N)$ and all $x \in M$ with $A(x) \geq s$, $B(x) \geq Ns$;*
- *for all $N > 0$ and all $x \in M$ with $A(x) \geq \eta(N)$, $B(x) \geq N \cdot A(x)$.*

Notice that the last two conditions do not mention other moduli.

SSSNE mappings

We may now say that a SSNE-mapping $T : X \to X$ is **supercoercively super strongly nonexpansive (SSSNE)** with a supercoercivity modulus ν if, for all $M > 0$ and all $x, y \in X$ with $\|x - y\|^2 - \|Tx - Ty\|^2 < M\|(x - y) - (Tx - Ty)\|$, we have that $\|(x - y) - (Tx - Ty)\| < \nu(M)$. Averaged mappings and contractions for large distances are SSSNE, so this is a nontrivial generalization. We may extend the correspondence from before:

Proposition

For every $\nu : (0, \infty) \to (0, \infty)$ and every $N > 0$, define

$$\eta_\nu(N) := \nu(2N)/2.$$

Let A be a maximally monotone operator on X which is inverse uniformly monotone, so R_A is super strongly nonexpansive. Let $\nu : (0, \infty) \to (0, \infty)$ be a supercoercivity modulus for R_A. Then A admits the supercoercivity modulus η_ν.

The last obstacle

We may also obtain a modulus of uniform rectangularity, which gives us hope that one could obtain asymptotic regularity. Still, the induction proof from before would seem to not work, as SSSNE mappings are not known to be closed under composition. However, the form of the theorem for two mappings makes that requirement unnecessary (as the R_1 in the statement is not required to be SSSNE):

Theorem

*Let χ, $\nu : (0, \infty) \to (0, \infty)$ and R_1, $R_2 : X \to X$ be super strongly nonexpansive mappings such that R_1 has SSNE-modulus χ and R_2 has supercoercivity modulus ν. Put $R := R_2 \circ R_1$. Let $K : (0, \infty) \to (0, \infty)$ be such that for all i and all $\varepsilon > 0$ there is a $p \in X$ with $\|p\| \leq K(\varepsilon)$ and $\|p - R_i p\| \leq \varepsilon$.
Then for all $\delta > 0$ there is a $p \in X$ with $\|p\| \leq \Phi(\chi, \nu, K, \delta)$ and $\|p - Rp\| \leq \delta$.*

Through the grid

For example, for a composition of three mappings

$$R_3 \circ R_2 \circ R_1,$$

we first apply the previous result for the composition $S := R_2 \circ R_1$, and then for $R_3 \circ S$, so we only need an SSNE-modulus for the composition, which we do have.

Thus, we obtain our final result, which we may also express in the following purely qualitative form.

Theorem

Let X be a Hilbert space, $m \geq 1$ and $R_1, \ldots, R_m : X \to X$ be supercoercively super strongly nonexpansive mappings which have the approximate fixed point property.
Then $R_m \circ \ldots \circ R_1$ is asymptotically regular.

These results may all be found in the following paper and the references given therein:

A. Sipoș, The computational content of super strongly nonexpansive mappings and uniformly monotone operators. arXiv:2303.02768 [math.OC], 2023. To appear in: *Israel Journal of Mathematics*.

Thank you for your attention.