# A proof mining case study on the unit interval

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# Proof mining

This, as the title says, is a talk on *proof mining*:

- an applied subfield of mathematical logic
- first suggested by G. Kreisel in the 1950s (under the name "unwinding of proofs"), then given maturity by U. Kohlenbach and his school starting in the 1990s
- goals: to find explicit and uniform witnesses or bounds and to remove superfluous premises from concrete mathematical statements by analyzing their proofs
- tools used: primarily proof interpretations (modified realizability, negative translation, functional interpretation)
- the adequacy of the tools to the goals is guaranteed by **general logical metatheorems**
- a survey of recent results may be found in Kohlenbach's contribution to ICM 2018
- also see my series of articles on the Proof Theory Blog, entitled **What proof mining is about**, for a short and hopefully accessible introduction

# Rates of convergence

For example, suppose we deal with a convergence theorem, whose conclusion looks like this:

$$\exists x \forall \varepsilon \, \exists N \, \forall n \geq N \; d(x_n, x) \leq \varepsilon.$$

An ideal goal would be to find a **rate of convergence**, i.e. a bound on the $N$ in terms of the $\varepsilon$ (and maybe also some other parameters of the problem).

In order to restrict ourselves to numerical quantifiers, we could eliminate the reference to the limit point by replacing convergence by Cauchyness, which is equivalent to it in a complete space and which can be written like

$$\forall \varepsilon \, \exists N \, \forall M \, \forall i, j \in [N, N+M] \; d(x_i, x_j) \leq \varepsilon.$$

It is immediate that a finding a rate for the above statement equates to finding a rate of convergence. Unfortunately, even in the case of quite elementary theorems this goal may be virtually unattainable. Let us see what we mean by this.

# Monotone sequences

Consider the following first-year real analysis theorem: a monotone sequence of reals in the unit interval is convergent. How would a rate of convergence look like? There are two drawbacks.

The first is that the convergence may be **arbitrarily slow** – for any candidate function one may cook up a sequence which converges slower than that, so this destroys the hope of a **uniform** rate. For example, if $\varphi : (0, \infty) \to \mathbb{N}$ is such a function, then if we set, for any $n \in \mathbb{N}$,

$$x_n := \begin{cases} \frac{2}{3}, & \text{if } n \leq \varphi(1/2), \\ \frac{1}{n+1}, & \text{if } n > \varphi(1/2), \end{cases}$$

then $(x_n)$ is a nonincreasing sequence in $[0, 1]$ with limit 0 and we have that

$$d(x_{\varphi(1/2)}, 0) = x_{\varphi(1/2)} = \frac{2}{3} > \frac{1}{2},$$

so $\varphi$ is not a rate of convergence for $(x_n)$.

# Monotone sequences

The second drawback is that even if we restrict ourselves to specific instances, there is this curious phenomenon that a monotone sequence of rationals in $[0, 1]$ may be **computable** while any function that serves as a rate of convergence for it is **uncomputable** (such sequences are called **Specker sequences**). Let us construct such a sequence.

Let $A \subseteq \mathbb{N}$ be a recursively enumerable set which is not decidable (for example, the set of codes of Turing machines that halt on empty input). Since $A$ is recursively enumerable, there exists a computable bijection $f : \mathbb{N} \to A$. Put, for any $n \in \mathbb{N}$,

$$x_n := \sum_{i=0}^{n} \frac{1}{2^{f(i)+1}} \in [0, 1] \cap \mathbb{Q}.$$

Then $(x_n)$ is nondecreasing and computable. We shall show that any rate of Cauchyness for it is uncomputable.

# Monotone sequences

Let $\varphi$ be a rate of Cauchyness for it. We claim that for any $k \in A$ there is an $i \leq \varphi(1/2^{k+2})$ such that $f(i) = k$.

Since $f$ is surjective, there is an $i$ such that $f(i) = k$. Suppose that $i > \varphi(1/2^{k+2})$, so $i - 1 \geq \varphi(1/2^{k+2})$. Then, on one hand,

$$|x_i - x_{i-1}| \leq \frac{1}{2^{k+2}} < \frac{1}{2^{k+1}}$$

and on the other

$$x_i - x_{i-1} = \frac{1}{2^{f(i)+1}} = \frac{1}{2^{k+1}},$$

so we have a contradiction. The claim is now proven.

If $\varphi$ were computable, then the above claim would give a decision procedure for $A$, contradicting our original assumption.

# The Herbrand normal form

Of course, there are situations where happy accidents may allow extraction of a rate of convergence, but in the general case one is forced to replace the Cauchy statement

$$\forall \varepsilon \; \exists N \; \forall M \; \forall i, j \in [N, N+M] \; d(x_i, x_j) \leq \varepsilon.$$

by one that logic calls its **Herbrand normal form**:

$$\forall \varepsilon \; \forall g : \mathbb{N} \to \mathbb{N} \; \exists N \; \forall i, j \in [N, N+g(N)] \; d(x_i, x_j) \leq \varepsilon.$$

We show that the two forms are equivalent. One direction, from Cauchyness to the Herbrand normal form, is easy (take $N := N$). For the other, we reason **non-constructively**. Suppose that Cauchyness does not hold, so

$$\exists \varepsilon \; \forall N \; \exists M \; \exists i, j \in [N, N+M] \; d(x_i, x_j) > \varepsilon.$$

But then we can choose for any $N$ a corresponding $M$ which we denote as $g(N)$ and so we obtain

$$\exists \varepsilon \; \exists g : \mathbb{N} \to \mathbb{N} \; \forall N \; \exists i, j \in [N, N+g(N)] \; d(x_i, x_j) > \varepsilon,$$

which is the negation of the Herbrand normal form.

# Metastability

The usefulness of this form was independently identified by Terence Tao in his work on multiple ergodic averages. After he highlighted it on his blog, it was dubbed **metastability** at the suggestion of Jennifer Chayes.

As this is a $\forall\exists$ statement (in a generalized sense, as we also allow e.g. functions), by the metatheorems of proof mining one can extract from its proof a computable **rate of metastability**, i.e. a bound $\Psi(\varepsilon, g, \ldots)$ on the $N$.

In particular, this shows that there cannot be a constructive proof of the equivalence between Cauchyness and metastability, because then that equivalence would lift to the level of rates, but we have just seen that a computable and uniform rate of convergence does not always exist.

# Rates of metastability for monotone sequences

We will now illustrate the concept by showing how monotone sequences in $[0,1]$ possess a uniform rate of metastability (this result will not only be useful as an example, but will actually be applied later in the talk).

For all $g : \mathbb{N} \to \mathbb{N}$, we define $\widetilde{g} : \mathbb{N} \to \mathbb{N}$, for all $n$, by $\widetilde{g}(n) := n + g(n)$. Also, for all $f : \mathbb{N} \to \mathbb{N}$ and all $n \in \mathbb{N}$, we denote by $f^{(n)}$ the $n$-fold composition of $f$ with itself. Note that for all $g$ and $n$, $\widetilde{g}^{(n)}(0) \leq \widetilde{g}^{(n+1)}(0)$.

**Finite Monotone Convergence Principle (Tao, 2007)**

Let $\varepsilon > 0$, $g : \mathbb{N} \to \mathbb{N}$. Let $(a_i)_{i=0}^{\widetilde{g}^{\left(\lceil \frac{1}{\varepsilon} \rceil + 1\right)}(0)}$ be a finite monotone sequence in $[0,1]$. Then there is an $N \leq \widetilde{g}^{\left(\lceil \frac{1}{\varepsilon} \rceil\right)}(0)$ with $N + g(N) \leq \widetilde{g}^{\left(\lceil \frac{1}{\varepsilon} \rceil + 1\right)}(0)$ such that for all $i, j \in [N, N + g(N)]$, $|a_i - a_j| \leq \varepsilon$.

# Proof of the principle

## Proof

Assume w.l.o.g. that $(a_i)$ is nondecreasing. Assume that the conclusion is false, hence in particular for all $i \leq \left\lceil \frac{1}{\varepsilon} \right\rceil$, $a_{\widetilde{g}^{(i+1)}(0)} - a_{\widetilde{g}^{(i)}(0)} > \varepsilon$. Then

$$a_{\widetilde{g}^{(\lceil \frac{1}{\varepsilon} \rceil + 1)}(0)} \geq a_{\widetilde{g}^{(\lceil \frac{1}{\varepsilon} \rceil + 1)}(0)} - a_0 = \sum_{i=0}^{\lceil \frac{1}{\varepsilon} \rceil} \left( a_{\widetilde{g}^{(i+1)}(0)} - a_{\widetilde{g}^{(i)}(0)} \right) > \left\lceil \frac{1}{\varepsilon} \right\rceil \cdot \varepsilon \geq 1,$$

a contradiction.

This immediately gives us a uniform and computable rate of metastability for monotone sequences in the unit interval.

## Corollary

Let $(a_n)$ be a monotone sequence in $[0,1]$. Then for all $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$ there is an $N \leq \widetilde{g}^{(\lceil \frac{1}{\varepsilon} \rceil)}(0)$ such that for all $i$, $j \in [N, N + g(N)]$, $|a_i - a_j| \leq \varepsilon$.

This was all known to Kreisel in the 1950s!

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## ON THE INTERPRETATION OF NON-FINITIST PROOFS

### PART II. INTERPRETATION OF NUMBER THEORY. APPLICATIONS.

G. KREISEL

*Example.* Consider the theorem that a bounded, monotone increasing sequence of reals $a_n$ converges. To simplify notation let them lie between 0 and 1, and let us use binary scale, in which rationals $n/2^m$ terminate.

$$\frac{a(n, m)}{2^m} \leq a_n < \frac{a(n, m) + 1}{2^m}.$$

Then convergence means:

$$(m)(En_0)(n)[n > n_0 \rightarrow a(n, m) = a(n_0, m)] \cdot$$
$$\vee (Er)(Es)[a(r + 1, s) < a(r, s) \vee a(r, 0) > 1 \vee a(r, 0) < 0].$$

A counter-example would be a number $\mathfrak{m}$, and a function $N(n_0)$ so that for all $n_0$

$$N(n_0) > n_0 \quad \text{and} \quad a[N(n_0), \mathfrak{m}] \neq a(n_0, \mathfrak{m}), \text{ also}$$
$$a(r, \mathfrak{m}) \leq a(r + 1, \mathfrak{m}) \quad \text{and} \quad 0 \leq a(r, 0) \leq 1. \qquad 39.3$$

This is impossible: take $n_0 = 0$, $n_1$ $(= N(n_0))$, $\cdots$, $n_{i+1}(= N(n_i))$; then if for all $r$, $a(r, \mathfrak{m}) \leq a(r + 1, \mathfrak{m})$,

$$a(0, \mathfrak{m}) + 1 \leq a(n_1, \mathfrak{m})$$
$$a(n_1, \mathfrak{m}) + 1 \leq a(n_2, \mathfrak{m})$$
$$\vdots$$
$$a(n_{2\mathfrak{m}}, \mathfrak{m}) + 1 \leq a(n_{2\mathfrak{m}+1}\ \mathfrak{m})$$

so that $a(n_{2\mathfrak{m}+1}, \mathfrak{m}) > 2^{\mathfrak{m}} + a(0, \mathfrak{m})$, hence $a(n_{2\mathfrak{m}+1}, 0) > 1$. In our notation, for some $n_0$, $0 \leq n_0 \leq \omega_x[N(x), 0, 2^{\mathfrak{m}} + 1]$, 39.3 breaks down.

# Krasnoselski-Mann iterations

We shall focus in this talk on sequences which are called **Krasnoselski-Mann iterations**. If $X$ is a metric space with some convex structure, $f : X \to X$ and $(t_n) \subseteq [0, 1]$ then a sequence $(x_n) \subseteq X$ is a Krasnoselski-Mann iteration of $f$ with parameter sequence $(t_n)$ if for each $n$, we have that

$$x_{n+1} = (1 - t_n)x_n + t_n f(x_n).$$

Such iterations are mainly used to find fixed points in the limit.

# The limit is a fixed point

A result that guarantees that the limit is a fixed point looks like the following.

**Proposition**

Let $f : [0,1] \to [0,1]$ be continuous and $(t_n) \subseteq [0,1]$. Let $(x_n)$ be a Krasnoselski-Mann iteration corresponding to $(t_n)$. If $\sum_{n \geq 0} t_n = \infty$ and $(x_n)$ is convergent then its limit is a fixed point of $f$.

**Proof**

Let $z$ be the limit and assume w.l.o.g. that $f(z) > z$. Set, for all $n$, $\varepsilon_n := f(x_n) - x_n \to f(z) - z > 0$. Thus, we also have that $\sum_{n \geq 0} t_n \varepsilon_n = \infty$. But for all $n$, $x_n - x_0 = \sum_{k=0}^{n-1} t_k \varepsilon_k$, contradicting the convergence of $(x_n)$.

Therefore, in this case one may only need to find sufficient conditions for convergence itself.

# The result of Borwein and Borwein

**Theorem (Borwein/Borwein, 1991)**

Let $L > 0$, $f : [0,1] \to [0,1]$ be $L$-Lipschitz and $(x_n)$, $(t_n) \subseteq [0,1]$ be such that for all $n$, $x_{n+1} = (1 - t_n)x_n + t_n f(x_n)$.

If there is a $\delta > 0$ such that for all $n$,

$$t_n \leq \frac{2 - \delta}{L + 1},$$

then the sequence $(x_n)$ converges.

The proof of this theorem relies on a completely new kind of argument, never analyzed before in proof mining. We managed to extract for $(x_n)$ in the above a rate of metastability having just $\delta$ as an extra parameter. Our plan today is to first go through the original proof, and then present the quantitative argument that gives the rate of metastability.

# A lemma

We start with the following preparatory lemma.

## Lemma

Let $L > 0$, $f : [0,1] \to [0,1]$ be $L$-Lipschitz, $x$, $x^* \in [0,1]$, $\delta \in (0,1)$ and $t \in [0,1]$ such that $t \leq \frac{2-\delta}{L+1}$ and $x^* = (1-t)x + tf(x)$. Let $p$ be a fixed point of $f$ which is located between $x$ and $x^*$. Then

$$|x^* - p| \leq (1-\delta)|x - p|.$$

## Proof

Assume w.l.o.g. $x \leq x^*$. Then

$$\begin{aligned} |x^* - p| = x^* - p &= (1-t)(x-p) + t(f(x) - f(p)) \\ &\leq (t-1)(p-x) + tL(p-x) \\ &= (t(1+L)-1)(p-x) \leq (1-\delta)|x-p|. \end{aligned}$$

# Direction

The proof will rely on identifying the points where the sequence changes direction. Therefore, we define these rigorously, taking care so that our definitions behave well with respect to the case where two subsequent elements of the sequence are equal.

## Definition

Let $(x_n) \subseteq [0,1]$ and $f : [0,1] \to [0,1]$.
We say that $(\sigma_n) \subseteq \{\pm 1\}$ is the **sign sequence** for $(x_n)$ relative to $f$ if $\sigma_0 = 1$ and for all $n$, if $f(x_n) - x_n \neq 0$, $\sigma_{n+1} = \text{sgn}(f(x_n) - x_n)$ and otherwise $\sigma_{n+1} = \sigma_n$ – note that for all $n$, if $\sigma_{n+1} = 1$ (respectively $-1$), then $f(x_n) - x_n \geq 0$ (respectively $\leq 0$).

We say that $(q_n) \subseteq \mathbb{N} \cup \{\infty\}$ is the **switching sequence** for $(x_n)$ relative to $f$ if, denoting by $(\sigma_n)$ the sign sequence for $(x_n)$ relative to $f$, $q_0 = 0$ and for all $n$, if $q_n = \infty$ then $q_{n+1} = \infty$ else if there is a $k > q_n$ with $\sigma_{k+1} = -\sigma_{q_n+1}$, $q_{n+1}$ is the least such $k$, else $q_{n+1} = \infty$ – note that for all $n$ with $q_{n+1} < \infty$, we have that $\sigma_{q_{n+1}+1} = -\sigma_{q_n+1}$ and that for all $l \in [q_n + 1, q_{n+1}]$, $\sigma_l = \sigma_{q_n+1}$.

# Another lemma

The proof of the following lemma is more involved, and we shall omit it, as it is not relevant to the proof analysis.

**Lemma**

Let $L > 0$, $f : [0,1] \to [0,1]$ be $L$-Lipschitz, $(t_n)$ and $(x_n)$ be sequences in $[0,1]$ such that for all $n$, $x_{n+1} = (1 - t_n)x_n + t_n f(x_n)$. Let $(q_n)$ be the switching sequence for $(x_n)$ relative to $f$. Let $r \geq 1$ with $q_{r+1} < \infty$ and put $n_1 := q_r - 1$ and $n_2 := q_{r+1} - 1$. Let $\delta \in (0,1)$ be such that for all $n$, $t_n \leq \frac{2-\delta}{L+1}$. Then:

1. for all $n \in [n_1 + 1, n_2 + 1]$, $x_n$ is located between $x_{n_1}$ and $x_{n_1+1}$;
2. $|x_{n_2} - x_{n_2+1}| \leq \left(1 - \frac{\delta}{2}\right) |x_{n_1} - x_{n_1+1}|$.

# Proof of the theorem

Armed with this lemma, we may now show the proof of the theorem itself.

Let $(q_n)$ be the switching sequence for $(x_n)$ relative to $f$. Note that $(q_n)$ is strictly increasing and for all $r$, $r \leq q_r$ and if $q_r < \infty$, then $(x_n)_{n \in [q_r, q_{r+1})}$ is monotone. We distinguish two cases.

**Case I.** There is an $r$ with $q_r = \infty$.

Take $r$ to be minimal with this property. Clearly, $r \geq 1$ and $q_{r-1} < \infty$, so $(x_n)_{n=q_{r-1}}^{\infty}$ is monotone and hence convergent.

# Proof of the theorem

**Case II.** For all $r$, $q_r < \infty$.

We first show that for all $r \geq 1$ and all $n \geq q_r$, $x_n$ is between $x_{q_r - 1}$ and $x_{q_r}$. Let $r \geq 1$. We prove that for all $s \geq r$ and all $n \in [q_s, q_{s+1}]$, $x_n$ is between $x_{q_r - 1}$ and $x_{q_r}$. If $s = r$, this follows immediately from the second lemma. Now let $s \geq r + 1$. By the induction hypothesis, for all $m \in [q_{s-1}, q_s]$, $x_m$ is between $x_{q_r - 1}$ and $x_{q_r}$ – in particular, $x_{q_s - 1}$ and $x_{q_s}$ are. By the second lemma, $x_n$ is between $x_{q_s - 1}$ and $x_{q_s}$, thus also between $x_{q_r - 1}$ and $x_{q_r}$.

# Proof of the theorem

Again by the second lemma, we get that for all $r \geq 1$, $|x_{q_{r+1}-1} - x_{q_{r+1}}| \leq \left(1 - \frac{\delta}{2}\right) |x_{q_r-1} - x_{q_r}|$ and thus, by an easy induction, for all $r \geq 1$, $|x_{q_r-1} - x_{q_r}| \leq \left(1 - \frac{\delta}{2}\right)^{r-1}$. Combining this with the result in the previous paragraph, we get that for all $r \geq 1$ and all $i, j \geq q_r$, $|x_i - x_j| \leq \left(1 - \frac{\delta}{2}\right)^{r-1}$.

Let $T := \left\lceil \log_{\left(1-\frac{\delta}{2}\right)} \varepsilon \right\rceil + 1$. Then we have that for all $i, j \geq q_T$, $|x_i - x_j| \leq \left(1 - \frac{\delta}{2}\right)^{T-1} \leq \varepsilon$. Thus, $(x_n)$ is Cauchy, hence convergent.

# The rate of metastability

**Theorem (A.S., 2021)**

Define, for any suitable $\varepsilon$, $g$, $\delta$, $m$, $n$:

$$\begin{aligned} h^g_m(n) &:= g(m+n) \\ P^{\varepsilon,g}_0 &:= 0 \\ P^{\varepsilon,g}_{n+1} &:= P^{\varepsilon,g}_n + \widetilde{h^g_{P^{\varepsilon,g}_n}}^{\left(\left\lceil \frac{1}{\varepsilon} \right\rceil + 1\right)}(0) \\ T_{\varepsilon,\delta} &:= \left\lceil \log_{\left(1-\frac{\delta}{2}\right)} \varepsilon \right\rceil + 1 \\ B_{\varepsilon,g,\delta} &:= T_{\varepsilon,\delta} + \widetilde{g}\left(P^{\varepsilon,g}_{T_{\varepsilon,\delta}}\right) + 1 \\ \Psi_\delta(\varepsilon, g) &:= P^{\varepsilon,g}_{B_{\varepsilon,g,\delta}}. \end{aligned}$$

Assume the framework of the Borwein/Borwein theorem. Put now $\delta \in (0,1)$ be the one introduced before such that for all $n$, $t_n \leq \frac{2-\delta}{L+1}$. Let $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$. Then there is an $N \leq \Psi_\delta(\varepsilon, g)$ such that for all $i, j \in [N, N + g(N)]$, $|x_i - x_j| \leq \varepsilon$.

# Proof of the quantitative statement

We may now drop $\varepsilon$, $g$, $\delta$ when they show up as indices or arguments. It is immediate that:

- for all $n$, $P_n \leq P_{n+1}$;
- $\left(1 - \frac{\delta}{2}\right)^{T-1} \leq \varepsilon$.

We distinguish again two cases, but in a **finitary** manner.

**Case I.** There is an $r \leq B$ with $q_r > P_r = P_{r-1} + \widetilde{h_{P_{r-1}}}^{\left(\lceil \frac{1}{\varepsilon} \rceil + 1\right)}(0)$.

Take $r$ to be minimal with this property. Clearly, $r \geq 1$ and $q_{r-1} \leq P_{r-1}$, so

$$\left(x_{P_{r-1}+i}\right)_{i=0}^{\widetilde{h_{P_{r-1}}}^{\left(\lceil \frac{1}{\varepsilon} \rceil + 1\right)}(0)}$$

is a subsequence of $(x_n)_{n \in [q_{r-1}, q_r)}$ and is thus monotone.

# Proof of the quantitative statement

By the Finite Monotone Convergence Principle, there is an $N' \leq \widetilde{h_{P_{r-1}}}^{(\lceil \frac{1}{\varepsilon} \rceil)}(0)$ such that for all $i$ and $j$ in the interval

$$[P_{r-1} + N', P_{r-1} + N' + h_{P_{r-1}}(N')],$$

$|x_i - x_j| \leq \varepsilon$.
Put $N := P_{r-1} + N'$. Then

$$N \leq P_{r-1} + \widetilde{h_{P_{r-1}}}^{(\lceil \frac{1}{\varepsilon} \rceil)}(0) \leq P_{r-1} + \widetilde{h_{P_{r-1}}}^{(\lceil \frac{1}{\varepsilon} \rceil + 1)}(0) = P_r \leq P_B = \Psi.$$

In addition,

$$h_{P_{r-1}}(N') = g(P_{r-1} + N') = g(N),$$

so for all $i$, $j \in [N, N + g(N)]$, $|x_i - x_j| \leq \varepsilon$.

# Proof of the quantitative statement

**Case II.** For all $r \leq B$, $q_r \leq P_r$.

We first show that for all $r \in [1, B-1]$ and all $n \in [q_r, q_B]$, $x_n$ is between $x_{q_r-1}$ and $x_{q_r}$. Let $r \in [1, B-1]$. We prove that for all $s \in [r, B-1]$ and all $n \in [q_s, q_{s+1}]$, $x_n$ is between $x_{q_r-1}$ and $x_{q_r}$. If $s = r$, this follows immediately from the second lemma. Now let $s \geq r+1$. By the induction hypothesis, for all $m \in [q_{s-1}, q_s]$, $x_m$ is between $x_{q_r-1}$ and $x_{q_r}$ – in particular, $x_{q_s-1}$ and $x_{q_s}$ are. By the second lemma, $x_n$ is between $x_{q_s-1}$ and $x_{q_s}$, thus also between $x_{q_r-1}$ and $x_{q_r}$.

## Proof of the quantitative statement

Again by the second lemma, we get that for all $r \in [1, B-1]$, $|x_{q_{r+1}-1} - x_{q_{r+1}}| \leq \left(1 - \frac{\delta}{2}\right) |x_{q_r-1} - x_{q_r}|$ and thus, by an easy induction, for all $r \in [1, B-1]$, $|x_{q_r-1} - x_{q_r}| \leq \left(1 - \frac{\delta}{2}\right)^{r-1}$. Combining this with the result in the previous paragraph, we get that for all $r \in [1, B-1]$ and all $i, j \in [q_r, q_B]$, $|x_i - x_j| \leq \left(1 - \frac{\delta}{2}\right)^{r-1}$.

Since $T \leq B - 1$, for all $i, j \in [q_T, q_B]$, $|x_i - x_j| \leq \varepsilon$. Take $N := P_T \leq P_B = \Psi$. Then on one hand $N = P_T \geq q_T$ (since $T \leq B$) and on the other

$$N + g(N) = \widetilde{g}(P_T) \leq T + \widetilde{g}(P_T) + 1 = B \leq q_B,$$

so $[N, N + g(N)] \subseteq [q_T, q_B]$, hence for all $i, j \in [N, N + g(N)]$, $|x_i - x_j| \leq \varepsilon$.

The proof is now finished.

This and some other results may be found in:

A. Sipoș, Rates of metastability for iterations on the unit interval, *Journal of Mathematical Analysis and Applications*, Volume 502, Issue 1, 125235 [11 pages], 2021.

Thank you for your attention.