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Monadic and Closure MV-Algebras

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The MV-algebras are the algebraic structures corresponding to the infinite valued calculus of Lukasiewicz. In this paper we investigate the monadic and the closure MV-algebras and we establish some connections between them. We also prove that monadic MV_n-algebras are polynomially equivalent to monadic n-valued Lukasiewicz–Moisil algebras for $n = 3$ and $n = 4$.

Keywords: MV-algebra; monadic algebra; closure operator; Lukasiewicz–Moisil algebra

The MV-algebras were introduced in [4] as algebraic structures corresponding to the infinite valued propositional calculus of Lukasiewicz. The quantifiers defined on an MV-algebra appear in [13, 14]. These two papers developed the algebraic logic for the infinite valued predicate calculus [2].

The aim of this paper is to investigate the monadic and the closure MV-algebras and some connections between them. Section 1 presents some important features of MV-algebras and Section 2 is concerned with some basic properties of monadic MV-algebras. Section 3 deals

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with closure MV-algebras and with their relationship with monadic MV-algebras. The closure MV-algebras reflect algebraically some many-valued modal propositional logic [11]. We study the concept of P-ideal in a closure MV-algebra. In Section 4 we establish some connections between monadic MV_n-algebras and monadic n-valued Lukasiewicz–Moisil algebras ([3, 7, 9]) for $n = 3$ and $n = 4$.

1. PRELIMINARIES ON MV-ALGEBRAS

In this section we shall briefly describe MV-algebras. For a better understanding one can see [4, 5].

An *MV-algebra* is an algebra $\langle A, \oplus, ^{-}, 0\rangle$ with a binary operation $\oplus$, a unary operation $^{-}$ and a constant 0 such that

0. $\langle A, \oplus, 0\rangle$ is an abelian monoid,
1. $\bar{\bar{x}} = x$ for every $x \in A$,
2. $x \oplus \bar{0} = \bar{0}$ for every $x \in A$,
3. $\overline{\bar{x} \oplus y} \oplus y = \overline{\bar{y} \oplus x} \oplus x$ for every $x, y \in A$.

On each MV-algebra we define the constant 1 and the binary operations $\odot$, $\rightarrow$ and $\ominus$ as follow:

$$1 \stackrel{\text{def}}{=} \bar{0},$$
$$x \odot y \stackrel{\text{def}}{=} \overline{\bar{x} \oplus \bar{y}},$$
$$x \rightarrow y \stackrel{\text{def}}{=} \bar{x} \oplus y,$$
$$x \ominus y \stackrel{\text{def}}{=} x \odot \bar{y}.$$

The following identities immediately follow:

4. $\bar{1} = 0$,
5. $x \oplus y = \overline{\bar{x} \odot \bar{y}}$ for every $x, y \in A$,
6. $x \oplus 1 = 1$ for every $x \in A$,
7. $x \oplus \bar{x} = 1, x \odot \bar{x} = 0$ for every $x \in A$.

Example 1 Let $n \geq 2$. If we denote $L_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ and we define the operations $x \oplus y = \min(1, x + y)$ and $\bar{x} = 1 - x$ for every $x, y \in L_n$ then $\langle L_n, \oplus, ^{-}, 0\rangle$ is an MV-algebra.

Let A be an MV-algebra. For any two elements x, y of A we say that $x \leq y$ iff $y = x \oplus (y \odot \bar{x})$. It follows that $\leq$ is a partial order, which is called the *natural order* of A. The natural order determines a lattice structure on A. The corresponding join and meet are given by the formulas:

$$x \vee y = x \oplus (\bar{x} \odot y),$$
$$x \wedge y = x \odot (\bar{x} \oplus y).$$

In every MV-algebra A the following equations hold:

8. $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$ for every $x, y, z \in A$.

The defined operations $\vee$ and $\wedge$ make A a distributive lattice with least element 0 and greatest element 1.

LEMMA 2 [4] *In every MV-algebra A the following properties hold:*

(*i*) $x \leq y$ *iff* $\bar{y} \leq \bar{x}$,
(*ii*) *if* $x \leq y$ *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,
(*iii*) $x \leq y$ *iff* $\bar{x} \oplus y = 1$ *iff* $x \odot \bar{y} = 0$,
(*iv*) $x \odot y \leq x \wedge y \leq x \leq x \vee y \leq x \oplus y$.

We say that an MV-algebra is *linearly ordered* if the order relation is total. An MV-algebra is complete if it is complete as a lattice.

LEMMA 3 [1] *In every MV-algebra A the following equalities hold:*

(*i*) $$a \oplus \bigvee_{i\in I} x_i = \bigvee_{i\in I}(a \oplus x_i), \quad a \oplus \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(a \oplus x_i),$$

(*ii*) $$a \odot \bigvee_{i\in I} x_i = \bigvee_{i\in I}(a \odot x_i), \quad a \odot \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(a \odot x_i),$$

(*iii*) *if A is linearly ordered then*

$$\bigvee_{i\in I}(x_i \oplus x_i) = \bigvee_{i\in I} x_i \oplus \bigvee_{i\in I} x_i, \quad \bigvee_{i\in I}(x_i \odot x_i) = \bigvee_{i\in I} x_i \odot \bigvee_{i\in I} x_i,$$

$$\bigwedge_{i\in I}(x_i \oplus x_i) = \bigwedge_{i\in I} x_i \oplus \bigwedge_{i\in I} x_i, \quad \bigwedge_{i\in I}(x_i \odot x_i) = \bigwedge_{i\in I} x_i \odot \bigwedge_{i\in I} x_i.$$

If we denote $C(A) = \{x \mid x \oplus x = x\}$ then for every $x, y \in C(A)$ the operations $\vee$ and $\oplus$ coincide and $C(A)$ is a Boolean algebra (we call it the *Boolean center* of A).

LEMMA 4 *In every MV-algebra the following properties hold*:

$(r1)$ $$y \odot z \leq x \quad iff \quad y \leq x \oplus \bar{z},$$

$(r2)$ $$(x \odot y) \vee z \geq (x \vee z) \odot (y \vee z), \quad (x \oplus y) \wedge z \leq (x \wedge z) \oplus (y \wedge z).$$

If $x \in A$ *we define*:

(i) $$0x = 0, \quad (n+1)x = nx \oplus x,$$

(ii) $$x^0 = 1, \quad x^{n+1} = x^n \odot x,$$

where n is a natural number.

The *order* of an element $x(ord(x))$ is the least natural number n such that $nx = 1$. If no such integer n exists then $ord(x) = \infty$.

We say that an MV-algebra is *locally finite* if every non-zero element has a finite order.

In [6] R. Grigolia defined the MV$_n$-algebras as corresponding algebraic structures of the n-valued propositional calculus. We say that an MV-algebra A is an MV$_n$-*algebra* if it satisfies:

9. $(n-1)\,x \oplus x = (n-1)\,x$ and $x^{n-1} \odot x = x^{n-1}$ for every $x \in A$
10. If $n \geq 4$ the following axioms are added:

$$[(jx) \odot (\bar{x} \oplus \overline{(j-1)x})]^{n-1} = 0 \text{ and } (n-1)[x^j \oplus (\bar{x} \odot \overline{x^{j-1}})] = 1$$

for every $x \in A$ and $1 < j < n-1$ such that j does not divide $n-1$.

Remark 5 The algebra L_n defined in Example 1 is an MV$_n$-algebra

In an MV$_n$-algebra A we define $x^{\bullet} = \overline{(n-1)x}$.

LEMMA 6 [6] *In every MV$_n$-algebra A the following equalities hold for every* $x \in A$:

(i) $x \wedge x^{\bullet} = 0,$
(ii) $(n-1)(x \vee x^{\bullet}) = 1.$

Let $\langle A, \oplus, ^{-}, 0\rangle$ be an MV-algebra. For every x and y in A the following properties hold:

(r3) $x \oplus (y \ominus x) = x \vee y,\ x \odot (x \to y) = x \wedge y,$
(r4) $x \leq y$ iff $x \ominus y = 0$ iff $x \to y = 1,$

(r5) $(x \ominus y) \ominus z = x \ominus (y \oplus z),\ x \to (y \to z) = (x \odot y) \to z,$
(r6) if $x \leq y$ then $z \ominus y \leq z \ominus x$ and $y \to z \leq x \to z.$

We finish our presentation with some properties of MV-ideals. If A is an MV-algebra then a nonempty subset I of A is an MV-*ideal* if for every $x,\ y \in A$ the following are satisfied:

(i1) $y \leq x,\ x \in I \Rightarrow y \in I$
(i2) $x,\ y \in I \Rightarrow x \oplus y \in I.$

In the sequel, for an MV-algebra A, we shall say that $I \subseteq A$ is an *ideal* if I is an MV-ideal of A.

If we denote $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ then we can associate with an ideal I the relation defined by

$$x \sim_I y \Leftrightarrow d(x, y) \in I$$

which is a *congruence* relation with respect to the MV-algebra operations. On the other side, if $\sim$ is a congruence on an MV-algebra A then $I = \{x \mid x \sim 0\}$ is an ideal. It is easy to see that there is a one to one correspondence between the set of all ideals of an MV-algebra and the set of its congruences [4]. If we denote by A/I the set of all congruence classes and by $[x]$ the congruence class of the element $x \in A$, then we can define the MV-operations in a natural way: $[x] \oplus [y] = [x \oplus y], \overline{[x]} = [\bar{x}]$ and $0_{A/I} = [0_A]$. It is easy to see that A/I becomes an MV-algebra.

2. MONADIC MV-ALGEBRAS

This section is a brief presentation of monadic MV-algebras (see [13, 14]). We remind some of their properties and we investigate the monadic ideals of a monadic MV-algebra.

DEFINITION 7 A *monadic MV-algebra* is a pair $\langle A, \exists\rangle$ where A is an MV-algebra and $\exists : A \longrightarrow A$ is a mapping which satisfies the following axioms:

M0. $\exists 0 = 0,$
M1. $x \leq \exists x,$

M2. $\exists(x \odot \exists y) = \exists x \odot \exists y$,
M3. $\exists(x \oplus \exists y) = \exists x \oplus \exists y$,
M4. $\exists(x \odot x) = \exists x \odot \exists x$,
M5. $\exists(x \oplus x) = \exists x \oplus \exists x$.

If we define $\forall x = \overline{\exists \bar{x}}$ for any $x \in A$ then the following properties hold in A:

M0°. $\forall 1 = 1$,
M1°. $\forall x \leq x$,
M2°. $\forall(x \oplus \forall y) = \forall x \oplus \forall y$,
M3°. $\forall(x \odot \forall y) = \forall x \odot \forall y$,
M4°. $\forall(x \oplus x) = \forall x \oplus \forall x$,
M5°. $\forall(x \odot x) = \forall x \odot \forall x$.

We can also define a monadic MV-algebra as a pair $\langle A, \forall \rangle$ where A is an MV-algebra and $\forall$ is a mapping which satisfies the above conditions M0°–M5°.

If A is an MV-algebra then an *existential quantifier* on A is a mapping $\exists: A \longrightarrow A$ which satisfies the axioms M0–M5. A *universal quantifier* on A is a mapping $\forall: A \longrightarrow A$ which satisfies the axioms M0°–M5°.

LEMMA 8 [13] *In every monadic MV-algebra the following properties are satisfied*:

(*i*) $\exists 1 = 1$,
(*ii*) $\exists\exists x = \exists x$,
(*iii*) $\exists(\overline{\exists x}) = \overline{\exists x}$,
(*iv*) $\exists(\exists x \odot \exists y) = \exists x \odot \exists y$,
(*v*) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,
(*vi*) $\exists(a \wedge \exists b) = \exists a \wedge \exists b$,
(*vii*) $\exists(a \vee b) = \exists a \vee \exists b$,
(*viii*) $x \leq y \Rightarrow \exists x \leq \exists y$ *and* $\forall x \leq \forall y$,
(*ix*) $\exists\forall x = \forall x$, $\forall\exists x = \exists x$.

LEMMA 9 *Let* $\langle A, \exists \rangle$ *be a monadic algebra. Then*

(*i*) $\exists(x \oplus y) \leq \exists x \oplus \exists y$,
(*ii*) $\exists x \ominus \exists y \leq \exists(x \ominus y)$,
(*iii*) $\exists(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} \exists x_i$.

Proof

(i) From $x \oplus y \leq x \oplus \exists y$ it follows that $\exists(x \oplus y) \leq \exists(x \oplus \exists y) = \exists x \oplus \exists y$.
(ii) From M1 and Lemma 2 it follows that $x \odot \overline{\exists y} \leq x \odot \bar{y}$. Using Lemma 8 and M2, we obtain $\exists x \ominus \exists y = \exists(x \odot \overline{\exists y}) \leq \exists(x \ominus y)$.
(iii) Let $x = \bigvee_{i\in I} x_i$, so $\exists x_i \leq \exists x$. Let $y \in A$ be such that $\exists x_i \leq y$ for all $i \in I$. We have that $x_i \leq \exists x_i = \forall\exists x_i \leq \forall y$ for all $i \in I$, so $x \leq \forall y$ and it follows that $\exists x \leq \exists\forall y = \forall y \leq y$.

Example 10 Consider a pair (X, R) where X is a nonempty set and R is an equivalence relation on X. Let L be a linear complete MV-algebra (for example $L = L_n$). Then L^X is an MV-algebra. Define the function $\exists : L^X \longrightarrow L^X$ by

$$(\exists p)(x) = \bigvee\{p(y) | xRy\} \quad \text{for any } p \in L^X \text{ and } x \in X.$$

Then $\langle L^X, \exists \rangle$ is a monadic MV-algebra. The axiom M0 is obvious and M1 follows from the fact that R is reflexive. We shall verify only the axioms M3 and M4 in Definition 7.

(M3) For $p, q \in L^X$ and $x \in X$ we obtain, using Lemma 3 (i), $\exists(p \oplus \exists q)(x) = \bigvee\{p(y) \oplus (\vee\{q(z) | yRz\}) | xRy\} = \bigvee\{p(y) \oplus q(z) | xRy, yRz\}$ $(\exists p \oplus \exists q)(x) = \bigvee\{p(y) \oplus q(z) | xRy, xRz\}$ But R is an equivalence relation, hence xRy and yRz iff xRy and xRz, therefore $\exists(p \oplus \exists q)(x) = (\exists p \oplus \exists q)(x)$.
(M4) $\exists(p \odot p)(x) = \bigvee\{p(y) \odot p(y) | xRy\}$ Using Lemma 3, (iii), we obtain $\exists(p \odot p)(x) = (\bigvee\{p(y) | xRy\}) \odot (\bigvee\{p(y) | xRy\}) = (\exists p \odot \exists p)(x)$.

DEFINITION 11 Let $\langle A, \exists \rangle$ be a monadic MV-algebra and $I \subseteq A$. We say that I is a *monadic ideal* if I is an ideal of A and for every $x \in A$ the following condition is satisfied:

$$x \in I \Rightarrow \exists x \in I.$$

Remark 12 Let $\langle A, \exists \rangle$ be a monadic MV-algebra and let X be a set of elements from A. The monadic ideal generated by X in $\langle A, \exists \rangle$ is

$$(X]_\exists = \{y \in A | y \leq \exists x_1 \oplus \cdots \oplus \exists x_n, x_1, \ldots, x_n \in X\}.$$

LEMMA 13 *If I is a monadic ideal then A/I is a monadic MV-algebra.*

Proof We have to prove that $d(x,y)\in I$ implies $d(\exists x, \exists y)\in I$. From $d(x,y)\in I$ it follows that $x\ominus y$, $y\ominus x\in I$, so $\exists(x\ominus y)$, $\exists(y\ominus x)\in I$. Using Lemma 9 (ii) we obtain $\exists x\ominus\exists y$, $\exists y\ominus\exists x\in I$ from which $d(\exists x, \exists y)\in I$. If we define $\exists([x]) = [\exists x]$ it follows that $\langle A/I, \exists\rangle$ is a monadic MV-algebra.

Remark 14 $\exists(A)$ is subalgebra of the monadic MV-algebra A (from M0 and Lemma 8 (ii), (iii), (v)).

PROPOSITION 15 *There is a one to one correspondence between the ideals of $\exists(A)$ and the monadic ideals of A. Moreover, maximal ideals of $\exists(A)$ correspond to monadic maximal ideals of A.*

Proof If J is an ideal of the MV-algebra $\exists(A)$ then $\bar{J} = \{x|\text{there is } y\in J,\ x\leq y\}$ (the MV-ideal generated by I in A) is a monadic ideal of A. If I is a monadic ideal of A then $I\cap\exists(A)$ is an ideal of $\exists(A)$. It is easy to see that $I=\overline{I\cap\exists(A)}$ and $J=\bar{J}\cap\exists(A)$ for every monadic ideal I of A and for every ideal J of $\exists(A)$.

DEFINITION 16 A monadic MV-algebra $\langle A, \exists\rangle$ is *simple* if $\{0\}$ is the only proper monadic ideal of A. An existential quantifier $\exists$ defined on A is *simple* if for every $x\neq 0$ the order of $\exists x$ is finite.

LEMMA 17 *A monadic MV-algebra $\langle A, \exists\rangle$ is simple if and only if $\exists$ is a simple quantifier.*

Proof Let $\langle A, \exists\rangle$ be simple and $x\neq 0$. Then the monadic ideal generated by x is not proper, so there is a natural number n such that $n\exists x = 1$ and $ord(\exists x) < \infty$.

Let $\exists$ be a simple quantifier and I a proper monadic ideal of A. If there is an element $x\neq 0$, $x\in I$ then $\exists x\in I$, so $1\in I$ which contradicts the fact that I is proper.

COROLLARY 18 *A monadic MV-algebra $\langle A, \exists\rangle$ is simple if and only if $\exists(A)$ is a locally finite MV-algebra.*

PROPOSITION 19 *Every monadic MV_n-algebra is isomorphic to a subdirect product of simple monadic algebras.*

Proof In [6] it is proved that the intersection of all the maximal ideals of an MV_n-algebra it is equal to $\{0\}$, so by Proposition 15 we obtain that the intersection of all maximal monadic ideals of a monadic MV_n-algebra is $\{0\}$. Let A be a monadic MV_n-algebra and Max(A) the set of monadic maximal ideals of A. We define a natural embedding from A to $\prod_{M\in \mathrm{Max}(A)}A/M$ which associates with an element from A the family of its classes. The intended result is now obvious.

3. CLOSURE MV-ALGEBRAS

A *closure Boolean algebra* is a pair $\langle B, P\rangle$ where $B=\langle B, \vee, \wedge, ^{-}, 0, 1\rangle$ is a Boolean algebra and P is a unary operation on B satisfying the following properties:

$$P0 = 0,$$
$$P(x\vee y) = Px\vee Py,$$
$$x\leq Px,$$
$$PPx\leq Px$$

for any x, y in B. The closure Boolean algebras are algebraic models for the classical modal propositional logic S_4, so they are also called S_4 modal Boolean algebras. If we denote $Qx=\overline{P\bar{x}}$ and we add the axiom

$$PQx\leq Qx$$

we say that $\langle B, P\rangle$ is an S_5 modal Boolean algebra. These structures reflect algebraically the classical modal propositional logic S_5.

In [11] there were studied some modal systems for n-valued Lukasiewicz logic. A natural problem is to consider the adequate algebraic structures for these modal logics.

In this section we shall define closure MV-algebras. Our definition is inspired by the system of axioms associated with the modal system S_4 for n-valued Lukasiewicz logic [11].

DEFINITION 20 Let $A=\langle A,\oplus,\odot,^{-},0,1\rangle$ be an MV-algebra and $P\colon A\longrightarrow A$ a map defined on A. We say that $\langle A, P\rangle$ is a *closure MV-algebra* (or S_4 *modal MV-algebra*, or *topological MV-algebra*) if P satisfies the following axioms:

A0. $P0 = 0$,
A1. $(Px\ominus Py)\leq P(x\ominus y)$,

A2. $P(x \odot y) \leq Px \odot Py$,
A3. $Px \odot Px \leq P(x \odot x)$,
A4. $Px \oplus Px \leq P(x \oplus x)$,
A5. $x \leq Px$,
A6. $PPx \leq Px$.

Let $Q: A \longrightarrow A$ be the map such that $Qx = \overline{P\bar{x}}$ for every $x \in A$. Then Q satisfies the following properties:

A0°. $Q1 = 1$,
A1°. $Q(x \to y) \leq (Qx \to Qy)$,
A2°. $Qx \oplus Qy \leq Q(x \oplus y)$,
A3°. $Q(x \oplus x) \leq Qx \oplus Qx$,
A4°. $Q(x \odot x) \leq Qx \odot Qx$,
A5°. $Qx \leq x$,
A6°. $Qx \leq QQx$.

LEMMA 21 *Let $\langle A, P\rangle$ be a closure MV-algebra and x, y arbitrary elements in A. The following properties are satisfied:*

(*i*) $x \leq y \Rightarrow Px \leq Py$ *and* $Qx \leq Qy$,
(*ii*) $P(x \oplus y) \leq Px \oplus Py, Q(x \odot y) \geq Qx \odot Qy$,
(*iii*) $P(x \odot x) = Px \odot Px$, $Q(x \oplus x) = Qx \oplus Qx$,
(*iv*) $P(x \oplus x) = Px \oplus Px$, $Q(x \odot x) = Qx \odot Qx$,
(*v*) $P(Px \odot Py) = Px \odot Py$, $Q(Qx \oplus Qy) = Qx \oplus Qy$,
(*vi*) $P(Px \oplus Py) = Px \oplus Py$, $Q(Qx \odot Qy) = Qx \odot Qy$.

Proof

(i) If $x \leq y$ then $x \ominus y = 0$; from A1 and A0 we have $Px \ominus Py = 0$, so $Px \leq Py$.
(ii) $P(x \oplus y) = P(x) \vee P(x \oplus y) = P(x) \oplus (P(x \oplus y) \ominus P(x))$; using A1 we obtain $P(x \oplus y) \leq Px \oplus P((x \oplus y) \ominus x) = Px \oplus P((x \oplus y) \odot \bar{x}) = Px \oplus P(y \wedge \bar{x}) \leq Px \oplus Py$.
(iii) From A2 and A3.
(iv) Using A4 and (ii) the intended relation is obvious.
(v) From A5 and A6 it is obvious that $P = PP$. From A2 we obtain $P(Px \odot Py) \leq PPx \odot PPy = Px \odot Py$. The converse inequality follows from A5.
(vi) follows in a similar way but using (ii) instead of A2.

We are able to give an equivalent definition for closure MV-algebras.

DEFINITION 22 Let A be an MV-algebra and $P: A \longrightarrow A$. The pair $\langle A, P\rangle$ is a *closure MV-algebra* if P satisfies the axioms A0, A2–A6 and

B1. $P(x \oplus y) \leq Px \oplus Py$,
B2. $x \leq y \Rightarrow Px \leq Py$,

for every $x, y \in A$.

LEMMA 23 *Definition* 20 *is equivalent to Definition* 22.

Proof From the previous lemma one can see that Definition 20 implies Definition 22. For the other implication it suffices to prove that A1 holds. Using (r5) we obtain $(Px \ominus Py) \ominus P(x \ominus y) = Px \ominus (Py \oplus P(x \ominus y))$. Using B1, (r6), (r3), B2 and (r4) it follows that $(Px \ominus Py) \ominus P(x \ominus y) \leq Px \ominus P(y \oplus (x \ominus y)) = Px \ominus P(x \vee y) = 0$, so $P(x \ominus y) \ominus (Px \ominus Py) = 0$ which, by (r4), is an equivalent form for A1.

Example 24 Consider a pair $\langle X, R\rangle$ where X is a nonempty set and $R \subseteq X^2$ a preorder relation. Let L be a linear and complete MV-algebra. If we define $P^*: L^X \to L^X$ by

$$P^*(p)(x) = \vee\{p(y) | xRy\} \quad \text{for every } p \in L^X \text{ and } x \in X$$

then $\langle L^X, P^*\rangle$ is a closure MV-algebra. This follows as in Example 10, by using (r1) for A1.

We remark that axioms A0–A4 hold for an arbitrary relation R, but for A5 we need the reflexivity and for A6 the transitivity of R.

PROPOSITION 25 *If $\langle A, P\rangle$ is a monadic MV-algebra then $\langle A, P\rangle$ is a closure algebra.*

Proof The axioms A0, A3, A4 and A5 follow from M0, M4, M5 and M1. The axiom A6 follows from Lemma 8 (ii).

A1. $x \leq x \vee y = (x \ominus y) \oplus y \leq (x \ominus y) \oplus y \leq (x \ominus y) \oplus Py$, so using M3 it follows that $Px \leq P(x \ominus y) \oplus Py$. From (r1) we get $Px \ominus Py \leq P(x \ominus y)$.
A2. Using M1, Lemma 8 (viii) and M2 we obtain $P(x \odot y) \leq P(x \odot Py) = Px \odot Py$.

LEMMA 26 *If $\langle A, P\rangle$ is a closure algebra then the following identities are equivalent:*

(*i*) $Px = QPx$,
(*ii*) $Qx = PQx$,
(*iii*) $P\overline{Px} = \overline{Px}$,
(*iv*) $Q\overline{Qx} = \overline{Qx}$.

Proof

(i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) by duality.
(ii) $\Rightarrow$ (iii) $P\overline{Px} = PQ\bar{x} = Q\bar{x} = \overline{Px}$.
(iii) $\Rightarrow$ (ii) $PQx = P\overline{P\bar{x}} = \overline{P\bar{x}} = Qx$.

DEFINITION 27 A closure MV-algebra $\langle A, P\rangle$ is an S_5 *modal MV-algebra* if the following axiom is added:

A7. $PQx \leq Qx$.

Remark 28 Let A be an MV-algebra and $C(A)$ the Boolean center of A. If x and y are in $C(A)$ then $x \odot y = x \wedge y$ and $x \oplus y = x \vee y$. It is easy to see that if $\langle A, P\rangle$ is an S_4 (S_5) modal MV-algebra then $\langle B(A), P|_{B(A)}\rangle$ is an S_4 (S_5) modal Boolean algebra.

PROPOSITION 29 *If* $\langle A, P\rangle$ *is a closure algebra, then the following assertions are equivalent*:

(*i*) *P is an existential quantifier on A*,
(*ii*) *P satisfies*

(1) $P\overline{Px} = \overline{Px}$,
(2) $Px \oplus Py \leq P(x \oplus Py)$.

Proof (i) $\Rightarrow$ (ii) (2) follows immediately from M3.

(1) From M1 we obtain $\overline{Px} \leq P\overline{Px}$. For the converse inequality we use the fact that $\overline{Px} \odot Px = 0$, so $P(\overline{Px} \odot Px) = 0$. From M2 we get $P\overline{Px} \odot Px = 0$. It follows from Lemma 2 (iii) that $P\overline{Px} \leq \overline{Px}$.

(ii) $\Rightarrow$ (i) The axioms M0 and M1 are A0 and A5, respectively. The axioms M4 and M5 follow from Lemma 21 (iii) and (iv).

M2: $x \odot Py \leq Px \odot Py$, so using Lemma 21 (v) we obtain $P(x \odot Py) \leq P(Px \odot Py) = Px \odot Py$. For the converse inequality we use (1) and Lemma 21 (ii). It follows that $P(x \odot Py) \oplus \overline{Py} = P(x \odot Py) \oplus P\overline{Py} \geq P((x \odot Py) \oplus \overline{Py} = P(x \vee \overline{Py}) \geq P(x)$, so (r1) implies $Px \odot Py \leq P(x \odot Py)$.

M3: $x \oplus Py \leq Px \oplus Py$, so using Lemma 21 (vi) we obtain $P(x \oplus Py) \leq P(Px \oplus Py) = Px \oplus Py$. Using hypothesis (2) we obtain the converse inequality.

In the sequel we shall denote by (*) the following inequality:

$$(^*) Px \oplus Py \leq P(x \oplus Py)$$

COROLLARY 30 *If* $\langle A, P\rangle$ *is a monadic MV-algebra then* $\langle A, P\rangle$ *is an* S_5 *modal MV-algebra.*

Proof By Propositions 25 and 29.

COROLLARY 31 *If* $\langle A, P\rangle$ *is an* S_5 *modal MV-algebra and P satisfies* (*) *then* $\langle A, P\rangle$ *is a monadic MV-algebra.*

Proof By Lemma 26 and Proposition 29.

PROPOSITION 32 *If* $\langle A, P\rangle$ *is a closure MV-algebra and P has the property* (*) *then the following are equivalent*:

(*i*) *P is an existential quantifier on A*,
(*ii*) $P(A)$ *is an MV-subalgebra of A*,
(*iii*) $P\overline{Px} = \overline{Px}$.

Proof

(i) $\Rightarrow$ (ii) From properties (iii), (iv) of Lemma 8.
(ii) $\Rightarrow$ (iii) It is obvious that $x \in P(A)$ iff $Px = x$. Because $P(A)$ is subalgebra we have that $\overline{Px} \in P(A)$, so $P(\overline{Px}) = \overline{Px}$.
(iii) $\Rightarrow$ (i) By Proposition 29.

Example 33

(a) Let $X = \{1, 2\}$ with the natural order relation. Following Example 24 we obtain that $\langle L_2^X, P\rangle$ is a closure Boolean algebra, where $P(x, y) = (x \vee y, y)$. It is obvious that (*) is satisfied, but (iii) from the above proposition is not, because $P\overline{P(1, 0)} = P(0, 1) = (1, 1) \neq \overline{P(1, 0)} = (0, 1)$.
(b) Let $X = \{1, 2, 3\}$ with the natural order. Then $\langle L_3^X, P\rangle$ with P defined as in Example 24, is a closure MV-algebra that satisfies neither (*) nor (iii) from the above proposition. If we take $x = (0, 1/2, 0)$ and $y = (1/2, 0, 0)$ then $Px = (1/2, 1/2, 0)$ and $Py = (1/2, 0, 0)$.

One can see that $Px \oplus Py = (1, 1/2, 0) \nleq (1/2, 1/2, 0) = P(x \oplus Py)$. If $x = (1, 1/2, 0)$ then $Px = (1, 1/2, 0)$, so $P\overline{Px} = P(0, 1/2, 1) = (1, 1, 1) \neq \overline{Px} = (0, 1/2, 1)$.

Open question. Find an example of a closure algebra satisfying (iii) and not (*).

In the sequel $\langle A, P \rangle$ will be a closure MV-algebra.

DEFINITION 34 An ideal I of A is said to be a *P-ideal* if for every $x \in A$ the following condition is satisfied:

$$x \in I \Rightarrow Px \in I.$$

Remark 35 The assertion stated in Remark 12 can be extended to closure MV-algebras: if X is a set in a closure algebra A then the P-ideal generated by X is

$$(X]_P = \{y \in A | y \leq Px_1 \oplus \cdots \oplus Px_n, x_1, \ldots, x_n \in X\}.$$

The proof uses Lemma 21 (vi).

Definitions of proper and maximal ideal are given as usual.

DEFINITION 36 Let I be a P-ideal in a closure algebra $\langle A, P \rangle$. We say that I is *P-prime* provided it is proper and for all $x, y \in A$ the condition $Px \wedge Py \in I$ implies that $Px \in I$ or $Py \in I$.

PROPOSITION 37 *Every maximal P-ideal is P-prime.*

Proof Let I be a maximal P-ideal and x, y two elements such that $Px \wedge Py \in I$ and $Px \notin I$. Because I is a maximal P-ideal it follows that $(I \cup \{Px\}]_P = A$, so there is an element $e \in I$ and a natural number k such that $Py \leq e \oplus kPx$. Using (r2) we obtain $Py = Py \wedge (e \oplus kPx) \leq (Py \wedge e) \oplus k(Py \wedge Px)$. We have that $Py \wedge e \in I$ (because $Py \wedge e \leq e \in I$) and $Py \wedge Px \in I$ from hypothesis, so $Py \in I$.

PROPOSITION 38 *Let I be a proper P-ideal of the closure MV-algebra $\langle A, P \rangle$. The following assertions are equivalent:*

(*i*) *I is a maximal P-ideal,*
(*ii*) *if $x \in A$ then $Px \in I$ or $\overline{kPx} \in I$ for some $k \in N$.*

Proof (i) $\Rightarrow$ (ii) Let $x \in A$ such that $Px \notin I$. It follows that $(I \cup \{Px\}]_P = A$, so there is an element $e \in I$ and a natural number k such that $e \oplus kPx = 1$. It follows that $\overline{kPx} \leq e$ so $\overline{kPx} \in I$.

(ii) $\Rightarrow$ (i) Suppose I is not maximal. Then there is a proper P-ideal U such that $I \subset U$. Let $x \in U - I$. Because both I and U are P-ideals we obtain $Px \in U - I$. From hypothesis there is a natural number k such that $\overline{kPx} \in I \subset U$ and because $Px \in U$ and U is an ideal we have that $kPx \in U$. We obtain that both an element and its negation are in U, so we contradict the fact that U is proper.

COROLLARY 39 *Let I be a proper P-ideal of the closure MV_n-algebra $\langle A, P \rangle$. The following assertions are equivalent:*

(*i*) *I is a maximal P-ideal,*
(*ii*) *if $x \in A$ then $Px \in I$ or $(Px)^\bullet \in I$.*

Proof We observe that if there is a natural number k such that $\overline{kPx} \in I$ then $(Px)^\bullet \in I$. For the properties of MV$_n$-algebras imply that $kPx \leq (n-1)Px$ for every natural number k (if $k \geq n-1$ then $kPx = (n-1)Px$), so $(Px)^\bullet \leq \overline{kPx}$ and $(Px)^\bullet \in I$. The rest of the proof follows from the previous proposition.

PROPOSITION 40 *Let $\langle A, P \rangle$ be a closure MV-algebra. Then for every $a \in A$, $a \neq 0$ there is a P-prime P-ideal I such that $a \notin I$.*

Proof Let I be a P-ideal that is maximal with respect to the property $a \notin I$. We shall prove that I is P-prime. Let x, $y \in A$ such that $Px \wedge Py \in I$. Suppose that $Px \notin I$ and $Py \notin I$. It follows that $a \in (I \cup \{Px\}]_P \cap (I \cup \{Py\}]_P$, so there are f, $g \in I$ and k, l natural numbers such that $a \leq f \oplus kPx$ and $a \leq g \oplus lPy$. Hence $a \leq (f \oplus kPx) \wedge (g \oplus lPy)$ and using (r2) we obtain $a \leq (f \wedge g) \oplus k(g \wedge Px) \oplus l(f \wedge Py) \oplus kl(Px \wedge Py) \in I$, so $a \in I$ which contradicts the choice of I. It follows that $Px \in I$ or $Py \in I$, so I is P-prime.

COROLLARY 41 *In every closure MV-algebra $\langle A, P \rangle$ the intersection of all P-prime P-ideals is $\{0\}$.*

DEFINITION 42 A closure MV-algebra $\langle A, P \rangle$ is called *semi-simple* if the intersection of all maximal P-ideals is $\{0\}$.

LEMMA 43 *Let $\langle A, P \rangle$ be a closure MV-algebra. If every P-prime P-ideal is maximal then $\langle A, P \rangle$ is semi-simple.*

Proof By Corollary 41.

PROPOSITION 44 *Let $\langle A, P\rangle$ be a closure MV_n-algebra. The following are equivalent:*

(*i*) *every P-prime P-ideal is maximal,*
(*ii*) *$\langle A, P\rangle$ is semi-simple.*

Proof (i) $\Rightarrow$ (ii) is the previous lemma.

(ii) $\Rightarrow$ (i) Suppose that $\langle A, P\rangle$ is semi-simple and there exists a P-prime P-ideal I that is not maximal. Then there is a maximal P-ideal U such that $I \subset U$. Consider $x \in U - I$, so $Px \in U - I(1)$. We infer that $(Px)^{\bullet} \notin U$ (otherwise U is not proper) and that $x \neq 0$. From the hypothesis there is a maximal P-ideal M such that $x \notin M$ and from Corollary 39 it follows that $(Px)^{\bullet} \in M$. So $(Px)^{\bullet} \in M - U$ and $P((Px)^{\bullet}) \in M - U$ (2). We observe that $Px \wedge P((Px)^{\bullet}) \neq 0$, otherwise $Px \wedge P((Px)^{\bullet}) \in I$ and because I is P-prime $Px \in I$ or $P((Px)^{\bullet}) \in I \subseteq U$ which contradicts (1) or (2). From the hypothesis it follows that there is a maximal P-ideal G such that $Px \wedge P((Px)^{\bullet}) \notin G$, so $Px \notin G$ and $P((Px)^{\bullet}) \notin G$ (3). But from Corollary 39 we know that if $Px \notin G$ then $(Px)^{\bullet} \in G$, so $P((Px)^{\bullet}) \in G$ (because G is a P-ideal) and we contradict (3). Thus the proposition is proved.

4. MONADIC MV$_n$-ALGEBRAS AND MONADIC LM$_n$-ALGEBRAS

In this section we shall establish some algebraic connections between the monadic MV$_n$-algebras and the monadic n-valued Lukasiewicz–Moisil algebras for $n = 3$ and $n = 4$. These are monadic versions of the results obtained for $n = 3$ in [10] and for $n = 4$ in [8].

Let n be a natural number, $n \geq 2$ and $I = \{1, \dots, n-1\}$.

An *n-valued Lukasiewicz–Moisil algebra* (or an LM$_n$-algebra) is a structure $\langle L, \vee, \wedge, 0, 1, {}^{-}, \varphi_1, \dots, \varphi_{n-1}\rangle$, where $\langle L, \vee, \wedge, 0, 1\rangle$ is a De Morgan algebra and $\varphi_1, \dots, \varphi_{n-1}$: $L \to L$ are bounded lattice morphisms such that:

1. $\varphi_i x \vee \overline{\varphi_i x} = 1$, $\varphi_i x \wedge \overline{\varphi_i x} = 0$ for all $i \in I$;
2. $\varphi_i \varphi_j x = \varphi_j x$ for all $i, j \in I$;
3. $\varphi_i \bar{x} = \overline{\varphi_j x}$ for all $i, j \in I$ with $i + j = n$;
4. $\varphi_1 x \leq \varphi_2 x \leq \dots \leq \varphi_{n-1} x$
5. *The determination principle*: $\varphi_i x = \varphi_i y$ for all $i \in I$ implies $x = y$. As consequences of the determination principle we obtain:
6. $x \leq y$ iff $\varphi_i x \leq \varphi_i y$ for all $i \in I$;
7. $\varphi_1 x \leq x \leq \varphi_{n-1} x$.

We denote by $C(L)$ the set of all complemented elements of L and we call it the *center* of L. It is easy to see that $(C(L), \vee, \wedge, N, 0, 1)$ is a Boolean algebra. It is also well known that $C(L) = \{x \mid \varphi_i x = x, i = 1, \dots, n-1\}$.

The following definition introduces the monadic LM$_n$-algebras. More informations can be founded in [9] (for the 3-valued case) and [3].

DEFINITION 45 A *monadic LM$_n$*-algebra is a pair $\langle L, \exists\rangle$, where L is an LM$_n$-algebra and $\exists$: $L \to L$ is a mapping which satisfies the following axioms:

LM0. $\exists 0 = 0$,
LM1. $x \leq \exists x$,
LM2. $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,
LM3. $\exists \varphi_i x = \varphi_i \exists x$, for $i = 1, \dots, n-1$.

LEMMA 46 [3] *In every monadic LM$_n$-algebra the following properties are satisfied:*

(*i*) $\exists 1 = 1$,
(*ii*) $\exists\exists x = \exists x$,
(*iii*) $x \in \exists(L) \Leftrightarrow x = \exists x$,
(*iv*) $x \leq y \Rightarrow \exists x \leq \exists y$,
(*v*) $x \in C(L) \Rightarrow \exists x \in C(L)$,
(*vi*) $\exists(x \vee y) = \exists x \vee \exists y$,
(*vii*) $\exists\overline{\exists x} = \overline{\exists x}$.

LEMMA 47 *In every monadic LM$_n$-algebra*

$$\exists(x \wedge \varphi_i x) = \exists x \wedge \exists \varphi_i x, \quad \textit{for every } i \in I.$$

Proof We shall use the determination principle. If $i \in I$ we shall prove that $\varphi_j(\exists(x \wedge \varphi_i x)) = \varphi_j(\exists x \wedge \exists \varphi_i x)$ for every $j \in I$ by using LM$_3$ and the definition of LM$_n$ algebras. If $j \leq i$ it follows that $\varphi_j(\exists(x \wedge \varphi_i x)) = \exists \varphi_j(x \wedge \varphi_i x) = \exists(\varphi_j x \wedge \varphi_i x) = \exists \varphi_j x = \varphi_j \exists x \wedge \varphi_i \exists x = \varphi_j(\exists x \wedge \varphi_i \exists x) = \varphi_j(\exists x \wedge \exists \varphi_i x)$.

If $j > i$ then $\varphi_j(\exists(x \wedge \varphi_i x)) = \exists\varphi_j(x \wedge \varphi_i x) = \exists(\varphi_j x \wedge \varphi_i x) = \exists\varphi_i x = \varphi_j \exists x \wedge \varphi_i \exists x = \varphi_j(\exists x \wedge \varphi_i \exists x) = \varphi_j(\exists x \wedge \exists\varphi_i x)$ and our proof is complete.

PROPOSITION 48 *Monadic MV_3-algebras are polynomially equivalent with monadic LM_3-algebras.*

Proof Let $A = \langle A, \oplus, \odot, ^-, 0, 1\rangle$ be an MV_3-algebra and $\langle A, \exists\rangle$ a monadic MV_3-algebra. If we define $\varphi_1 x = x \odot x$ and $\varphi_2 x = x \oplus x$ then from [10] it follows that $\langle A, \vee, \wedge, 0, 1, ^-, \varphi_1, \varphi_2\rangle$ is an LM_3-algebra. From Definition 7 (M0, M1, M4, M5) and Lemma 8 (vi) it is quite obvious that $\langle A, \exists\rangle$ is a monadic LM_3-algebra.

For the converse affirmation let $\langle A, \vee, \wedge, 0, 1, ^-, \varphi_1, \varphi_2\rangle$ be an LM_3-algebra and $\langle A, \exists\rangle$ a monadic LM_3-algebra. If we define

$$x \oplus y = (\varphi_2 x \vee y) \wedge (x \vee \varphi_2 y) = \varphi_2(x \wedge y) \vee x \vee y \quad \text{and}$$
$$x \odot y = (\varphi_1 x \wedge y) \vee (x \wedge \varphi_1 y) = \varphi_1(x \vee y) \wedge x \wedge y$$

it follows that $\langle A, \oplus, \odot, ^-, 0, 1\rangle$ is an MV_3-algebra [10]. From Definition 45 (LM0, LM1, LM3) and the easy remark that $x \odot x = \varphi_1 x$ and $x \oplus x = \varphi_2 x$, we see that in order to prove that $\langle A, \exists\rangle$ is a monadic MV_3-algebra it remains to verify that M2 and M3 from Definition 7 are satisfied:

$$\exists(x \oplus \exists y) = \exists(\varphi_2(x \wedge \exists y) \vee x \vee \exists y) = \exists\varphi_2(x \wedge \exists y) \vee \exists x \vee \exists\exists y = \varphi_2(\exists x \wedge \exists y) \vee \exists x \vee \exists y = \exists x \oplus \exists y,$$
$$\exists(x \odot \exists y) = \exists((\varphi_1 x \wedge \exists y) \vee (x \wedge \varphi_1 \exists y)) = \exists(\varphi_1 x \wedge \exists y) \vee \exists(x \wedge \varphi_1 \exists y) = (\exists\varphi_1 x \wedge \exists y) \vee (\exists x \wedge \exists\varphi_1 \exists y) = (\varphi_1 \exists x \wedge \exists y) \vee (\exists x \wedge \varphi_1 \exists y) = \exists x \odot \exists y.$$

PROPOSITION 49 *Every monadic MV_4-algebra has a structure of monadic LM_4-algebra with respect to $\varphi_1 x = x^3$, $\varphi_2 x = (2x)^3 = 3x^2$ and $\varphi_3 x = 3x$.*

Proof Let $A = \langle A, \oplus, \odot, ^-, 0, 1\rangle$ be an MV_4-algebra and $\langle A, \exists\rangle$ a monadic MV_4-algebra. From [8] it follows that $\langle A, \vee, \wedge, 0, 1, ^-, \varphi_1, \varphi_2, \varphi_3\rangle$ is an LM_4-algebra. To show that $\langle A, \exists\rangle$ is a monadic LM_4-algebra it remains to prove LM3 from Definition 45:

$$\varphi_1 \exists x = (\exists x)^3 = (\exists x)^2 \odot (\exists x)^2 = \exists(x)^2 \odot \exists(x^2) = \exists(x^4) = \exists(x^3) = \exists\varphi_1 x,$$
$$\varphi_2 \exists x = (\exists x \oplus \exists x)^3 = (\exists 2x)^3 = (\exists 2x)^4 = \exists(2x)^4 = \exists(2x)^3 = \exists\varphi_2 x,$$
$$\varphi_3 \exists x = 3\exists x = 2\exists x \oplus 2\exists x = \exists(2x) \oplus \exists(2x) = \exists 4x = \exists 3x = \exists\varphi_3 x.$$

We have used the fact that in every MV_4-algebra $4x = 3x$ and $x^4 = x^3$.

Let $\langle A, \vee, \wedge, 0, 1, ^-, \varphi_1, \varphi_2, \varphi_3\rangle$ be an LM_4-algebra and $\langle A, \exists\rangle$ a monadic LM_4-algebra. If we define

$$x \oplus y = (x \vee \varphi_3 y) \wedge (y \vee \varphi_3 x) \wedge (\bar{x} \vee \bar{y} \vee \varphi_2 x \vee \varphi_2 y) \quad \text{and}$$
$$x \odot y = (x \wedge \varphi_1 y) \vee (y \wedge \varphi_1 x) \vee (\bar{x} \wedge \bar{y} \wedge \varphi_2 x \wedge \varphi_2 y)$$

then it follows that $\langle A, \oplus, \odot, ^-, 0, 1\rangle$ is an MV_4-algebra [8].

LEMMA 50 *Let $\langle A, \exists\rangle$ be a monadic LM_4-algebra. For the above operations $\oplus$ and $\odot$ the following properties hold for every $x, y \in A$:*

(*i*) $\exists 0 = 0$,
(*ii*) $x \leq \exists x$,
(*iii*) $\exists(x \odot \exists y) = \exists x \odot \exists y$,
(*iv*) $\exists(x \odot x) = \exists x \odot \exists x$,
(*v*) $\exists(x \oplus x) = \exists x \oplus \exists x$,
(*vi*) $\exists(x \oplus \exists y) \leq \exists x \oplus \exists y$,
(*vii*) $\overline{\exists x} \leq \exists\bar{x}$,
(*viii*) $\exists x \odot \overline{\exists y} \leq \exists(x \odot \bar{y})$,
(*ix*) $\exists x \wedge \overline{\exists x} \leq \exists(x \wedge \bar{x})$.

Proof (i) and (ii) hold be definition. In order to prove (iii)–(v) we establish some preliminary results.

(a) $\exists(\exists x \odot \exists y) = \exists x \odot \exists y$

$\exists(\exists x \odot \exists y) = \exists(\exists x \wedge \exists\varphi_1 y) \vee \exists(\exists y \wedge \exists\varphi_1 x) \vee \exists(\overline{\exists x} \wedge \overline{\exists y} \wedge \exists\varphi_2 x \wedge \varphi_2 y) = (\exists x \wedge \varphi_1 \exists y) \vee (\exists y \wedge \varphi_1 \exists x) \vee (\overline{\exists x} \wedge \overline{\exists y} \wedge \varphi_2 \exists x \wedge \varphi_2 \exists y) = \exists x \odot \exists y.$

(b) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$

$\exists(\exists x \oplus \exists y) = \exists[(\exists x \vee \varphi_3 \exists y) \wedge (\exists y \vee \varphi_3 \exists x) \wedge (\overline{\exists x} \vee \overline{\exists y} \vee \varphi_2 \exists x \vee \varphi_2 \exists y)] = \exists[\exists(x \vee \varphi_3 y) \wedge \exists(y \vee \varphi_3 x) \wedge \exists(\overline{\exists x} \vee \overline{\exists y} \vee \varphi_2 x \vee \varphi_2 y)] = \exists(x \vee \varphi_3 y) \wedge \exists(y \vee \varphi_3 x) \vee (\overline{\exists x} \vee \overline{\exists y} \vee \varphi_2 \exists x \vee \varphi_2 \exists y) = \exists x \oplus \exists y$

(iii) From $x \odot \exists y \leq \exists x \odot \exists y$ it follows that $\exists(x \odot \exists y) \leq \exists(\exists x \odot \exists y) = \exists x \odot \exists y$. For the converse inequality we have:

$$\begin{aligned}\exists x \odot \exists y &= (\exists x \wedge \varphi_1 \exists y) \vee (\exists y \wedge \varphi_1 \exists x) \vee (\overline{\exists x} \wedge \overline{\exists y} \wedge \varphi_2 \exists x \wedge \varphi_2 \exists y)\\ &= \exists(x \wedge \varphi_1 \exists y) \vee (\exists y \wedge \varphi_1 x) \vee (\exists\overline{\exists x} \wedge \exists\varphi_2 x \wedge \overline{\exists y} \wedge \varphi_2 \exists y)\\ &= \exists(x \wedge \varphi_1 \exists y) \vee \exists(\exists y \wedge \varphi_1 x) \vee (\exists(\overline{\exists x} \wedge \varphi_2 x) \wedge \overline{\exists y} \wedge \varphi_2 \exists y)\end{aligned}$$

$$\leq \exists(x \wedge \varphi_1\exists y) \vee \exists(\exists y \wedge \varphi_1 x) \vee (\exists(\bar{x} \wedge \varphi_2 x) \wedge \exists\overline{\exists y} \wedge \varphi_2\exists y)$$
$$= \exists(x \wedge \varphi_1\exists y) \vee \exists(\exists y \wedge \varphi_1 x) \vee \exists(\bar{x} \wedge \varphi_2 x \wedge \overline{\exists y} \wedge \varphi_2\exists y)$$
$$= \exists[(x \wedge \varphi_1\exists y) \vee (\exists y \wedge \varphi_1 x) \vee (\bar{x} \wedge \varphi_2 x \wedge \overline{\exists y} \wedge \varphi_2\exists y)]$$
$$= \exists(x \odot \exists y)$$

(iv) From $x\odot x \leq \exists x \odot \exists x$ it follows that $\exists(x\odot x)\leq\exists(\exists x\odot\exists x) = \exists x\odot\exists x$. For the converse inequality we observe that $x\odot x = \varphi_1 x \vee (\bar{x} \wedge \varphi_2 x)$. We obtain $\exists x\odot\exists x = \varphi_1\exists x \vee (\overline{\exists x} \wedge \varphi_2\exists x) = \varphi_1\exists x \vee \exists(\overline{\exists x}\wedge\varphi_2 x) = \exists\ [\varphi_1 x \vee (\overline{\exists x}\wedge\varphi_2 x)]$, so $\exists x\odot\exists x \leq \exists[\varphi_1 x\vee(\bar{x}\wedge\varphi_2 x)] = \exists(x\odot x)$.

(v) From $x\oplus x\leq \exists x\oplus\exists x$ it follows that $\exists(x\oplus x)\leq\exists(\exists x\oplus\exists x) = \exists x\oplus\exists x$. For the converse inequality we observe that $x\oplus x = \varphi_3 x\wedge(\bar{x}\vee\varphi_2 x) = \varphi_2 x\vee(\bar{x}\wedge\varphi_3 x)$. We have $\exists x\oplus\exists x = \varphi_2\exists x\vee(\overline{\exists x}\wedge\varphi_3\exists x) = \varphi_2\exists x\vee\exists\ (\overline{\exists x}\wedge\varphi_3 x) = \exists[\varphi_2 x\vee(\overline{\exists x}\wedge\varphi_3 x)]$, so $\exists x\oplus\exists x\leq\exists[\varphi_2 x\vee(\bar{x}\wedge\varphi_3 x)] = \exists(x\oplus x)$.

(vi) From $x\oplus\exists y\leq\exists x\oplus\exists y$ we obtain $\exists(x\oplus\exists y)\leq\exists(\exists x\oplus\exists y) = \exists x\oplus\exists y$.

(vii) From (ii) we have $x\leq\exists x$, so $\overline{\exists x}\leq\bar{x}$. We apply (ii) for $\bar{x}$ and we obtain $\bar{x}\leq\exists\bar{x}$, hence $\overline{\exists x}\leq\exists\bar{x}$ by transitivity.

(viii) From $x\odot\overline{\exists y}\leq x\odot\bar{y}$ it follows that $\exists x\odot\overline{\exists y} = \exists(x\odot\overline{\exists y}) \leq \exists(x\odot\bar{y})$.

(ix) $\exists x\wedge\overline{\exists x} = (\exists x\oplus\exists x)\odot\overline{\exists x} = \exists(x\oplus x)\odot\overline{\exists x}$. From (viii) it follows that $\exists x\wedge\overline{\exists x}\leq\exists((x\oplus x)\odot\bar{x}) = \exists(x\wedge\bar{x})$.

PROPOSITION 51 *Every monadic LM_4-algebra has a structure of monadic MV_4-algebra with respect to the operations $\oplus$ and $\odot$ defined by*

$$x\oplus y = (x\vee\varphi_3 y)\wedge(y\vee\varphi_3 x)\wedge(\bar{x}\vee\bar{y}\vee\varphi_2 x\vee\varphi_2 y) \quad \textit{and}$$
$$x\odot y = (x\wedge\varphi_1 y)\vee(y\wedge\varphi_1 x)\vee(\bar{x}\wedge\bar{y}\wedge\varphi_2 x\wedge\varphi_2 y).$$

Proof Let $\langle A,\vee,\wedge,0,1,^{-},\varphi_1,\varphi_2,\varphi_3\rangle$ be an LM_4-algebra and $\langle A,\exists\rangle$ a monadic LM_4-algebra. From [8] it follows that $\langle A,\oplus,\odot,^{-},0,1\rangle$ is an MV_4-algebra. From Lemma 50(i–v) it follows that axioms M0–M2, M4, M5 are satisfied. To prove that $\langle A,\exists\rangle$ is a monadic MV_4-algebra we must show that axiom M3 is also satisfied. One inequality is established in Lemma 50(vi). It remains to prove that $\exists x\oplus\exists y\leq\exists(x\oplus\exists y)$.

Using distributivity, the Property 4 from the definition of LM_n-algebra and the consequence 7 it follows that

$$\exists x\oplus\exists y = (\exists x\vee\varphi_3\exists y)\wedge(\exists y\vee\varphi_3\exists x)\wedge(\overline{\exists x}\vee\overline{\exists y}\vee\varphi_2\exists x\vee\varphi_2\exists y)$$
$$= (\exists x\wedge\exists y\wedge(\overline{\exists x}\vee(\overline{\exists y}))\vee(\exists x\wedge\exists y\vee\varphi_2(\exists x\vee\exists y))\vee$$
$$(\exists x\wedge(\overline{\exists x}\vee\overline{\exists y}))\vee(\exists x\wedge\varphi_2(\exists x\vee\exists y))\vee$$
$$(\exists y\wedge(\overline{\exists x}\vee\overline{\exists y}))\vee(\exists y\wedge\varphi_2(\exists x\vee\exists y))\vee$$
$$(\varphi_3(\exists x\wedge\exists y)\wedge(\overline{\exists x}\vee\overline{\exists y}))\vee$$
$$(\varphi_3\exists y\wedge\varphi_2\exists x)\vee(\varphi_3\exists x\wedge\varphi_2\exists y).$$

By absorption and again distributivity we obtain $\exists x\oplus\exists y = a\vee b\vee c\vee d$ where

$$a = (\exists x\vee\exists y)\wedge(\overline{\exists x}\vee\overline{\exists y})$$
$$= (\exists x\wedge\overline{\exists x})\vee(\exists x\wedge\overline{\exists y})\vee(\exists y\wedge\overline{\exists x})\vee(\exists y\wedge\overline{\exists y}),$$
$$b = (\exists x\vee\exists y)\wedge\varphi_2(\exists x\vee\exists y),$$
$$c = \varphi_3(\exists x\wedge\exists y)\wedge(\overline{\exists x}\vee\overline{\exists y}),$$
$$d = (\varphi_2\exists x\wedge\varphi_3\exists y)\vee(\varphi_3\exists x\wedge\varphi_2\exists y).$$

In the sequel we shall decompose in the same manner the expression $\exists(x\oplus\exists y)$.

$$x\oplus\exists y = (x\vee\varphi_3\exists y)\wedge(\exists y\vee\varphi_3 x)\wedge(\bar{x}\vee\overline{\exists y}\vee\varphi_2 x\vee\varphi_2\exists y)$$
$$= (x\vee\exists y\vee\varphi_3(x\wedge\exists y))\wedge(\bar{x}\vee\overline{\exists y}\vee\varphi_2(x\vee\exists y))$$
$$= ((x\vee\exists y)\wedge(\bar{x}\vee\overline{\exists y}))\vee(\varphi_3(x\wedge\exists y)\wedge(\bar{x}\vee\overline{\exists y}))\vee$$
$$((x\vee\exists y)\wedge\varphi_2(x\vee\exists y))\vee(\varphi_3(x\wedge\exists y)\wedge\varphi_2(x\vee\exists y)).$$

If we denote

$$a' = \exists[(x\vee\exists y)\wedge(\bar{x}\vee\overline{\exists y})]$$
$$b' = \exists[(x\vee\exists y)\wedge\varphi_2(x\vee\exists y)],$$
$$c' = \exists[\varphi_3(x\wedge\exists y)\wedge(\bar{x}\vee\overline{\exists y})],$$
$$d' = \exists[\varphi_3(x\wedge\exists y)\wedge\varphi_2(x\vee\exists y)],$$

then Lemma 46(vi) implies $\exists(x\oplus\exists y) = a'\vee b'\vee c'\vee d'$.

We shall prove that $a\leq a'$, $b = b'$, $c\leq c'$ and $d = d'$.

Using distributivity, Lemma 46(vi), (vii) and LM2. We obtain

$$a' = \exists[x \wedge \bar{x}) \vee (\bar{x} \wedge \exists y) \vee (x \wedge \overline{\exists y}) \vee (\exists y \wedge \overline{\exists y})]$$
$$= \exists(x \wedge \bar{x}) \vee (\exists\bar{x} \wedge \exists y) \vee (\exists x \wedge \overline{\exists y}) \vee (\exists y \wedge \overline{\exists y}).$$

By Lemma 50 (ix), (vii) it follows that

$$\exists x \wedge \overline{\exists x} \leq \exists(x \wedge \bar{x}),$$
$$\exists y \wedge \overline{\exists x} \leq \exists y \wedge \exists\bar{x},$$

so $a \leq a'$.

Lemma 47 implies that

$$b' = \exists[(x \vee \exists y) \wedge \varphi_2(x \vee \exists y)]$$
$$= \exists(x \vee \exists y) \wedge \exists\varphi_2(x \vee \exists y)$$
$$= (\exists x \vee \exists y) \wedge \varphi_2(\exists x \vee \exists y)$$
$$= b.$$

Using Lemma 46 (vii), (vi), LM2, LM1 and Lemma 2(i) it follows that

$$c = \varphi_3(\exists x \wedge \exists y) \wedge (\overline{\exists x} \wedge \overline{\exists y})$$
$$= \exists\varphi_3(x \wedge \exists y) \wedge \exists(\overline{\exists x} \vee \overline{\exists y})$$
$$= \exists[\varphi_3(x \wedge \exists y) \wedge \exists(\overline{\exists x} \vee \overline{\exists y})]$$
$$= \exists[\varphi_3(x \wedge \exists y) \wedge (\overline{\exists x} \vee \overline{\exists y})]$$
$$\leq \exists[\varphi_3(x \wedge \exists y) \wedge (\bar{x} \vee \overline{\exists y})]$$
$$= c'.$$

Using distributivity, the Property 4 from the definition of LM_n-algebra, LM3 and Lemma 46 (ii), (vi) we obtain

$$d' = \exists[\varphi_3(x \wedge \exists y) \wedge \varphi_2(x \vee \exists y)]$$
$$= \exists[(\varphi_2 x \wedge \varphi_3\exists y) \vee (\varphi_3 x \wedge \varphi_2\exists y)]$$
$$= [(\varphi_2\exists x \wedge \varphi_3\exists y) \vee (\varphi_3\exists x \wedge \varphi_2\exists y)]$$
$$= d.$$

It follows that $\exists x \oplus \exists y = a \vee b \vee c \vee d \leq a' \vee b' \vee c' \vee d' = \exists(x \oplus \exists y)$, so our proof is finished.

COROLLARY 52 *Monadic MV_4-algebras are polynomially equivalent with monadic LM_4-algebras.*

Proof From Propositions 49 and 51.

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