Scientific report for the period December 2011 - November 2012

1 Publications

1.1 Published papers

P1 U. Kohlenbach, L. Leuştean, Effective metastability of Halpern iterates in CAT(0) spaces, Advances in Mathematics, Vol. 231 (2012), 2526-2556.

P2 U. Kohlenbach, L. Leuştean, On the computational content of convergence proofs via Banach limits, Philosophical Transactions of the Royal Society A, Vol. 370 (2012), 3449–3463. Theme Issue *The foundations of computation, physics and mentality: the Turing legacy*.

1.2 Papers accepted for publication

P3 D. Ariza-Ruiz, L. Leuştean, G. López-Acedo, Firmly nonexpansive mappings in classes of geodesic spaces, arXiv:1203.1432v3 [math.FA], 2012; accepted for publication in Transactions of the American Mathematical Society.

1.3 Submitted papers

P4 M. Buliga, Local and global moves on locally planar trivalent graphs, lambda calculus and lambda-Scale, arXiv:1207.0332v1 [cs.LO], 2012.

P5 M. Buliga, Sub-riemannian geometry from intrinsic viewpoint, arXiv:1206.3093v1 [math.MG], 2012.

P6 M. Buliga, Emergent algebras, arXiv:0907.1520v3 [math.RA], 2011.

P7 A. Fernandez-Leon, A. Nicolae, Averaged alternating reflections in geodesic spaces, arXiv:1211.4958v2 [math.OC], 2012.

P8 A. Nicolae, Asymptotic behavior of averaged and firmly nonexpansive mappings in geodesic spaces, arXiv:1210.2105 [math.FA], 2012.

1.4 Preprints

P9 M. Buliga, Graphic lambda calculus and knot diagrams, arXiv:1211.1604 [math.GT], 2012.

P10 M. Buliga, λ-Scale, a lambda calculus for spaces with dilations, arXiv:1205.0139v3 [cs.LO], 2012.

2 Directions of research

2.1 Effective rates of metastability for nonlinear ergodic averages

The papers **P1** and **P2** present effective rates of metastability (in the sense of Tao [15, 16]) for nonlinear generalizations of the classical von Neumann ergodic theorem, obtained by Saejung [13] and Shioji and Takahashi [14]. These results constitute a significant extension of the actual context of proof mining, as both Saejung's and Shioji-Takahashi's proofs make use of Banach limits, whose existence requires the use of the axiom of choice. In **P1** we develop a method to convert such proofs into elementary ones which no longer use Banach limits.

Let X be a CAT(0) space, $C \subseteq X$ a convex subset X and $T : C \to C$ be nonexpansive. The *Halpern iteration* starting from $x \in C$ is defined as follows:

$$x_0 := x, \quad x_{n+1} := \lambda_{n+1} u \oplus (1 - \lambda_{n+1}) T x_n,$$

where $x, u \in C$ and $(\lambda_n)_{n \geq 1}$ is a sequence in $[0, 1]$. In a geodesic space (X, d), given a geodesic segment $[x, y]$ and $\alpha \in [0, 1]$, we denote by $(1 - \alpha)x \oplus \alpha y$ the unique point $z \in [x, y]$ satisfying $d(x, z) = \alpha d(x, y)$ and $d(y, z) = (1 - \alpha) d(x, y)$.

One can see easily that if X is a Hilbert space, T is linear and $\lambda_n = \frac{1}{n+1}$, then (x_n) coincides with the Cesàro mean. The most important result on the convergence of Halpern iterations in Hilbert spaces was obtained by Wittmann [17]. The following theorem, proved by Saejung [13] using Banach limits, generalizes Wittmann's theorem to CAT(0) spaces.

Teorema 2.1. *Let C be a bounded closed convex subset of a complete CAT(0) space X and $T : C \to C$ a nonexpansive mapping. Assume that (λ_n) satisfies*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_{n+1} = \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| \textit{ converges.}$$

Then for any $u, x \in C$, (x_n) converges to a fixed point of T.

While one cannot expect to get effective rates of convergence for the sequence (x_n) in the above theorem, an effective and highly uniform rate of metastability is guaranteed to exist, after the elimination of Banach limits from the proof, by [10, Theorem 3.7.3].

Teorema 2.2. *In the hypotheses of Theorem 2.1, let α be a rate of convergence of (λ_n), β be a Cauchy modulus of $s_n := \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$ and θ be a rate of divergence of $\sum_{n=1}^{\infty} \lambda_{n+1}$. Then for all $\varepsilon \in (0, 2)$ and $g : \mathbb{N} \to \mathbb{N}$,*

$$\exists N \leq \Sigma(\varepsilon, g, M, \theta, \alpha, \beta)\ \forall m, n \in [N, N + g(N)]\ (d(x_n, x_m) \leq \varepsilon),$$

where $M \in \mathbb{Z}_+$ is an upper bound on the diameter of C.

The rate of metastability Σ, extracted in Theorem 4.2 from **P1** does not depend on T, the starting point $x \in C$ and depends weakly on C, via its diameter. We remark that in practical cases, such as $\lambda_n = \frac{1}{n+1}$, the rates α, β, θ are easy to compute.

In the paper **P2** we apply the same method of eliminating Banach limits from the proof given by Shioji and Takahashi to a generalization of Wittmann's theorem to Banach spaces with a uniformly Gâteaux differentiable norm. Furthermore, we prove a logical metatheorem for a class of Banach spaces introduced in **P2** under the name of *spaces with a uniformly continuous duality selection map.*

2.2 Asymptotic behavior of classes of nonlinear mappings

In the papers **P3**, **P7** and **P8** we study important classes of mappings (firmly nonexpansive, averaged and reflections) in different classes of geodesic spaces and we apply proof mining methods to obtain effective results on the asymptotic behaviour of the associated Picard iteration.

Firmly nonexpansive mappings, introduced by Browder [3] in Hilbert spaces and by Bruck [5] in Banach spaces, play an important role in nonlinear analysis and optimization due to their correspondence with maximal monotone operators. Bruck's definition extends immediately to W-hyperbolic spaces X (in the sense of Kohlenbach [10]): a mapping $T : C \subseteq X \to C$ is said to be firmly nonexpansive if for all $x, y \in C$ and all $\lambda \in (0, 1)$,

$$d(Tx, Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty) \quad \text{for all } x, y \in C.$$

A first main result of the paper **P3** is a fixed point theorem for firmly nonexpansive mappings defined on unions of convex closed subsets of a complete UCW-hyperbolic space. UCW-hyperbolic spaces [11] are a class of uniformly convex spaces generalizing both CAT(0) spaces and uniformly convex Banach spaces. A second main result of the paper **P3** generalizes results obtained by Reich and Shafrir [12] in Banach spaces or in the Hilbert ball.

Teorema 2.3. *Let C be a subset of a W-hyperbolic space X and $T : C \to C$ be a firmly nonexpansive mapping. Then for all $x \in C$ and $k \in \mathbb{Z}_+$,*

$$\lim_{n\to\infty} d(T^{n+1}x, T^n x) = \frac{1}{k} \lim_{n\to\infty} d(T^{n+k}x, T^n x) = \lim_{n\to\infty} \frac{d(T^n x, x)}{n} = r_C(T),$$

where $r_C(T) := \inf\{d(x, Tx) \mid x \in C\}$.

Asymptotic regularitaty is a very important concept in the study of the asymptotic behaviour of nonlinear mappings, introduced by Browder and Petryshyn [4]: T is asymptotically regular if $\lim\limits_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in C$. A consequence of the above theorem is the fact that any mapping T with bounded orbits is asymptotically regular.

In **P8** a quantitative version of Theorem 2.3 is proved, having as an immediate consequence an exponential rate of asymptotic regularity for the Picard iteration, the only known even for Banach spaces. Using different methods, rates of asymptotic regularity for UCW-hyperbolic spaces are computed in **P3**, which turn out to be quadratic for CAT(0) spaces or polynomial for L_p spaces, $1 < p < \infty$.

The papers **P7** and **P8** study different known algorithms for the convex feasibility problem in geodesic spaces. Thus, effective rates of asymptotic regularity are obtained in **P8** for the well-known alternating projections method introduced by von Neumann, as well as for a method defined in terms of weighted averages of nonexpansive retractions [8]. The paper **P7** studies the convergence, in spaces of constant curvature, of the algorithm AAR (*Averaged Alternating Reflection*), introduced by Bauschke, Combettes and Luke [1].

2.3 Dilation structures in sub-riemanian geometry and emergent algebras

The paper **P5** was prepared for the course "Metric spaces with dilations and sub-riemannian geometry from intrinsic point of view", CIMPA Research School "Sub-riemannian geometry", Beirut, Liban, January 30 - February 9, 2012. Marius Buliga could not give this course due to weather conditions which prevented him to reach Beirut. The paper is based mainly on [6], with numerous improvements: new theorems (for example Theorem 8.10 giving an intrinsic characterization of riemannian metric spaces), new notions (for example Section 2.5 Curvdimension and curvature), extended proofs (for example the proof of Theorem 8.8 on the gamma-convergence of the length functionals for tempered dilation structures). The paper is 58 pages long and is the most advanced exposition of the subject analysis on metric spaces with dilation structures.

In the paper **P6**, emergent algebras (uniform idempotent right quasigroups) are defined and studied. An emergent algebra is a collection of idempotent right quasigroup operations, called dilations, indexed by a parameter $\varepsilon \in \Gamma$ in an abelian topological group, with the property that certain constructions (compositions of dilations), called approximate operations, converg uniformly, with ε, to an operation of a conical group, or nilpotent. Emergent algebras are an example of approximate algebraic structures, different but related to approximate groups of Breuillard, Green, Tao [2]. On the other hand, dilation structures are examples of local emergent algebras, according to [7, Section 7].

In the paper **P6**, the main results are the structure Theorem 5.2, Theorem 6.1 characterizing distributive emergent algebras algebrelor emergente as being identical with contractible groups and Proposition 6.8, which associates to every symmetric riemannian space an emergent algebra.

2.4 Graphic lambda calculus: dilation structures and logic

The proofs using dilation structures and emergent algebras become much clearer and simpler if we use a graphic formalism; two such formalisms were proposed, one based on trivalent digraphs and another one based on oriented knots diagrams (the second formalism is not surprising, since emergent algebras are related to racks and quandles, algebraic structures appearing in knot theory). The papers **P10**, **P4** and **P9** explore the logical content of these formalisms.

The paper **P10** proposes an extension of untyped lambda calculus, called lambda-Scale, which contains both emergent algebras (Theorem 3.4) and untyped lambda calculus (Theorem 3.5).

In the paper **P4**, lambda-Scale calculus is reformulated in a graphic formalism based on trivalent locally planar graphs and it is proved that certain sets of such graphs, together with certain subsets of the set of rules (called sectors) are equivalent respectively to emergent algebras, lambda-Scale and untyped lambda calculus (Theorems 3.1 and 4.1).

Theorem 4.2 from the paper **P9** proves that a sector of this formalism is equivalent to oriented knot diagrams. The result is interesting in the light of previous efforts (for example Kauffman [9]) to link untyped lambda calculus (under the equivalent form of combinatory algebras) with manipulations of knot diagrams.

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