Scientific report for the period October 2011 - October 2016

1 Publications

1.1 Published papers

1.1.1 U. Kohlenbach, L. Leuştean, Effective metastability of Halpern iterates in CAT(0) spaces, Advances in Mathematics 231 (2012), 2526-2556.

1.1.2 U. Kohlenbach, L. Leuştean, On the computational content of convergence proofs via Banach limits, Philosophical Transactions of the Royal Society A 370 (2012), 3449-3463. Theme Issue *The foundations of computation, physics and mentality: the Turing legacy.*

1.1.3 A. Fernandez-Leon, A. Nicolae, Averaged alternating reflections in geodesic spaces, Journal of Mathematical Analysis and Applications 402 (2013), 558-566.

1.1.4 A. Nicolae, Asymptotic behavior of averaged and firmly nonexpansive mappings in geodesic spaces, Nonlinear Analysis - Theory, Methods & Applications 87 (2013), 102-115.

1.1.5 M. Buliga, Graphic lambda calculus, Complex Systems 22 (2013), 311-360.

1.1.6 L. Leuştean, A. Nicolae, Effective results on compositions of nonexpansive mappings, Journal of Mathematical Analysis and Applications 410 (2014), 902-907.

1.1.7 A. Fernandez-Leon, A. Nicolae, Best proximity pair results for relatively nonexpansive mappings in geodesic spaces, Numerical Functional Analysis and Optimization 35 (2014), 1399-1418.

1.1.8 D. Ariza-Ruiz, L. Leuştean, G. Lopez-Acedo, Firmly nonexpansive mappings in classes of geodesic spaces, Transactions of the American Mathematical Society 366 (2014), 4299-4322.

1.1.9 U. Kohlenbach, L. Leuştean, Addendum to "Effective metastability of Halpern iterates in CAT(0) spaces" [Adv.Math. 231 (5) (2012) 2526-2556], Advances in Mathematics 250 (2014) 650-651.

1.1.10 D. Ivan, L. Leuştean, A rate of asymptotic regularity for the Mann iteration of κ-strict pseudo-contractions, Numerical Functional Analysis and Optimization 36 (2015), 792-798.

1.1.11 L. Leuştean, A. Nicolae, A note on an ergodic theorem in weakly uniformly convex geodesic spaces, Archiv der Mathematik 105 (2015), 467-477.

1.1.12 L. Leuştean, A. Nicolae, A note on an alternative iterative method for nonexpansive mappings, Journal of Convex Analysis 24 (2017).

1.2 Papers accepted for publication

1.2.1 L. Leuştean, A. Nicolae, Effective results on nonlinear ergodic averages in $CAT(\kappa)$ spaces, accepted for publication in Ergodic Theory and Dynamical Systems, DOI: 10.1017/etds.2015.31, 2015.

1.2.2 A. Sipoş, Codensity and Stone spaces, accepted for publication in Mathematica Slovaca, arXiv:1409.1370v3 [math.CT], 2016.

1.2.3 A. Sipoş, A note on the Mann iteration for κ-strict pseudocontractions in Banach spaces, accepted for publication in Numerical Functional Analysis and Optimization, arXiv:1605.02237 [math.FA], 2016.

1.2.4 A. Sipoş, Effective results on a fixed point algorithm for families of nonlinear mappings, accepted for publication in Annals of Pure and Applied Logic, arXiv:1606.03895 [math.FA], 2016.

1.2.5 L. Leuştean, V. Radu, A. Sipoş, Quantitative results on the Ishikawa iteration of Lipschitz pseudo-contractions, accepted for publication in Journal of Nonlinear and Convex Analysis, arXiv:1608.05969 [math.FA], 2016.

1.2.6 U. Kohlenbach, L. Leuştean, A. Nicolae, Quantitative results on Fejér monotone sequences, accepted for publication in Communications in Contemporary Mathematics, arXiv:1412.5563v2 [math.LO], 2016.

1.3 Published abstracts

1.3.1 L. Leuştean (joint with U. Kohlenbach), Recent developments in proof mining, Mathematisches Forschungsinstitut Oberwolfach, Report No. 52/2011, 2982-2984.

1.3.2 L. Leuştean (joint with U. Kohlenbach and A. Nicolae), Proof-theoretic methods in nonlinear analysis III: Quantitative results on Fejér monotone sequences, Mathematisches Forschungsinstitut Oberwolfach, Report No. 52/2014, 2960-2961.

1.3.3 L. Leuştean, Effective results on the asymptotic behavior of nonexpansive iterations, Bulletin of Symbolic Logic 21 (2015), No. 1, p. 79.

1.4 Submitted papers and preprints

1.4.1 M. Buliga, Sub-riemannian geometry from intrinsic viewpoint, arXiv:1206.3093v1 [math.MG], 2012.

1.4.2 A. Sipoş, Ultraproducts and uniform rates of asymptotic regularity and metastability, preprint, 2015.

1.4.3 A. Sipoş, Proof mining in L^p spaces, arXiv:1609.02080 [math.LO], 2016.

1.4.4 L. Leuştean, A. Nicolae, A. Zaharescu, Barycenters in uniformly convex geodesic spaces, arXiv:1609.02589 [math.MG], 2016.

1.4.5 M. Buliga, Geometric Ruzsa triangle inequality in metric spaces with dilations, arXiv:1304.3358 [math.CO], 2016.

1.4.6 L. Leuştean, A. Nicolae, A. Sipoş, Quantiative results on generalizations of the proximal point algorithm, draft, 2016.

2 Conference and Seminar Talks

- L. Leuştean, Proof mining and applications, Middlesex University London, 19.09.2016.
- A. Sipoş, Proof mining and positive-bounded logic, Colloquium Logicum 2016, Hamburg, 10 - 12.09.2016.
- A. Sipoş, Proof mining and positive-bounded logic, Logic Colloquium 2016, Leeds, 31.07. - 06.08.2016.
- A. Sipoş, Effective results on algorithms using strict pseudo-contractions, IMUS Analysis Seminar, University of Seville, 30.06.2016.
- A. Sipoş, Proof mining and families of mappings, PhDs in Logic VIII, Darmstadt, 09 - 11.05.2016.
- A. Sipoş, Codensity and Stone spaces, Logic Colloquium 2015, Helsinki, 03 - 08.08 2015.
- A. Sipoş, Codensity and Stone spaces, The Eighth Congress of Romanian Mathematicians (special session Logic in Computer Science), Iaşi, 26.06 - 01.07 2015.
- L. Leuştean, Proof theoretic methods in nonlinear analysys III. Quantitative results on Fejér monotone sequences, Oberwolfach Workshop 1147: Mathematical Logic: Proof Theory, Type Theory and Constructive Mathematics, Mathematisches Forschungsinstitut Oberwolfach, 16 - 22.11.2014.
- L. Leuştean, An invitation to proof mining, Topics in Geometric Group Theory, Bucharest, 29.09 - 05.10 2014.
- L. Leuştean, Effective results on the asymptotic behavior of nonexpansive iterations, Logic Colloquium 2014, Vienna, 14.07 - 19.07.2014.
- L. Leuştean, An invitation to proof mining II (Effective results on the mean ergodic theorem), Logic in Computer Science Seminar, University of Bucharest, 22.05.2014.
- L. Leuştean, An invitation to proof mining I, Logic in Computer Science Seminar, University of Bucharest, 16.05.2014.

- L. Leuştean, Effective methods in geodesic spaces, IRTG 1529 Research Seminar "Proof Mining and Nonlinear Analysis", Technische Universität Darmstadt, 26.02 - 28.02.2014.
- L. Leuştean, Proof mining and applications in nonlinear ergodic theory, Anniversary Conference "Faculty of Sciences - 150 years", University of Bucharest, 29.08 - 01.09.2013.
- L. Leuştean, Effective methods in geodesic spaces, Workshop on Operator Theory in Metric Spaces, Sevilla, 03 - 04.04.2013.
- A. Nicolae, Asymptotic behavior of averaged and firmly nonexpansive mappings in geodesic spaces, Workshop on Operator Theory in Metric Spaces, Sevilla, 03 - 04.04.2013.
- L. Leuştean, Proof mining in nonlinear analysis, Logic Seminar, Technische Universität Darmstadt, 09.11.2012.
- A. Nicolae, Reflecting in classes of geodesic spaces, IMUS Analysis Seminar, University of Seville, 23.10.2012.
- L. Leuştean, Firmly nonexpansive mappings in classes of geodesic spaces, Invited Session: Variational inequalities and optimization problems on Riemannian manifolds, 21st International Symposium on Mathematical Programming (ISMP 2012), Berlin, 19 - 24.08.2012.
- L. Leuştean, Recent developments in proof mining, Oberwolfach Workshop 1145: Mathematical Logic: Proof Theory, Type Theory and Constructive Mathematics, Mathematisches Forschungsinstitut Oberwolfach, 06 - 12.11.2011.

3 Participations in Summer Schools

- A. Sipoş participated at the Scandinavian Logic Society Summer School in Logic, University of Helsinki, 27 - 31.07 2015.

4 Workshops/Conferences/ Scientific seminars organized

- L. Leuştean was a co-organizer of the Special Session Logic in Computer Science at The Eighth Congress of Romanian Mathematicians, Alexandru Ioan Cuza University of Iaşi, 26.06 - 01.07 2015.
- L. Leuştean and A. Sipoş organized Logic Seminar, FMI/IMAR (2012 - present).
- M. Buliga and L. Leuştean organised the scientific seminar Effective methods in metric analysis (2011 - 2013).

5 A description of the results

5.1 Effective results in nonlinear ergodic theory

Let us recall the Hilbert space formulation of the celebrated von Neumann mean ergodic theorem.

Theorem 1. *Let H be a Hilbert space and $U : H \to H$ be a unitary operator. Then for all $x \in H$, the ergodic average $x_n = \frac{1}{n}\sum_{i=0}^{n-1} U^i x$ converges strongly to $P_{Fix(U)}x$, the projection of x on the set of fixed points of U.*

Avigad, Gerhardy and Towsner [2] showed that we can not obtain in general computable rates of convergence. In this situation, one can consider the following equivalent reformulation of the Cauchy property of (x_n):

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall i, j \in [N, N+g(N)] \ \left(\|x_i - x_j\| < 2^{-k}\right). \quad (1)$$

This is known in logic as the *no-counterexample* interpretation [37, 38] of the Cauchy property and it was popularized in the last years under the name of *metastability* by Tao [60, 61]. In [61], Tao generalized the mean ergodic theorem to multiple commuting measure-preserving transformations, by deducing it from a finitary norm convergence result, expressed in terms of metastability. Recently, Walsh [63] used again metastability to show the L^2-convergence of multiple polynomial ergodic averages arising from nilpotent groups of measure-preserving transformations. Logical metatheorems [31] show that from wide classes of mathematical proofs one can extract upper bounds $\Phi(\varepsilon, g)$ on $\exists N$ in (1). Thus, taking $\varepsilon > 0$ instead of 2^{-k}, we define a *rate of metastability* as a functional $\Phi : (0, \infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfying

$$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists N \leq \Phi(\varepsilon, g) \, \forall i, j \in [N, N + g(N)] \ (\|x_i - x_j\| < \varepsilon) \, . \quad (2)$$

In [33] we obtained a quantitative version of the mean ergodic theorem for uniformly convex Banach spaces [8], computing an effective uniform rate of metastability for the ergodic average. Immediate consequences are the results obtained by Avigad, Gerhardy and Towsner [2] for Hilbert spaces and by Tao [61] for a particular dynamical system.

An important generalization of the von Neumann mean ergodic theorem was obtained by Wittmann [64] in 1992.

Theorem 2. *[64] Let C be a bounded closed convex subset of a Hilbert space X, $T : C \to C$ a nonexpansive mapping and $(\lambda_n)_{n\geq 1}$ be a sequence in $[0, 1]$. Assume that (λ_n) satisfies*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = \infty. \quad (3)$$

For any $x, u \in C$, define

$$x_0 := u, \quad x_{n+1} := \lambda_{n+1} u + (1 - \lambda_{n+1}) T x_n. \quad (4)$$

Then (x_n) converges to $P_{Fix(T)} u$.

One can easily see that (x_n) coincides with the ergodic average when T is linear and $\lambda_n = \frac{1}{n+1}$. The iteration (x_n) is known as the Halpern iteration, as it was introduced by Halpern [23] for the special case $u = 0$. The Halpern iteration can be defined similarly in more general spaces, like the geodesic ones.

5.1.1 Effective uniform rates of asymptotic regularity

The first step towards proving the weak or strong convergence of an iteration consists in obtaining the so-called *asymptotic regularity* and this can be done in a very general setting. Asymptotic regularity is a very important concept, introduced by Browder and Petryshyn [10] in the 60's for the Picard iteration, but it can be defined in general for any iteration (x_n) associated with a mapping T on a metric space (X, d): (x_n) is asymptotically regular if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ for all $x \in C$. A rate of convergence of the sequence $(d(x_n, Tx_n))$ towards 0 will be called a *rate of asymptotic regularity*.

In the paper 1.1.1 we obtained quantitative results on the asymptotic regularity of the Halpern iteration in CAT(0) spaces for general (λ_n), by considering as a hypothesis both $\sum_{n=1}^{\infty} \lambda_{n+1} = \infty$ and the equivalent condition $\prod_{n=1}^{\infty}(1-\lambda_{n+1}) = 0$. As an immediate corollary, one obtains a quadratic rate of asymptotic regularity, generalizing in this way the result proved for Hilbert spaces by Kohlenbach [32].

Theorem 3. *Assume that* $\lambda_n = \frac{1}{n+1}, n \geq 1$. *Then for every* $\varepsilon \in (0,1)$,

$$\forall n \geq \Psi(\varepsilon, M) \ \ (d(x_n, Tx_n) \leq \varepsilon)\,,$$

where $\Psi(\varepsilon, M) = \left\lceil \frac{4M}{\varepsilon} + \frac{16M^2}{\varepsilon^2} \right\rceil - 1$, *with* $M \in \mathbb{Z}_+$ *such that* $M \geq d_C$.

The method used in 1.1.1 for CAT(0) spaces can not be applied in the case of CAT(κ) spaces (with $\kappa > 0$). For these spaces, we computed an exponential rate of asymptotic regularity in 1.2.1.

In the paper 1.1.6 this result is extended to finite families of nonexpansive mappings and to (r, δ)-convex spaces, introduced by us as a generalization of CAT(κ) spaces and of the metric spaces with a convex geodesic bicombing (examples of such spaces are the normed ones, Busemann spaces, hyperconvex spaces or W-hyperbolic spaces in the sense of [30]). Immediate consequences of our extension are the results in [42, 43].

The paper 1.1.12 shows that results on the asymptotic behaviour of an alternative iterative method are immediate consequences of corresponding results on the Halpern iteration. This alternative iterative method was introduced for Banach spaces in [65] as a discretization of an approximating curve considered in [7].

5.1.2 Effective uniform rates of metastability

In the papers 1.1.1, 1.1.2, 1.1.9 and 1.2.1 we obtain finitary versions, with effective and highly uniform rates of metastability for generalizations of Theorem 2 proved in [58, 57, 53].

These results constitute a significant extension of the actual context of proof mining, as the proofs in [57, 58] make use of Banach limits, inspired by Lorentz' seminal paper [46], in which almost convergence was introduced.

Reich [54] initiated the use of almost convergence in nonlinear ergodic theory, while Bruck and Reich [13] applied for the first time Banach limits to the subject of Halpern iteration. The existence of Banach limits is either proved by applying the Hahn-Banach theorem to l^∞ or via ultralimits, in both cases the axiom of choice being needed.

In the paper 1.1.1 we develop a method to convert such proofs into more elementary proofs which no longer rely on Banach limits and can be analyzed by the existing logical machinery. The way Banach limits are used in these proofs seems to be rather typical for other proofs in nonlinear ergodic theory. Therefore, our method can be used to obtain similar results in those cases too.

The following theorem, proved by Saejung [57] using Banach limits, generalizes Wittmann's theorem to CAT(0) spaces.

Theorem 4. *Let C be a bounded closed convex subset of a complete* CAT(0) *space X and $T : C \to C$ a nonexpansive mapping. Assume that (λ_n) satisfies (3). Then for any $u, x \in C$, (x_n) converges to a fixed point of T.*

While we can not expect to obtain effective rates of convergence for (x_n), the existence of an effective and highly uniform rate of metastability is guaranteed, after the elimination of Banach limits, by [30, Teorema 3.7.3].

Theorem 5. *In the hypotheses of Theorem 4, let α be a rate of convergence of (λ_n), β be a Cauchy modulus of $s_n := \sum_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$ and θ be a rate of divergence of $\sum_{n=1}^{\infty} \lambda_{n+1}$.*

Then for all $\varepsilon \in (0, 2)$ and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \leq \Sigma(\varepsilon, g, M, \theta, \alpha, \beta)\ \forall m, n \in [N, N + g(N)]\ (d(x_n, x_m) \leq \varepsilon),$$

where $M \in \mathbb{Z}_+$ is an upper bound on the diameter of C.

The rate of metastability Σ, extracted in Theorem 4.2 from 1.1.1 does not depend on T, the starting point $x \in C$ and depends weakly on C, via its diameter. We remark that in practical cases, such as $\lambda_n = \frac{1}{n+1}$, the rates α, β, θ are easy to compute. In the paper 1.1.9 we remark that the quantitative analysis of Saejung's proof has as a result the complete elimination of any contribution of the use of Banach limits, which results in simpler bounds.

In 1.1.2 we apply the same method of eliminating Banach limits from the proof given by Shioji and Takahashi [58] to a generalization of Wittmann's theorem to Banach spaces with a uniformly Gâteaux differentiable norm. Furthermore, we prove a logical metatheorem for uniformly smooth Banach spaces.

The paper 1.2.1 is dedicated to the extraction of a uniform rate of metastability for the generalization of Wittmann's theorem to CAT(κ) spaces (with $\kappa > 0$). We use the same general method as in 1.1.1, but the proofs from this paper are much more involved than the ones from 1.1.1. In the case $\lambda_n = \frac{1}{n+1}$, we obtain a rate having a very simple logical form, similar with the one described in [35]:

$$\Sigma(\varepsilon, g, \kappa, M) = A_{\varepsilon,\kappa,M}\left(\widetilde{f^*}^{B_{\varepsilon,\kappa,M}}(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil\right),$$

computed in Corollary 3.5 from 1.2.1. Thus, the function g appears only in the definition of $\widetilde{f}^*$, the mappings $A_{\varepsilon,\kappa,M}, B_{\varepsilon,\kappa,M}$ do not depend at all on g.

5.2 Quantitative results in nonlinear analysis and convex optimization

In a series of papers, important classes of mappings (firmly nonexpansive, averaged, reflections and relatively nonexpansive) are studied and proof mining methods are applied to obtain effective results on the asymptotic behaviour of the associated Picard or Krasnoselskii iterations.

Firmly nonexpansive mappings, introduced by Browder [9] in Hilbert spaces and by Bruck [12] in Banach spaces, play a very important role in nonlinear analysis and convex optimization, due to their correspondence with maximal monotone operators proved by Minty [49]. Bruck's definition extends immediately to W-hyperbolic spaces X. A mapping $T : C \subseteq X \to C$ is called firmly nonexpansive if for all $x, y \in C$ and for all $\lambda \in (0, 1)$,

$$d(Tx, Ty) \leq d((1-\lambda)x + \lambda Tx, (1-\lambda)y + \lambda Ty). \tag{5}$$

A first main result of the paper 1.1.8 is a fixed point theorem for firmly nonexpansive mappings defined on unions of closed convex subsets of complete UCW-hyperbolic spaces. These spaces [41, 44] are a class of uniformly convex spaces generalizing both CAT(0) spaces and uniformly convex Banach spaces. A second main result of the paper 1.1.8 generalizes results obtained by Reich and Shafrir [55] in Banach spaces or in the Hilbert ball.

Theorem 6. *Let C be a subset of a W-hyperbolic space X and $T : C \to C$ be a firmly nonexpansive mapping. Then for all $x \in X$ and $k \in \mathbb{Z}_+$,*

$$\lim_{n\to\infty} d(T^{n+1}x, T^n x) = \frac{1}{k}\lim_{n\to\infty} d(T^{n+k}x, T^n x) = \lim_{n\to\infty} \frac{d(T^n x, x)}{n} = r_C(T),$$

where $r_C(T) := \inf\{d(x, Tx) \mid x \in C\}$.

In the paper 1.1.4, a quantitative version of Theorem 6 is proved, having as an immediate consequence an exponential rate of asymptotic regularity for the Picard iteration, the only one known even for Banach space. Using different methods, inspired by [41], rates of asymptotic regularity for UCW-hyperbolic spaces are computed in 1.1.8, which turn out to be quadratic for CAT(0) spaces or polynomial for L_p spaces, $1 < p < \infty$.

In the papers 1.1.3 and 1.1.4 different algorithms for the convex feasibility problem in geodesic spaces are studied. Thus, effective rates of asymptotic regularity are obtained in 1.1.4 for the well-known alternating projections method introduced by von Neumann, as well as for a method defined in terms of weighted averages of nonexpansive retractions [19]. The paper 1.1.3 studies the convergence, in spaces of constant curvature, of the algorithm AAR (*Averaged Alternating Reflection*), introduced by Bauschke, Combettes and Luke [5].

The paper 1.1.7 obtains existence results of best proximity points for cyclic and noncyclic relatively nonexpansive mappings in the context of Busemann convex reflexive metric spaces. Moreover, polynomial bounds on the existence of approximate fixed points for such mappings in UCW-hyperbolic spaces are computed.

In 1.4.6 we initiate the study of quantitative versions of different generalisations of the proximal point algorithm.

5.2.1 Fejér monotone sequences

The paper 1.2.6 provides in a unified way quantitative forms of strong convergence results for numerous iterative procedures which satisfy a general type of Fejér monotonicity where the convergence uses the compactness of the underlying set. Fejér monotonicity is a key notion employed in the study of many problems in convex optimization and programming, fixed point theory and the study of inverse problems (see, for example, [4, 6, 62]).

Let (X, d) be a metric space and $F \subseteq X$ be a nonempty subset of X. An iteration (x_n) is Fejér monotone w.r.t. F if

$$d(x_{n+1}, p) \leq d(x_n, p) \qquad \text{for all } n \in \mathbb{N} \text{ and all } p \in F.$$

We represent F as an intersection $F = \bigcap_{k \in \mathbb{N}} AF_k$ of approximations to F with the property that $AF_{k+1} \subseteq AF_k \subseteq X$ for all $k \in \mathbb{N}$. One prime example is $F := Fix(T)$ and $AF_k := \{p \in X \mid d(p, Tp) \leq 1/(k+1)\}$, where $Fix(T)$ denotes the fixed point set of some selfmap $T : X \to X$. We say that (x_n) possesses approximate F-points if for all $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $x_n \in AF_k$.

The set F is called *explicitly closed* (w.r.t. the representation AF_k) if

$$\forall p \in X \ \left(\forall N, M \in \mathbb{N}(AF_M \cap \overline{B}\,(p, 1/(N+1)) \neq \emptyset) \to p \in F\right).$$

One can easily see that F is explicitly closed when all the sets AF_k are closed and that if F is explicitly closed, then F is closed.

The main contributions of the paper 1.2.6 are quantitative versions of the following generalization of strong convergence results for Fejér monotone sequences.

Theorem 7. *Let X be compact, F explicitly closed and (x_n) a sequence in X which is Fejér monotone w.r.t. F and possesses approximate F-points. Then (x_n) converges to a point in F.*

These quantitative versions are in the form of explicit rates of metastabili-ty. The approach introduced in the paper 1.2.6 covers examples ranging from the proximal point algorithm for maximal monotone operators to various fixed point iterations for firmly nonexpansive, asymptotically nonexpansive, strictly pseudo-contractions and other types of mappings. Many of these results hold for geodesic spaces as W-hyperbolic spaces, UCW-hyperbolic spaces and CAT(0) spaces.

5.3 The asymptotic behaviour of pseudo-contractions

Let H be a real Hilbert space, $C \subseteq H$ a nonempty convex subset and $T : C \to C$ be a mapping.

We say that T is a *pseudo-contraction* if for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - Tx) - (y - Ty)\|^2. \tag{6}$$

This class of nonlinear mappings was introduced in the 1960s by Browder and Petryshyn [11]. Its significance lies in the following fact: an operator T is a pseudo-contraction if and only if its complement $U := Id - T$ is monotone, i.e. for all $x, y \in C$ we have that

$$\langle Ux - Uy, x - y \rangle \geq 0.$$

Monotone operators arise naturally in the study of partial differential equations: often such an equation can be written in the form $U(x) = 0$ (or $0 \in U(x)$ when considering multi-valued operators). Finding a zero of U is equivalent to finding a fixed point of its complement $T := Id - U$, which is a pseudo-contraction.

Let $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ be sequences in $[0, 1]$. The *Ishikawa iteration* [25] starting with an $x \in C$ is defined by:

$$x_0 := x, \quad x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T(\beta_n T x_n + (1 - \beta_n)x_n). \tag{7}$$

In the special case where $\beta_n := 0$ for all $n \in \mathbb{N}$, we obtain the well-known Mann iteration [47, 21]:

$$x_0 := x, \quad x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T(x_n). \tag{8}$$

If, furthermore, $\alpha_n = \alpha \in [0, 1]$ for all $n \in \mathbb{N}$, the Mann iteration becomes the Krasnoselskii iteration [36].

Ishikawa proved the following strong convergence result.

Theorem 8. *[25] Let H be a Hilbert space, $C \subseteq H$ a convex compact subset and $T : C \to C$ be an L-Lipschitzian pseudo-contraction. Suppose that (α_n), (β_n) satisfy the following conditions:*

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty \quad \text{and} \quad \alpha_n \leq \beta_n \text{ for all } n \in \mathbb{N}.$$

Then, for all $x \in C$, the Ishikawa iteration starting with x converges strongly to a fixed point of T.

The main result of the paper 1.2.5 is a finitary, quantitative version of Theorem 8, which computes a uniform rate of metastability for the Ishikawa iteration. In order to obtain this result, we apply methods developed in 1.2.6 for the strong convergence of Fejér monotone sequences.

Let $b \in \mathbb{N}$ be an upper bound on the diameter of C and γ be a modulus of total boundedness for C (defined in the paper 1.2.6). In the particular case $\alpha_n = \beta_n = \frac{1}{\sqrt{n+1}}$, we obtain that for all $k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}$, there exists $N \leq \Sigma_{b,\gamma,L}(k, g)$ such that

$$\forall i, j \in [N, N + g(N)] \ \left(\|x_i - x_j\| \leq \frac{1}{k+1} \text{ and } \|x_i - Tx_i\| \leq \frac{1}{k+1} \right),$$

where $\Sigma_{b,\gamma,L} : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is defined as follows:

$$\Sigma_{b,\gamma,L}(k, g) := K_0 + (\Sigma_0)_{b,L}(P_0, k, h),$$

where $(\Sigma_0)_{b,L}(0, k, g) := 0$ and

$$(\Sigma_0)_{b,L}(n+1, k, g) \ := \ 4^{2(b^2+1)\left(\max\{2k+1, 8b(8k_0^2+16k_0+10)g^M((\Sigma_0)_{b,L}(n,k,g))\}+1\right)^2},$$

with $K_0 := \left(\left\lceil 1 + \sqrt{2L^2+4} \right\rceil + 1\right)^2$, $h(n) := g(K_0 + n)$, $g^M(n) = \max\limits_{0 \leq i \leq n} g(i)$, $k_0 := \left\lceil \frac{\lceil L \rceil (4k+4) - 1}{2} \right\rceil$ şi $P_0 := \gamma\left(\left\lceil \sqrt{8k_0^2 + 16k_0 + 9} \right\rceil\right)$.

A very important class of pseudo-contractions are the strict pseudo-contractions, introduced also in [11]. If $0 \leq \kappa < 1$, then T is a *κ-strict pseudo-contraction* if for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|x - Tx - (y - Ty)\|^2. \qquad (9)$$

Nonexpansive mappings coincide with 0-strict pseudo-contractions.

In the paper 1.1.10 we apply methods of proof mining to obtain a uniform effective rate of asymptotic regularity for the Mann iteration associated to κ-strict pseudo-contractions on convex subsets of Hilbert spaces. The rate of of asymptotic regularity is quadratic for the Krasnoselskii iteration.

The Mann iteration associated to a κ-strict pseudo-contraction in the more general class of uniformly convex and 2-uniformly smooth Banach spaces is studied in the paper 1.2.3. Using a generalization of a result of Browder and Petryshyn [11, Theorem 2], one obtains much simpler proofs of the weak convergence theorems due to Marino and Xu [48] and Zhou [66]. A rate of asymptotic regularity for this iterations is also computed.

The paper 1.2.4 studies the parallel algorithm, used by Lopez-Acedo and Xu [45] to approximate common fixed points of finite families of k-strict pseudo-contractions. The main result of this paper is an effective rate of asymptotic regularity for this algorithm. It is also proved that this result is guaranteed by logical metatheorems for classical and semi-intuitionistc systems [30, 20].

5.4 Proof mining in L^p spaces

The paper 1.4.3 obtains an equivalent characterization of L^p spaces that is used to axiomatize these spaces into a higher-order logical system. This axiomatization allows us to prove a logical metatheorem for L^p spaces, an application of this metatheorem being the derivation of the standard modulus of uniform convexity. The research from 1.4.3 has as a point of departure the recent results of Günzel and Kohlenbach [22] on proof mining and the positive-bounded logic introduced by Henson [24].

5.5 Dilation structures

The paper 1.4.1 presents a description of sub-riemannian geometry with the help of dilation structures and it is based mainly on [14], with numerous improvements, as Theorem 8.10 giving an intrinsic characterization of riemannian metric spaces, Section 2.5 Curvdimension and curvature or the extended proof of Theorem 8.8 on the Γ-convergence of the length functionals for tempered dilation structures.

The paper 1.1.5 introduces and studies graphic lambda calculus, which consists in a class of graphs endowed with moves between them. Graphic lambda calculus can be used for representing terms and reductions from untyped lambda calculus, its main move being called graphic beta move for its relation to the beta reduction in lambda calculus. This formalism can also be used for computations in emergent algebras [15] or for tangle diagrams. The paper 1.1.5 is a massive revision of the descriptions from [16, 17, 18] and Section 5 of the paper is based on [15].

The paper 1.4.5 presents a geometric Ruzsa triangle inequality in metric spaces with dilations.

5.6 Uniformly convex geodesic spaces in geometric group theory and ergodic theory

A geodesic space X is said to be *weakly uniformly convex* if there exists a mapping $\eta : X \times (0,\infty) \times (0,2] \to (0,1]$ such that for any $a \in X, r > 0$, $\varepsilon \in (0,2]$, every $x, y \in X$ and all geodesic segments $[x,y]$ we have that,

$$\left.\begin{array}{l} d(a,x) \le r \\ d(a,y) \le r \\ d(x,y) \ge \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d\left(a, m(x,y)\right) \le (1-\eta(a,r,\varepsilon))r. \tag{10}$$

Such a mapping η is referred to as *a modulus of weak uniform convexity.* This notion was used by Reich and Shafrir [56] in the setting of hyperbolic spaces. If the modulus η does not depend on $a \in X$, hence $\eta : (0,\infty) \times (0,2] \to (0,1]$, X is called *uniformly convex.* This class of spaces was studied, for example, in [41, 34, 44].

Karlsson and Margulis [26] proved, in the setting of complete Busemann convex geodesic spaces satisfying a uniform convexity condition, an ergodic theorem that focuses on the asymptotic behavior of integrable cocycles of nonexpansive mappings over an ergodic measure-preserving transformation. Their result generalises the multiplicative ergodic theorem of Oseledec [52]. The paper 1.1.11 shows that one can relax the uniform convexity assumption used in [26] to the one of weak uniform convexity, obtaining thus a more general result.

Barycenters in geodesic spaces have been studied by various authors assuming different regularity conditions on the space [51, 27, 39]. Applications of different notions of barycenters to ergodic theory were given by Austin [1] and Navas [50]. The paper 1.4.4 extends to uniformly convex geodesic spaces a result on the existence of barycenters, proved by Sturm [59] for CAT(0) spaces.

5.7 Codensity and Stone spaces

In the paper 1.2.2, the author explicitly computes some categories of topological spaces important for algebraic logic as images of canonical codensity monads. The method was introduced by Kock [28] and applied by Leinster [40] to obtain the category of ultraproducts.

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