Notes on applicative matching logic

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1 Introduction

Matching logic (ML) [10, 9] was developed by Grigore Roşu and collaborators as a logic for defining the formal semantics of programming languages and for specifying and reasoning about the behavior of programs. Applicative Matching Logic (AML), a functional variant of ML, was introduced recently [3] and developed further in [5, 2]

These lecture notes present basic definitions and results on AML, they were written initially as a theoretical foundation for the implementation of AML in Lean [6]. As such, they are very rigorous, the proofs are extremely detailed. They can also be used as an introductory text in the theory of AML. We point out that the definition of patterns in these notes is more general than the one from [3], as we do not require the positivity of the pattern in the definition of the μ binder.

Monk's textbook on mathematical logic [8] has an enormous influence on the notes. A number of notations and results for first-order logic from Monk's book are adapted to AML. The set-theoretic notions and properties used in the notes are given in Appendix A.

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2 Language

Definition 2.1. *An* ***applicative matching logic (AML) signature*** *(or simply* ***signature****) is a triple* $\tau = (EVar, SVar, \Sigma)$*, where*

(i) $EVar = \{v_n \mid n \in \mathbb{N}\}$ *is a countable set of* ***element variables****.*

(ii) $SVar = \{V_n \mid n \in \mathbb{N}\}$ *is a countable set of* ***set variables****.*

(iii) Σ *is a set of* ***constants****.*

The sets $EVar$*,* $SVar$ *and* Σ *are pairwise disjoint.*

Notation 2.2. *We denote element variables by* $x, y, z, x_1, x_2, \ldots$*, set variables by* $X, Y, Z, X_1, X_2, \ldots$ *and symbols by* $\sigma, f, g, \ldots$*.*

In the sequel, $\tau = (EVar, SVar, \Sigma)$ is a signature.

Definition 2.3. *The set* Sym_τ *of* τ***-symbols*** *is defined as*

$$Sym_\tau = EVar \cup SVar \cup \{\rightarrow, \exists, \mu, Appl\} \cup \Sigma.$$

Definition 2.4. *The set* $Expr_\tau$ *of* τ***-expressions*** *is the set of all expressions over* Sym_τ*.*

Definition 2.5. *An* ***atomic*** τ***-pattern*** *is an element variable, a set variable or a constant. We shall use the notation* $AtomicPattern_\tau$ *for the set of atomic* τ*-patterns.*

Definition 2.6. *The* τ***-patterns*** *are the* τ*-expressions inductively defined as follows:*

(i) Every atomic τ*-pattern is a* τ*-pattern.*

(ii) If φ *and* ψ *are* τ*-patterns, then* $Appl\varphi\psi$ *is a* τ*-pattern.*

(iii) If φ *and* ψ *are* τ*-patterns, then* $\rightarrow \varphi\psi$ *is a* τ*-pattern.*

(iv) If φ *is a* τ*-pattern and* x *is an element variable, then* $\exists x\varphi$ *is a* τ*-pattern.*

(v) If φ *is a* τ*-pattern and* X *is a set variable, then* $\mu X\varphi$ *is a* τ*-pattern.*

(vi) Only the expressions obtained by applying the above rules are τ*-patterns.*

We use the Polish notation in the definition of τ-patterns as this notation allows us to obtain the unique readability of τ-patterns (see Proposition 2.10), a fundamental property.
The set of τ-patterns is denoted by $Pattern_\tau$ and τ-patterns are denoted by $\varphi, \psi, \chi, \ldots$.
For any τ-pattern φ, we denote by $EVar(\varphi)$ the set of element variables occuring in φ and by $SVar(\varphi)$ the set of set variables occuring in φ.

Remark 2.7. *The definition of* τ*-patterns can be written using the BNF notation:*

$$\begin{aligned}\varphi ::= & x \in EVar \mid X \in SVar \mid \sigma \in \Sigma \mid Appl\varphi\varphi \mid \rightarrow \varphi\varphi \\ & \mid \exists x\varphi \text{ if } x \in EVar \\ & \mid \mu X\varphi \text{ if } x \in SVar.\end{aligned}$$

Definition 2.8 (Alternative definition for τ-patterns)**.**
The set of τ***-patterns*** *is the intersection of all sets* Γ *of* τ*-expressions that have the following properties:*

(i) Γ *contains all atomic* τ*-patterns.*

(ii) Γ *is closed to* $Appl$*,* $\rightarrow$*,* $\exists x$ *(for any element variable* x*), and* μX *(for any set variable* X*), that is:*

if $\varphi, \psi \in \Gamma$, *then* $Appl\varphi\psi$, $\to \varphi\psi$, $\exists x\varphi$, $\mu X\varphi \in \Gamma$.

When the signature τ is clear from the context, we shall write simply expression(s), pattern(s) and we shall denote the set of expressions by $Expr$, the set of patterns by $Pattern$, the set of atomic patterns by $AtomicPattern$, etc..

Proposition 2.9 (Induction principle on patterns)**.**
Let Γ *be a set of patterns satisfying the following properties:*

(i) Γ *contains all atomic patterns.*

(ii) Γ *is closed to* $Appl$, $\to$, $\exists x$ *(for any element variable* x*), and* μX *(for any set variable* X*).*

Then $\Gamma = Pattern$.

Proof. By hypothesis, $\Gamma \subseteq Pattern$. By Definition 2.8, we get that $Pattern \subseteq \Gamma$. □

Induction principle on patterns is used to prove that all patterns have a property $\mathcal{P}$: we define Γ as the set of all patterns satisfying $\mathcal{P}$ and apply induction on patterns to obtain that $\Gamma = Pattern$.

2.1 Unique readability

Proposition 2.10 (Unique readability of patterns)**.**

(i) Any pattern has a positive length.

(ii) If φ *is a pattern, then one of the following hold:*

  (a) $\varphi = x$, *where* $x \in EVar$.

  (b) $\varphi = X$, *where* $X \in SVar$.

  (c) $\varphi = \sigma$, *where* $\sigma \in \Sigma$.

  (d) $\varphi = Appl\psi\chi$, *where* ψ, χ *are patterns.*

  (e) $\varphi = \to \psi\chi$, *where* ψ, χ *are patterns.*

  (f) $\varphi = \exists x\psi$, *where* x *is an element variable and* ψ *is a pattern.*

  (g) $\varphi = \mu X\psi$, *where* X *is a set variable and* ψ *is a pattern.*

(iii) Any proper initial segment of a pattern is not a pattern.

(iv) If φ *is a pattern, then exactly one of the cases from* (ii) *holds. Moreover,* φ *can be written in a unique way in one of these forms.*

Proof. (i) Let Γ be the set of patterns of positive length. We prove that $\Gamma = Pattern$ using the Induction principle on patterns (Proposition 2.9).

  (a) If φ is an atomic pattern, then its length is 1, so $\varphi \in \Gamma$.

  (b) If $\varphi, \psi \in \Gamma$, hence they have positive length, then obviously the patterns $Appl\varphi\psi$, $\to \varphi\psi$, $\exists x\varphi$, $\mu X\varphi \in \Gamma$ have positive length, hence they are in Γ.

(ii) Let $\Gamma_1 = \{Appl\psi\chi \mid \psi, \chi \in Pattern\}$, $\Gamma_2 = \{\to \psi\chi \mid \psi, \chi \in Pattern\}$, $\Gamma_3 = \{\exists x\psi \mid x \in EVar \text{ and } \psi \in Pattern\}$ and $\Gamma_4 = \{\mu X\psi \mid X \in SVar \text{ and } \psi \in Pattern\}$. Define

$$\Gamma = EVar \cup SVar \cup \Sigma \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4.$$

Then obviously, $\Gamma \subseteq Pattern$. We prove that $\Gamma = Pattern$ using the Induction principle on patterns (Proposition 2.9).

  (a) As $EVar \cup SVar \cup \Sigma \subseteq \Gamma$, we have that Γ contains all atomic patterns.

(b) Let $\varphi, \psi \in \Gamma$, $x \in EVar$ and $X \in SVar$. Then $Appl\varphi\psi \in \Gamma_1 \subseteq \Gamma$, $\to \varphi\psi \in \Gamma_2 \subseteq \Gamma$, $\exists x\varphi \in \Gamma_3 \subseteq \Gamma$, and $\mu X\varphi \in \Gamma_4 \subseteq \Gamma$. Thus, Γ is closed to $Appl$, $\to$, $\exists x$, and μX.

(iii) As, by (i), patterns have positive length, it follows that we have to prove that for all $n \geq 1$,

(P) if $\varphi = \varphi_0 \dots \varphi_{n-1}$ is a pattern of length n, then for any $0 \leq i < n-1$,
$\varphi = \varphi_0 \dots \varphi_i$ is not a pattern.

The proof is by induction on n.

$n = 1$: Then one cannot have $0 \leq i < 0$, hence (P) holds.

Assume that $n > 1$ and that (P) holds for any pattern of length $< n$. Let $\varphi = \varphi_0 \dots \varphi_{n-1}$ be a pattern of length n. By (ii), we have the following cases:

(a) $\varphi = \star\psi\chi$, where $\star \in \{Appl, \to\}$ and ψ, χ are patterns. Thus, $\varphi_0 = \star$, $\psi = \varphi_1 \dots \varphi_{k-1}$ and $\chi = \varphi_k \dots \varphi_{n-1}$, where $2 \leq k \leq n-1$.

Let $0 \leq i < n-1$ and assume, by contradiction, that $\varphi_0 \dots \varphi_i$ is a pattern. Applying again (ii), it follows that $\varphi_0 \dots \varphi_i = \star\psi^1\chi^1$, where ψ^1, χ^1 are patterns. Thus, $\psi^1 = \varphi_1 \dots \varphi_{p-1}$, $\chi^1 = \varphi_p \dots \varphi_i$, where $2 \leq p \leq i$. We have the following cases:

(1) $p < k$. Then ψ^1 is a proper initial segment of ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get that ψ^1 is not a pattern. We have obtained a contradiction.

(2) $p = k$. Then $\psi^1 = \psi$ and χ^1 is a proper initial segment of χ. As the length of χ is $< n$, we can apply the induction hypothesis to get that χ^1 is not a pattern. We have obtained a contradiction.

(3) $p > k$. Then ψ is a proper initial segment of ψ^1. As the length of ψ^1 is $< n$, we can apply the induction hypothesis to get that ψ is not a pattern. We have obtained a contradiction.

(b) $\varphi = \theta\psi$, where $\theta \in \{\exists x, \mu X\}$ for some $x \in EVar$ and $X \in SVar$ and ψ is a pattern. Then $\varphi_0\varphi_1 = \theta$ and $\psi = \varphi_2 \dots \varphi_{n-1}$. Let $0 \leq i < n-1$ and assume, by contradiction, that $\varphi_0 \dots \varphi_i$ is a pattern. Applying again (ii), it follows that $\varphi_0 \dots \varphi_i = \theta\psi^1$, where $\psi^1 = \varphi_2 \dots \varphi_i$ is a pattern. Then ψ^1 is a proper initial segment of ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get that ψ^1 is not a pattern. We have obtained a contradiction.

(iv) is an immediate consequece of (ii) and (iii).

□

The following results will be useful in Subsections 2.4 and 2.5.

Proposition 2.11. *Let $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ be a pattern and suppose that $\varphi_i \in \{\exists, \mu\}$ for some $i = 0, \dots, n-1$. Then there exists a unique j such that $i < j \leq n-1$ and $\varphi_i \dots \varphi_j$ is a pattern.*

Proof. Let us prove first the uniqueness. Assume, by contradiction, that $i < j < k \leq n-1$ are such that $\varphi_i \dots \varphi_j$ and $\varphi_i \dots \varphi_k$ are both patterns. As $\varphi_i \dots \varphi_j$ is a proper initial segment of $\varphi_i \dots \varphi_k$, it follows, by Proposition 2.10.(iii) that $\varphi_i \dots \varphi_j$ is not a pattern. We have obtained a contradiction.
Let us prove in the sequel the existence.
As, by Proposition 2.10.(i), patterns have positive length, it follows that we have to prove that for all $n \geq 1$,

(P) if $\varphi = \varphi_0 \dots \varphi_{n-1}$ is a pattern of length n and $\varphi_i \in \{\exists, \mu\}$ for some $i = 0, \dots, n-1$,
then there exists j such that $i < j \leq n-1$ and $\varphi_i \dots \varphi_j$ is a pattern.

The proof is by induction on n.
$n = 1$. Then $\varphi = \varphi_0$ is an atomic pattern, so there exists no i satisfying the premise in (P), hence (P) holds.

Assume that $n > 1$ and that (P) holds for any pattern of length $< n$. Let $\varphi = \varphi_0 \ldots \varphi_{n-1}$ be a pattern of length n such that $\varphi_i \in \{\exists, \mu\}$ for some $i = 0, \ldots, n-1$. By Proposition 2.10.(ii), we have the following cases:

(i) $\varphi = \circ\psi\chi$, where $\circ \in \{Appl, \to\}$ and ψ, χ are patterns. Thus, $\varphi_0 = \circ$, $\psi = \varphi_1 \ldots \varphi_{k-1}$ and $\chi = \varphi_k \ldots \varphi_{n-1}$, where $2 \leq k \leq n-1$. We have the following cases:

  (a) $i \leq k-1$. Then φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get the existence of j such that $i < j \leq k-1$ and $\varphi_i \ldots \varphi_j$ is a pattern.

  (b) $i \geq k$. Then φ_i occurs in χ. As the length of χ is $< n$, we can apply the induction hypothesis to get the existence of j such that $i < j \leq n-1$ and $\varphi_i \ldots \varphi_j$ is a pattern.

(ii) $\varphi = \theta\psi$, where $\theta \in \{\exists x, \mu X\}$ for some $x \in EVar$ or $X \in SVar$ and ψ is a pattern. Then $\varphi_0\varphi_1 = \theta$ and $\psi = \varphi_2 \ldots \varphi_{n-1}$. We have the following cases:

  (a) $i = 0$. Then $j = n-1$ and $\varphi_i \ldots \varphi_j = \varphi$ is a pattern.

  (b) $2 \leq i \leq n-1$. Then φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get the existence of j such that $i < j \leq n-1$ and $\varphi_i \ldots \varphi_j$ is a pattern.

□

Proposition 2.12. *Let $\varphi = \varphi_0\varphi_1 \ldots \varphi_{n-1}$ be a pattern and suppose that $\varphi_i \in \{Appl, \to\}$ for some $i = 0, \ldots, n-1$. Then there exist unique j, l such that $i < j < l \leq n-1$ and $\varphi_{i+1} \ldots \varphi_j$, $\varphi_{j+1} \ldots \varphi_l$ are patterns.*

Proof. Let us prove first the uniqueness. Assume, by contradiction, that $i < j < l \leq n-1$ and $i < j_1 < l_1 \leq n-1$ are such that $\psi = \varphi_{i+1} \ldots \varphi_j$, $\chi = \varphi_{j+1} \ldots \varphi_l$, $\psi^1 = \varphi_{i+1} \ldots \varphi_{j_1}$, $\chi = \varphi_{j+1} \ldots \varphi_{l_1}$ are patterns. If $j \neq j_1$, then either $j < j_1$ or $j_1 < j$, hence one of ψ, ψ^1 is a proper initial segment of the other one. By Proposition 2.10.(iii), we get that one of ψ, ψ^1 is not a pattern. We have obtained a contradiction. Thus, we must have $j = j_1$. We prove similarly that we must have $l = l_1$.
Let us prove in the sequel the existence.
As, by Proposition 2.10.(i), patterns have positive length, it follows that we have to prove that for all $n \geq 1$,

(P) if $\varphi = \varphi_0 \ldots \varphi_{n-1}$ is a pattern of length n and $\varphi_i \in \{Appl, \to\}$ for some $i = 0, \ldots, n-1$, then there exist j, l such that $i < j < l \leq n-1$ and $\varphi_{i+1} \ldots \varphi_j$, $\varphi_{j+1} \ldots \varphi_l$ are patterns.

The proof is by induction on n.
$n = 1$. Then $\varphi = \varphi_0$ is an atomic pattern, so there exists no i satisfying the premise in (P), hence (P) holds.
Assume that $n > 1$ and that (P) holds for any pattern of length $< n$. Let $\varphi = \varphi_0 \ldots \varphi_{n-1}$ be a pattern of length n such that $\varphi_i \in \{Appl, \to\}$ for some $i = 0, \ldots, n-1$. By Proposition 2.10.(ii), we have the following cases:

(i) $\varphi = \circ\psi\chi$, where $\circ \in \{Appl, \to\}$ and ψ, χ are patterns. Thus, $\varphi_0 = \circ$, $\psi = \varphi_1 \ldots \varphi_{k-1}$ and $\chi = \varphi_k \ldots \varphi_{n-1}$, where $2 \leq k \leq n-1$. We have the following cases:

  (a) $i = 0$. Then we can take $j = k-1$ and $l = n-1$.

  (b) $i \in \{1, \ldots, k-1\}$. Then φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get the conclusion.

  (c) $i \geq k$. Then φ_i occurs in χ. As the length of χ is $< n$, we can apply the induction hypothesis to get the conclusion

(ii) $\varphi = \theta\psi$, where $\theta \in \{\exists x, \mu X\}$ for some $x \in EVar$ and $X \in SVar$ and ψ is a pattern. Then $\varphi_0\varphi_1 = \theta$ and $\psi = \varphi_2 \ldots \varphi_{n-1}$. As φ_i does not occur in θ, it follows that φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get the conclusion.

□

2.2 Recursion principle on patterns

Proposition 2.13 (Recursion principle on patterns)**.** *Let D be a set and the mappings*

$$G_0 : \mathit{AtomicPattern} \to D, \quad G_{\mathit{Appl}}, G_{\to} : D^2 \times \mathit{Pattern}^2 \to D, \tag{1}$$

$$G_{\exists} : D \times \mathit{EVar} \times \mathit{Pattern} \to D, \quad G_{\mu} : D \times \mathit{SVar} \times \mathit{Pattern} \to D. \tag{2}$$

Then there exists a unique mapping

$$F : \mathit{Pattern} \to D$$

that satisfies the following properties:

(i) $F(\varphi) = G_0(\varphi)$ *for any atomic pattern* φ*.*

(ii) $F(\mathit{Appl}\varphi\psi) = G_{\mathit{Appl}}(F(\varphi), F(\psi), \varphi, \psi)$ *for any patterns* φ*,* ψ*.*

(iii) $F(\to \varphi\psi) = G_{\to}(F(\varphi), F(\psi), \varphi, \psi)$ *for any patterns* φ*,* ψ*.*

(iv) $F(\exists x\varphi) = G_{\exists}(F(\varphi), x, \varphi)$ *for any pattern* φ *and any element variable* x*.*

(v) $F(\mu X\varphi) = G_{\mu}(F(\varphi), X, \varphi)$ *for any pattern* φ *and any set variable* X*.*

Proof. Apply Proposition 2.10. □

2.3 Subpatterns

Definition 2.14. *Let φ be a pattern. A subpattern of φ is a pattern ψ that occurs in φ.*

Notation 2.15. *We denote by $\mathit{SubPattern}(\varphi)$ the set of subpatterns of φ.*

Remark 2.16 (Alternative definition)**.**
The mapping

$$\mathit{SubPattern} : \mathit{Pattern} \to 2^{\mathit{Pattern}}, \quad \varphi \mapsto \mathit{SubPattern}(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcl}
\mathit{SubPattern}(\varphi) & = & \{\varphi\} \quad \text{if } \varphi \text{ is an atomic pattern,} \\
\mathit{SubPattern}(\mathit{Appl}\varphi\psi) & = & \mathit{SubPattern}(\varphi) \cup \mathit{SubPattern}(\psi) \cup \{\mathit{Appl}\varphi\psi\}, \\
\mathit{SubPattern}(\to \varphi\psi) & = & \mathit{SubPattern}(\varphi) \cup \mathit{SubPattern}(\psi) \cup \{\to \varphi\psi\}, \\
\mathit{SubPattern}(\exists x\varphi) & = & \mathit{SubPattern}(\varphi) \cup \{\exists x\varphi\}, \\
\mathit{SubPattern}(\mu X\varphi) & = & \mathit{SubPattern}(\varphi) \cup \{\mu X\varphi\}.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.13) with $D = 2^{\mathit{Pattern}}$ and

$$G_0(\varphi) = \{\varphi\}, \quad G_{\circ}(\Gamma, \Delta, \varphi, \psi) = \Gamma \cup \Delta \cup \{\circ\varphi\psi\} \text{ for } \circ \in \{\mathit{Appl}, \to\},$$

$$G_{\exists}(\Gamma, x, \varphi) = \Gamma \cup \{\exists x\varphi\} \quad G_{\mu}(\Gamma, X, \varphi) = \Gamma \cup \{\mu X\varphi\}.$$

Then

(i) $\mathit{SubPattern}(\varphi) = \{\varphi\} = G_0(\varphi)$ if φ is an atomic pattern.

(ii) For $\circ \in \{\mathit{Appl}, \to\}$, we have that

$$\begin{aligned}
\mathit{SubPattern}(\circ\varphi\psi) &= \mathit{SubPattern}(\varphi) \cup \mathit{SubPattern}(\psi) \cup \{\circ\varphi\psi\} \\
&= G_{\circ}(\mathit{SubPattern}(\varphi), \mathit{SubPattern}(\psi), \varphi, \psi).
\end{aligned}$$

(iii) $\mathit{SubPattern}(\exists x\varphi) = \mathit{SubPattern}(\varphi) \cup \{\exists x\varphi\} = G_{\exists}(\mathit{SubPattern}(\varphi), x, \varphi)$.

(iv) $\mathit{SubPattern}(\mu X\varphi) = \mathit{SubPattern}(\varphi) \cup \{\mu X\varphi\} = G_{\mu}(\mathit{SubPattern}(\varphi), X, \varphi)$.

Thus, $\mathit{SubPattern} : \mathit{Pattern} \to 2^{\mathit{Pattern}}$ is the unique mapping given by Proposition 2.13. □

2.4 Free and bound variables

Definition 2.17. *Let $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ be a pattern and x be an element variable.*

*(i) We say that $\exists$ is **a quantifier on x at the ith place with scope** ψ if $\varphi_i = \exists$, $\varphi_{i+1} = x$ and $\psi = \varphi_i \dots \varphi_j$ is the unique pattern given by Proposition 2.11.*

*(ii) We say that x **occurs bound at the kth place of** φ if $\varphi_k = x$ and there exist $0 \le i, j \le n-1$ such that $i < k \le j$ and $\exists$ is a quantifier on x at the ith place with scope $\psi = \varphi_i \dots \varphi_j$.*

*(iii) If $\varphi_k = x$ but x does not occur bound at the kth place of φ, we say that x **occurs free at the kth place of** φ.*

*(iv) x is a **bound variable** of φ if there exists k such that x occurs bound at the kth place of φ.*

*(v) x is a **free variable** of φ if there exists k such that x occurs free at the kth place of φ.*

Definition 2.18. *Let $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ be a pattern and X be a set variable.*

*(i) We say that μ is **a binder on X at the ith place with scope** ψ if $\varphi_i = \mu$, $\varphi_{i+1} = X$ and $\psi = \varphi_i \dots \varphi_j$ is the unique pattern given by Proposition 2.11.*

*(ii) We say that X **occurs bound at the kth place of** φ if $\varphi_k = X$ and there exist $0 \le i, j \le n-1$ such that $i < k \le j$ and μ is a binder on X at the ith place with scope $\psi = \varphi_i \dots \varphi_j$.*

*(iii) If $\varphi_k = X$ but X does not occur bound at the kth place of φ, we say that X **occurs free at the kth place of** φ.*

*(iv) X is a **bound variable** of φ if there exists k such that X occurs bound at the kth place of φ.*

*(v) X is a **free variable** of φ if there exists k such that X occurs free at the kth place of φ.*

Notation 2.19. *$FV(\varphi) =$ the set of free element and set variables of φ.*

Remark 2.20 (Alternative definition)**.**
The mapping

$$FV : Pattern \to 2^{EVar \cup SVar}, \quad \varphi \mapsto FV(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll} FV(\varphi) & = & EVar(\varphi) \cup SVar(\varphi) & \text{if } \varphi \text{ is an atomic pattern,} \\ FV(\circ\varphi\psi) & = & FV(\varphi) \cup FV(\psi) & \text{for } \circ \in \{Appl, \to\}, \\ FV(\exists x\varphi) & = & FV(\varphi) \setminus \{x\}, & \\ FV(\mu X\varphi) & = & FV(\varphi) \setminus \{X\}. & \end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.13) with $D = 2^{EVar \cup SVar}$ and

$$G_0(\varphi) = EVar(\varphi) \cup SVar(\varphi), \quad G_\circ(V_1, V_2, \varphi, \psi) = V_1 \cup V_2,$$
$$G_\exists(V, x, \varphi) = V \setminus \{x\}, \quad G_\mu(V, X, \varphi) = V \setminus \{X\}.$$

Then

(i) $FV(\varphi) = EVar(\varphi) \cup SVar(\varphi) = G_0(\varphi)$ if φ is an atomic pattern.

(ii) $FV(\circ\varphi\psi) = FV(\varphi) \cup FV(\psi) = G_\circ(FV(\varphi), FV(\psi), \varphi, \psi)$ for $\circ \in \{Appl, \to\}$.

(iii) $FV(\exists x\varphi) = FV(\varphi) \setminus \{x\} = G_\exists(FV(\varphi), x, \varphi)$.

(iv) $FV(\mu X\varphi) = FV(\varphi) \setminus \{X\} = G_\mu(FV(\varphi), X, \varphi)$.

Thus, $FV : Pattern \to 2^{EVar \cup SVar}$ is the unique mapping given by Proposition 2.13. ∎

2.5 Positive and negative occurences of set variables

Definition 2.21. *Let $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ be a pattern and X be a set variable.*

*(i) We say that $\to$ is **an implication at the ith place of φ with left scope ψ and right scope χ** if $\varphi_i =\to$ and $\psi = \varphi_{i+1} \dots \varphi_j$, $\chi = \varphi_{j+1} \dots \varphi_l$ are the unique patterns given by Proposition 2.12.*

*(ii) X **occurs left at the kth place of φ** if X occurs free at the kth place of φ and there exist $0 \leq i < k \leq j \leq n-1$ such that $\psi = \varphi_{i+1} \dots \varphi_j$ is the left scope of an implication $\to$ at the ith place of φ.*

Definition 2.22. *Let X be a set variable. We define the mapping*

$$N_{X,L} : Pattern \to Fun(\mathbb{N}, \mathbb{N})$$

by recursion on patterns as follows:

(i) φ is an atomic pattern. Then $N_{X,L}(\varphi)(k) = 0$ for every $k \in \mathbb{N}$.

(ii) $\varphi = Appl\psi\chi$, that is $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ with $\varphi_0 = Appl$, $\psi = \varphi_1 \dots \varphi_j$ and $\chi = \varphi_{j+1} \dots \varphi_{n-1}$ for some $1 \leq j < n-1$. We have the following cases:

- *(a) $k = 0$ or $k \geq n$. Then $N_{X,L}(\varphi)(k) = 0$.*
- *(b) $1 \leq k \leq j$. Then $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-1)$.*
- *(c) $j+1 \leq k \leq n-1$. Then $N_{X,L}(\varphi)(k) = N_{X,L}(\chi)(k-j-1)$.*

(iii) $\varphi =\to \psi\chi$, that is $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ with $\varphi_0 =\to$, $\psi = \varphi_1 \dots \varphi_j$ and $\chi = \varphi_{j+1} \dots \varphi_{n-1}$ for some $1 \leq j < n-1$. We have the following cases:

- *(a) $k = 0$ or $k \geq n$. Then $N_{X,L}(\varphi)(k) = 0$.*
- *(b) $1 \leq k \leq j$. Then $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-1) + 1$.*
- *(c) $j+1 \leq k \leq n-1$. Then $N_{X,L}(\varphi)(k) = N_{X,L}(\chi)(k-j-1)$.*

(iv) $\varphi = \exists x\psi$, that is $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ with $\varphi_0 = \exists$, $\varphi_1 = x$ and $\psi = \varphi_2 \dots \varphi_{n-1}$. We have the following cases:

- *(a) $k \in \{0, 1\}$ or $k \geq n$. Then $N_{X,L}(\varphi)(k) = 0$.*
- *(b) $2 \leq k \leq n-1$. Then $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-2)$.*

(v) $\varphi = \mu Z\psi$, that is $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ with $\varphi_0 = \mu$, $\varphi_1 = Z$ and $\psi = \varphi_2 \dots \varphi_{n-1}$. If $Z = X$, then $N_{X,L}(\varphi)(k) = 0$ for every $k \in \mathbb{N}$. If $Z \neq X$, we have the following cases:

- *(a) $k \in \{0, 1\}$ or $k \geq n$. Then $N_{X,L}(\varphi)(k) = 0$.*
- *(b) $2 \leq k \leq n-1$. Then $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-2)$.*

Proof. Apply Recursion principle on patterns (Proposition 2.13) with $D = Fun(\mathbb{N}, \mathbb{N})$ and

$$G_0(\varphi)(k) = 0 \qquad \text{if } \varphi \text{ is an atomic pattern,}$$

$$G_{Appl}(f, g, \psi, \chi)(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq n, \\ f(k-1) & \text{if } 1 \leq k \leq j, \\ g(k-j-1) & \text{if } j+1 \leq k \leq n-1, \end{cases},$$

$$G_{\rightarrow}(f, g, \psi, \chi)(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq n, \\ f(k-1)+1 & \text{if } 1 \leq k \leq j, \\ g(k-j-1) & \text{if } j+1 \leq k \leq n-1, \end{cases},$$

$$G_{\exists}(f, x, \psi)(k) = \begin{cases} 0 & \text{if } k \in \{0,1\} \text{ or } k \geq n, \\ f(k-2) & \text{if } 2 \leq k \leq n-1, \end{cases},$$

$$G_{\mu}(f, Z, \psi)(k) = \begin{cases} 0 & \text{if } X = Z, \\ 0 & \text{if } X \neq Z \text{ and } (k \in \{0,1\} \text{ or } k \geq n), \\ f(k-2) & \text{if } X \neq Z \text{ and } 2 \leq k \leq n-1. \end{cases}$$

Then

(i) $N_{X,L}(\varphi)(k) = 0 = G_0(\varphi)(k)$ for every $k \in \mathbb{N}$ if φ is an atomic pattern.

(ii) $\varphi = Appl\psi\chi$. Then

$$\begin{aligned} N_{X,L}(\rightarrow \psi\chi)(k) &= \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq n, \\ N_{X,L}(\psi)(k-1) & \text{if } 1 \leq k \leq j, \\ N_{X,L}(\chi)(k-j-1) & \text{if } j+1 \leq k \leq n-1 \end{cases} \\ &= G_{Appl}(N_{X,L}(\psi), N_{X,L}(\chi), \psi, \chi)(k). \end{aligned}$$

(iii) $\varphi = \rightarrow \psi\chi$. Then

$$\begin{aligned} N_{X,L}(\rightarrow \psi\chi)(k) &= \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq n, \\ N_{X,L}(\psi)(k-1)+1 & \text{if } 1 \leq k \leq j, \\ N_{X,L}(\chi)(k-j-1) & \text{if } j+1 \leq k \leq n-1 \end{cases} \\ &= G_{Appl}(N_{X,L}(\psi), N_{X,L}(\chi), \psi, \chi)(k). \end{aligned}$$

(iv) $\varphi = \exists x\psi$. Then

$$\begin{aligned} N_{X,L}(\exists x\psi)(k) &= \begin{cases} 0 & \text{if } k \in \{0,1\} \text{ or } k \geq n, \\ N_{X,L}(\psi)(k-2) & \text{if } 2 \leq k \leq n-1 \end{cases} \\ &= G_{\exists}(N_{X,L}(\psi), x, \psi)(k). \end{aligned}$$

(v) $\varphi = \mu Z\psi$. If $X = Z$, then $N_{X,L}(\mu Z\psi)(k) = 0 = G_{\mu}(N_{X,L}(\psi), Z, \psi)(k)$. Assume that $X \neq Z$. Then

$$\begin{aligned} N_{X,L}(\mu Z\psi)(k) &= \begin{cases} 0 & \text{if } k \in \{0,1\} \text{ or } k \geq n, \\ N_{X,L}(\psi)(k-2) & \text{if } 2 \leq k \leq n-1 \end{cases} \\ &= G_{\mu}(N_{X,L}(\psi), Z, \psi)(k). \end{aligned}$$

$\square$

Notation 2.23. *Let X be a set variable, $k \in \mathbb{N}$ and φ be a pattern. We denote $N_{X,L}(\varphi)(k)$ $N_L(\varphi, X, k)$.*

Definition 2.24. *Let X be a set variable, $k \in \mathbb{N}$ and φ be a pattern such that X occurs free at the kth place of φ.*

(i) We say that X ***occurs positively at the kth place of*** *φ (or that X* ***has a positive occurence at the kth place of*** *φ) if $N_L(\varphi, X, k) = 0$ or $N_L(\varphi, X, k)$ is an even natural number.*

(ii) We say that X ***occurs negatively at the kth place of*** *φ (or that X* ***has a negative occurence at the kth place of*** *φ) if $N_L(\varphi, X, k)$ is an odd natural number.*

Definition 2.25. *We say that φ is* ***positive in*** *X if one of the following is true:*

(i) X does not occur free in φ.

(ii) For every $k \in \mathbb{N}$, if X occurs free at the kth place of φ, then X occurs positively at the kth place of φ.

Definition 2.26. *We say that φ is* ***negative in*** *X if one of the following is true:*

(i) X does not occur free in φ.

(ii) For every $k \in \mathbb{N}$, if X occurs free at the kth place of φ, then X occurs negatively at the kth place of φ.

Example 2.27. (i) $\varphi = \to X \to X\bot$. Then $N_L(X, \varphi, 1) = N_L(X, X, 0) + 1 = 0 + 1 = 1$. Furthermore, $N_L(X, \varphi, 3) = N_L(X, \to X\bot, 1) = N_L(X, X, 0) + 1 = 1$. Thus, φ is negative in X.

(ii) $\varphi = \to\to X\bot\bot$. Then $N_L(X, \varphi, 2) = N_L(X, \to X\bot, 1) + 1 = N_L(X, X, 0) + 2 = 2$. Thus, φ is positive in X.

(iii) $\varphi = Appl{\to} X \to X\bot{\to\to} X\bot\bot$. Then φ is neither positive nor negative in X.

Remark 2.28 (Alternative definition)**.** *The property that φ is positive in X can be defined by recursion on patterns as follows:*

(i) If φ is atomic, then φ is positive in X.

(ii) If $\varphi = Appl\psi\chi$, then φ is positive in X iff both ψ, χ are positive in X.

(iii) If $\varphi = \to \psi\chi$, then φ is positive in X iff ψ is negative in X and χ is positive in X.

(iv) If $\varphi = \exists x\psi$, then φ is positive in X iff ψ is positive in X.

(v) If $\varphi = \mu X\psi$, then φ is positive in X.

(vi) If $\varphi = \mu Z\psi$ with $Z \neq X$, then φ is positive in X iff ψ is positive in X.

Remark 2.29 (Alternative definition)**.** *The property that φ is negative in X can be defined by recursion on patterns as follows:*

(i) If φ is atomic, then φ is negative in X iff $\varphi \neq X$.

(ii) If $\varphi = Appl\psi\chi$, then φ is negative in X iff both ψ, χ are negative in X.

(iii) If $\varphi = \to \psi\chi$, then φ is negative in X iff ψ is positive in X and χ is negative in X.

(iv) If $\varphi = \exists x\psi$, then φ is negative in X iff ψ is negative in X.

(v) If $\varphi = \mu X\psi$, then φ is negative in X.

(vi) If $\varphi = \mu Z\psi$ with $Z \neq X$, then φ is negative in X iff ψ is negative in X.

2.6 Substitution of element variables

Let x be an element variable and φ, δ be patterns.

Definition 2.30. *We define $Subf^x_\delta\varphi$ to be the expression obtained from φ by replacing every free occurence of x in φ with δ.*

Remark 2.31 (Alternative definition)**.**
The mapping

$$Subf^x_\delta : Pattern \to Expr, \quad Subf^x_\delta(\varphi) = Subf^x_\delta\varphi$$

can be defined by recursion on patterns as follows:

$$
\begin{array}{lcll}
Subf^x_\delta(z) & = & \begin{cases} \delta \text{ if } x = z \\ z \text{ if } x \neq z \end{cases} & \text{if } z \in EVar, \\
Subf^x_\delta(\varphi) & = & \varphi & \text{if } \varphi \in SVar \cup \Sigma, \\
Subf^x_\delta(\circ\psi\chi) & = & \circ Subf^x_\delta(\psi) Subf^x_\delta(\chi) & \text{for } \circ \in \{Appl, \to\}, \\
Subf^x_\delta(\exists z\psi) & = & \begin{cases} \exists z\psi & \text{if } x = z \\ \exists z Subf^x_\delta(\psi) & \text{if } x \neq z \end{cases}, & \\
Subf^x_\delta(\mu X\psi) & = & \mu X Subf^x_\delta(\psi). &
\end{array}
$$

Proof. Apply Recursion principle on patterns (Proposition 2.13) with $D = Expr$ and

$$
\begin{aligned}
G_0(\varphi) &= \begin{cases} \delta & \text{if } \varphi = x \\ \varphi & \text{if } \varphi \in (EVar \setminus \{x\}) \cup SVar \cup \Sigma \end{cases} \\
G_\circ(\theta, \tau, \psi, \chi) &= \circ\theta\tau \text{ for } \circ \in \{Appl, \to\}, \\
G_\exists(\theta, z, \psi) &= \begin{cases} \exists z\psi & \text{if } x = z \\ \exists z\theta & \text{if } x \neq z \end{cases}, \\
G_\mu(\theta, X, \psi) &= \mu X\theta.
\end{aligned}
$$

Then

(i) If φ is an atomic pattern, we have the following cases:

  (a) $\varphi = x$. Then $Subf^x_\delta(\varphi) = Subf^x_\delta(x) = \delta = G_0(\varphi)$.

  (b) $\varphi \in (EVar \setminus \{x\}) \cup SVar \cup \Sigma$. Then $Subf^x_\delta(\varphi) = \varphi = G_0(\varphi)$.

(ii) For $\circ \in \{Appl, \to\}$, we have that

$$Subf^x_\delta(\circ\psi\chi) = \circ Subf^x_\delta(\psi) Subf^x_\delta(\chi) = G_\circ(Subf^x_\delta(\psi), Subf^x_\delta(\chi), \psi, \chi).$$

(iii) $Subf^x_\delta(\exists z\psi) = \begin{cases} \exists z\psi & \text{if } x = z \\ \exists z Subf^x_\delta(\psi) & \text{if } x \neq z \end{cases} = G_\exists(Subf^x_\delta(\psi), z, \psi)$

(iv) $Subf^x_\delta(\mu X\psi) = \mu X Subf^x_\delta(\psi) = G_\mu(Subf^x_\delta(\psi), X, \psi)$.

Thus, $Subf^x_\delta$ is the unique mapping given by Proposition 2.13. □

Proposition 2.32. *$Subf^x_\delta\varphi$ is a pattern.*

Proof. The proof is immediate by induction on φ, using the alternative definition of $Subf^x_\delta\varphi$. □

Example 2.33. [TRIVIAL CASES]

(i) $Subf^x_x\varphi = \varphi$.

(ii) If x does not occur free in φ, then $Subf^x_\delta \varphi = \varphi$.

Definition 2.34. *We say that x is **free for** δ in φ or that δ is **substitutable for** x in φ if the following hold:*

(i) if z is an element variable occuring free in δ and $\exists$ is a quantifier on z in φ with scope θ, then x does not occur free in θ.

(ii) if Z is a set variable occuring free in δ and μ is a binder on Z in φ with scope θ, then x does not occur free in θ.

Example 2.35. x is free for δ in φ in any of the following cases:

(i) x does not occur free in φ.

(ii) Any element or set variable of δ does not occur bound in φ.

(iii) $EVar(\delta) = SVar(\delta) = \emptyset$.

Remark 2.36. *Let $y \neq x$ be an element variable.*

(i) x is free for y in φ if the following holds: if $\exists$ is a quantifier on y in φ with scope θ, then x does not occur free in θ.

(ii) x is not free for y in φ if $\exists$ is a quantifier on y in φ with scope θ and x occurs free in θ.

2.6.1 Bounded substitution

Let φ be a pattern and x, y be element variables.

Definition 2.37. *We define $Subb^x_y \varphi$ to be the expression obtained from φ by replacing every bound occurence of x in φ with y.*

Remark 2.38 (Alternative definition)**.**
If $x = y$, then obviously $Subb^x_y \varphi = \varphi$. Assume that $x \neq y$. Then the mapping

$$Subb^x_y : Pattern \to Expr, \quad Subb^x_y(\varphi) = Subb^x_y \varphi$$

can be defined by recursion on patterns as follows:

$$\begin{aligned}
Subb^x_y(\varphi) &= \varphi && \text{if } \varphi \text{ is an atomic pattern,}\\
Subb^x_y(\circ\psi\chi) &= \circ Subb^x_y(\psi) Subb^x_y(\chi) && \text{for } \circ \in \{Appl, \to\},\\
Subb^x_y(\exists z\psi) &= \begin{cases} \exists y Subf^x_y\big(Subb^x_y(\psi)\big) & \text{if } x = z \\ \exists z Subb^x_y(\psi) & \text{if } x \neq z \end{cases}, &&\\
Subb^x_y(\mu X\psi) &= \mu X Subb^x_y(\psi). &&
\end{aligned}$$

Proof. Apply Recursion principle on patterns (Proposition 2.13) with $D = Expr$ and

$$G_0(\varphi) = \varphi, \quad G_\circ(\theta, \tau, \psi, \chi) = \circ\theta\tau \text{ for } \circ \in \{Appl, \to\},$$

$$G_\exists(\theta, z, \psi) = \begin{cases} \exists y Subf^x_y(\theta) & \text{if } x = z \\ \exists z\theta & \text{if } x \neq z \end{cases}, \quad G_\mu(\theta, X, \psi) = \mu X\theta.$$

Then

(i) If φ is an atomic pattern, $Subb^x_y(\varphi) = \varphi = G_0(\varphi)$.

(ii) For $\circ \in \{Appl, \to\}$, we have that

$$Subb^x_y(\circ\psi\chi) = \circ Subb^x_y(\psi) Subb^x_y(\chi) = G_\circ(Subb^x_y(\psi), Subb^x_y(\chi), \psi, \chi).$$

(iii) $Subb^x_y(\exists z\psi) = \begin{cases} \exists y Subf^x_y\big(Subb^x_y(\psi)\big) & \text{if } x = z \\ \exists z Subb^x_y(\psi) & \text{if } x \neq z \end{cases} = G_\exists(Subb^x_y(\psi), z, \psi)$

(iv) $Subb^x_y(\mu X\psi) = \mu X Subb^x_y(\psi) = G_\mu(Subb^x_y(\psi), X, \psi)$.

Thus, $Subb^x_y$ is the unique mapping given by Proposition 2.13. □

Proposition 2.39. *$Subb^x_y\varphi$ is a pattern.*

Proof. The proof is immediate by induction on φ, using the alternative definition of $Subb^x_y\varphi$. □

Proposition 2.40. *Assume that $x \neq y$ and y does not occur in φ. Then x is free for y in $Subb^x_y\varphi$.*

Proof. The proof is by induction on φ.

(i) φ is an atomic pattern. Then $Subb^x_y\varphi = \varphi$ and x is free for y in φ, as y does not occur in φ.

(ii) $\varphi = \circ\psi\chi$ for $\circ \in \{Appl, \to\}$. Then $Subb^x_y\varphi = \circ Subb^x_y(\psi) Subb^x_y(\chi)$. As y does not occur in ψ, χ, we can apply the induction hypothesis to get that x is free for y in $Subb^x_y\psi$, $Subb^x_y\chi$. It follows that x is free for y in $Subb^x_y\varphi$.

(iii) $\varphi = \exists z\psi$. We have two cases:

  (a) $x = z$. Then $Subb^x_y\varphi = \exists y Subf^x_y\big(Subb^x_y(\psi)\big)$. It is obvious that x does not occur in $Subb^x_y\varphi$, so x is free for y in $Subb^x_y\varphi$.

  (b) $x \neq z$. Then $Subb^x_y\varphi = \exists z Subb^x_y(\psi)$. As y does not occur in ψ, we can apply the induction hypothesis to get that x is free for y in $Subb^x_y\psi$. As y does not occur in φ, we must have that $y \neq z$. Then, obviously x is free for y in $Subb^x_y\varphi$.

(iv) $\varphi = \mu X\psi$. Then $Subb^x_y\varphi = \mu X Subb^x_y(\psi)$. As y does not occur in ψ, we can apply the induction hypothesis to get that x is free for y in $Subb^x_y\psi$. Then, obviously x is free for y in $Subb^x_y\varphi$.

□

2.7 Substitution of set variables

Let X be a set variable and φ, δ be patterns.

Definition 2.41. *We define $Subf^X_\delta\varphi$ to be the expression obtained from φ by replacing every free occurence of X in φ with δ.*

Remark 2.42 (Alternative definition)**.**
The mapping

$$Subf^X_\delta : Pattern \to Expr, \quad Subf^X_\delta(\varphi) = Subf^X_\delta\varphi$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
Subf^X_\delta(Z) & = & \begin{cases} \delta \text{ if } X = Z \\ Z \text{ if } X \neq Z \end{cases} & \text{if } Z \in SVar, \\
Subf^X_\delta(\varphi) & = & \varphi & \text{if } \varphi \in EVar \cup \Sigma, \\
Subf^X_\delta(\circ\psi\chi) & = & \circ Subf^X_\delta(\psi) Subf^X_\delta(\chi) & \text{for } \circ \in \{Appl, \to\}, \\
Subf^X_\delta(\exists x\psi) & = & \exists x Subf^X_\delta(\psi), & \\
Subf^X_\delta(\mu Z\psi) & = & \begin{cases} \mu Z\psi & \text{if } X = Z \\ \mu Z Subf^X_\delta(\psi) & \text{if } X \neq Z. \end{cases} &
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.13) with $D = Expr$ and

$$G_0(\varphi) = \begin{cases} \delta & \text{if } \varphi = X \\ \varphi & \text{if } \varphi \in EVar \cup (SVar \setminus \{X\}) \cup \Sigma \end{cases}$$
$$G_\circ(\theta, \tau, \psi, \chi) = \circ\theta\tau \text{ for } \circ \in \{Appl, \to\},$$
$$G_\exists(\theta, x, \psi) = \exists x\theta,$$
$$G_\mu(\theta, Z, \psi) = \begin{cases} \mu Z\psi & \text{if } X = Z \\ \mu Z\theta & \text{if } X \neq Z. \end{cases}$$

Then

(i) If φ is an atomic pattern, we have the following cases:

  (a) $\varphi = X$. Then $Subf_\delta^X(\varphi) = Subf_\delta^X(X) = \delta = G_0(\varphi)$.

  (b) $\varphi \in EVar \cup (SVar \setminus \{X\}) \cup \Sigma$. Then $Subf_\delta^X(\varphi) = \varphi = G_0(\varphi)$.

(ii) For $\circ \in \{Appl, \to\}$, we have that

$$Subf_\delta^X(\circ\psi\chi) = \circ Subf_\delta^X(\psi) Subf_\delta^X(\chi) = G_\circ(Subf_\delta^X(\psi), Subf_\delta^X(\chi), \psi, \chi).$$

(iii) $Subf_\delta^X(\exists x\psi) = \exists x Subf_\delta^X(\psi) = G_\exists(Subf_\delta^X(\psi), x, \psi)$.

(iv) $Subf_\delta^X(\mu Z\psi) = \begin{cases} \mu Z\psi & \text{if } X = Z \\ \mu Z Subf_\delta^X(\psi) & \text{if } X \neq Z. \end{cases} = G_\mu(Subf_\delta^X(\psi), Z, \psi)$

Thus, $Subf_\delta^X$ is the unique mapping given by Proposition 2.13. □

Proposition 2.43. *$Subf_\delta^X\varphi$ is a pattern.*

Proof. The proof is immediate by induction on φ, using the alternative definition of $Subf_\delta^X\varphi$. □

Definition 2.44. *We say that X is **free for** δ in φ or that δ is **substitutable for** X in φ if the following hold:*

(i) if z is an element variable occuring free in δ and $\exists$ is a quantifier on z in φ with scope θ, then X does not occur free in θ.

(ii) if Z is a set variable occuring free in δ and μ is a binder on Z in φ with scope θ, then X does not occur free in θ.

Example 2.45. X is free for δ in φ in any of the following cases:

(i) X does not occur free in φ.

(ii) Any element or set variable of δ does not occur bound in φ.

(iii) $EVar(\delta) = SVar(\delta) = \emptyset$.

2.7.1 Bounded substitution

Let φ be a pattern and X, Y be set variables.

Definition 2.46. *We define $Subb_Y^X\varphi$ to be the expression obtained from φ by replacing every bound occurence of X in φ with Y.*

Remark 2.47 (Alternative definition)**.**
If $X = Y$, then obviously $Subb^X_Y\varphi = \varphi$. Assume that $X \neq Y$. Then the mapping

$$Subb^X_Y : Pattern \to Expr, \quad Subb^X_Y(\varphi) = Subb^X_Y\varphi$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll} Subb^X_Y(\varphi) & = & \varphi & \textit{if } \varphi \textit{ is an atomic pattern,} \\ Subb^X_Y(\circ\psi\chi) & = & \circ Subb^X_Y(\psi)Subb^X_Y(\chi) & \textit{for } \circ \in \{Appl, \to\}, \\ Subb^X_Y(\exists x\psi) & = & \exists x Subb^X_Y(\psi), & \\ Subb^X_Y(\mu Z\psi) & = & \begin{cases} \mu Y Subf^X_Y\big(Subb^X_Y(\psi)\big) & \textit{if } X = Z \\ \mu Z Subb^X_Y(\psi) & \textit{if } X \neq Z \end{cases}. & \end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.13) with $D = Expr$ and

$$G_0(\varphi) = \varphi, \quad G_\circ(\theta, \tau, \psi, \chi) = \circ\theta\tau \text{ for } \circ \in \{Appl, \to\},$$
$$G_\exists(\theta, x, \psi) = \exists x\theta, \quad G_\mu(\theta, Z, \psi) = \begin{cases} \mu Y Subf^X_Y(\theta) & \text{if } X = Z \\ \mu Z\theta & \text{if } X \neq Z. \end{cases}$$

Then

(i) If φ is an atomic pattern, then $Subb^X_Y(\varphi) = \varphi = G_0(\varphi)$.

(ii) For $\circ \in \{Appl, \to\}$, we have that

$$Subb^X_Y(\circ\psi\chi) = \circ Subb^X_Y(\psi)Subb^X_Y(\chi) = G_\circ(Subb^X_Y(\psi), Subb^X_Y(\chi), \psi, \chi).$$

(iii) $Subb^X_Y(\exists x\psi) = \exists x Subb^X_Y(\psi) = G_\exists(Subb^X_Y(\psi), x, \psi)$.

(iv) $Subb^X_Y(\mu Z\psi) = \begin{cases} \mu Y Subf^X_Y\big(Subb^X_Y(\psi)\big) & \text{if } X = Z \\ \mu Z Subb^X_Y(\psi) & \text{if } X \neq Z \end{cases} = G_\mu(Subb^X_Y(\psi), Z, \psi)$.

Thus, $Subb^X_Y$ is the unique mapping given by Proposition 2.13. ☐

Proposition 2.48. *$Subb^X_Y\varphi$ is a pattern.*

Proof. The proof is immediate by induction on φ, using the alternative definition of $Subb^X_Y\varphi$. ☐

Proposition 2.49. *Assume that $X \neq Y$ and Y does not occur in φ. Then X is free for Y in $Subb^X_Y\varphi$.*

Proof. The proof is by induction on φ.

(i) φ is an atomic pattern. Then $Subb^X_Y\varphi = \varphi$ and X is free for Y in φ, as Y does not occur in φ.

(ii) $\varphi = \circ\psi\chi$ for $\circ \in \{Appl, \to\}$. Then $Subb^X_Y\varphi = Appl Subb^X_Y\psi Subb^X_Y\chi$. As Y does not occur in ψ, χ, we can apply the induction hypothesis to get that X is free for Y in $Subb^X_Y\psi$, $Subb^X_Y\psi$. It follows that X is free for Y in $Subb^X_Y\varphi$.

(iii) $\varphi = \exists x\psi$. Then $Subb^X_Y\varphi = \exists x Subb^X_Y\psi$. As Y does not occur in ψ, we can apply the induction hypothesis to get that X is free for Y in $Subb^X_Y\psi$. Then, obviously X is free for Y in $Subb^X_Y\varphi$.

(iv) $\varphi = \mu Z\psi$. We have two cases:

(a) $X = Z$. Then $Subb^X_Y \varphi = \mu Y Subf^X_Y \big(Subb^X_Y \psi\big)$. It is obvious that X does not occur in $Subb^X_Y \varphi$, so X is free for Y in $Subb^X_Y \varphi$.

(b) $X \neq Z$. Then $Subb^X_Y \varphi = \mu Z Subb^X_Y \psi$. As Y does not occur in ψ, we can apply the induction hypothesis to get that X is free for Y in $Subb^X_Y \psi$. As Y does not occur in φ, we must have that $Y \neq Z$. Then, obviously X is free for Y in $Subb^X_Y \varphi$.

□

2.8 Free substitution

Definition 2.50. *Let x be an element variable and φ, δ be patterns. We define $\varphi_x(\delta)$ as follows:*

(i) If x is free for δ in φ, then $\varphi_x(\delta) = Subf^x_\delta \varphi$.

(ii) Assume that x is not free for δ in φ and let $u_1, \ldots, u_k$ be the element variables and $U_1, \ldots, U_p$ be the set variables that occur bound in φ and also occur in δ. Let $z_1, \ldots, z_k$ be new element variables and $Z_1, \ldots, Z_p$ be new set variables, that do not occur in φ or δ. Consider the pattern

$$\theta = Subb^{U_1}_{Z_1} Subb^{U_2}_{Z_2} \ldots Subb^{U_p}_{Z_p} Subb^{u_1}_{z_1} Subb^{u_2}_{z_2} \ldots Subb^{u_k}_{z_k} \varphi.$$

Then $\varphi_x(\delta) = Subf^x_\delta \theta$.

Definition 2.51. *Let X be a set variable and φ, δ be patterns. We define $\varphi_X(\delta)$ as follows:*

(i) If X is free for δ in φ, then $\varphi_X(\delta) = Subf^X_\delta \varphi$.

(ii) Assume that X is not free for δ in φ and let $u_1, \ldots, u_k$ be the element variables and $U_1, \ldots, U_p$ be the set variables that occur bound in φ and also occur in δ. Let $z_1, \ldots, z_k$ be new element variables and $Z_1, \ldots, Z_p$ be new set variables, that do not occur in φ or δ. Consider the pattern

$$\theta = Subb^{U_1}_{Z_1} Subb^{U_2}_{Z_2} \ldots Subb^{U_p}_{Z_p} Subb^{u_1}_{z_1} Subb^{u_2}_{z_2} \ldots Subb^{u_k}_{z_k} \varphi.$$

Then $\varphi_X(\delta) = Subf^X_\delta \theta$.

Remark 2.52. *In the above definitions, we could define $z_1, \ldots, z_k$ and $Z_1, \ldots, Z_p$ as follows:*

(i) Let $l \in \mathbb{N}$ be maximum such that v_l occurs in φ or δ. Then define $z_i = v_{l+i}$ for all $i = 1, \ldots, k$.

(ii) Let $m \in \mathbb{N}$ be maximum such that V_m occurs in φ or δ. Then define $Z_i = V_{m+i}$ for all $i = 1, \ldots, p$.

2.9 Contexts

Let $\Box$ be a new symbol and let us denote

$$Sym_\Box = Sym_\tau \cup \{\Box\} \cup \{Appl_\Box\}.$$

Definition 2.53. *The τ-**contexts** are the expressions over $Sym_\Box$ inductively defined as follows:*

(i) $\Box$ is a τ-context.

(ii) If C is a τ-context and φ is a τ-pattern, then $Appl_\Box C\varphi$ and $Appl_\Box \varphi C$ are τ-contexts.

(iii) Only the expressions over $Sym_\Box$ obtained by applying the above rules are τ-contexts.

The set of τ-contexts is denoted by $\mathcal{C}_\tau$ and τ-contexts are denoted by C, C_1, C_2, ...

Remark 2.54. *The definition of τ-contexts can be written using the BNF notation:*

$$C ::= \Box \mid Appl_\Box C\varphi \mid Appl_\Box \varphi C.$$

Definition 2.55 (Alternative definition for τ-contexts).
*The set of τ-**contexts** is the intersection of all sets Γ of expressions over $Sym_\square$ that have the following properties:*

(i) $\square \in \Gamma$.

(ii) If $C \in \Gamma$ and φ is a τ-pattern, then $Appl_\square C\varphi, Appl_\square \varphi C \in \Gamma$.

Proposition 2.56. *[Induction principle on contexts]*
Let Γ be a set of τ-contexts satisfying the following properties:

(i) $\square \in \Gamma$.

(ii) If $C \in \Gamma$ and φ is a pattern, then $Appl_\square C\varphi, Appl_\square \varphi C \in \Gamma$.

Then $\Gamma = \mathcal{C}$.

Proof. By hypothesis, $\Gamma \subseteq \mathcal{C}$. By Definition 2.55, we get that $\mathcal{C} \subseteq \Gamma$. ☐

When the signature τ is clear from the context, we shall say simply context(s) and we shall denote the set of contexts by $\mathcal{C}$.

Proposition 2.57. *$\square$ occurs exactly once in every context C.*

Proof. The proof is by induction on the context C:

(i) $C = \square$. Obviously.

(ii) $C = Appl_\square C_1\psi$ and, by the induction hypothesis, $\square$ occurs exactly once in C_1. As ψ is a pattern, $\square$ does not occur in ψ. Thus, $\square$ occurs exactly once in C.

(iii) $C = Appl_\square \psi C_1$ and, by the induction hypothesis, $\square$ occurs exactly once in C_1. As ψ is a pattern, $\square$ does not occur in ψ. Thus, $\square$ occurs exactly once in C.

☐

2.9.1 Unique readability and recursion principle

Proposition 2.58 (Unique readability of contexts).

(i) Any context has a positive length.

(ii) If C is a context, then one of the following hold:

(a) $C = \square$.

(b) $C = Appl_\square C_1\varphi$, where C_1 is a context and φ is a pattern.

(c) $C = Appl_\square \varphi C_1$, where C_1 is a context and φ is a pattern.

(iii) Any proper initial segment of a context is not a context.

(iv) If C is a context, then exactly one of the cases from (ii) holds. Moreover, C can be written in a unique way in one of these forms.

Proof. Similarly with the proof of Proposition 2.10. ☐

Proposition 2.59 (Recursion principle on contexts).
Let A be a set, $\square^A \in A$ and the mappings

$$G_1, G_2 : A \times Pattern \to A.$$

Then there exists a unique mapping

$$F : \mathcal{C} \to A$$

that satisfies the following properties:

(i) $F(\Box) = \Box^A$ *for any atomic pattern* φ.

(ii) $F(Appl_\Box C_1 \varphi) = G_1(F(C_1), \varphi)$ *for any context* C_1 *and any pattern* φ.

(iii) $F(Appl_\Box \varphi C_1) = G_2(F(C_1), \varphi)$ *for any context* C_1 *and any pattern* φ.

Proof. Apply Proposition 2.58. □

2.9.2 Context substitution

Definition 2.60. *Let* C *be a context and* δ *be a pattern. We denote by* $C[\delta]$ *the* τ*-expression obtained by replacing all the occurences of* $\Box$ *with* δ *and all occurences of* $Appl_\Box$ *with* $Appl$.

Remark 2.61 (Alternative definition)**.**
$C[\delta]$ can be defined by recursion on contexts as follows:

$$\begin{array}{lcl} \Box[\delta] & = & \delta, \\ (Appl_\Box C_1 \varphi)[\delta] & = & Appl C_1[\delta]\varphi, \\ (Appl_\Box \varphi C_1)[\delta] & = & Appl \varphi C_1[\delta]. \end{array}$$

Proof. Let $F : \mathcal{C} \to Fun(Pattern, Expr_\tau)$ be defined by $F(C)(\delta) = C[\delta]$. We show in the sequel that F can be defined by recursion on patterns. The above definitions become

$$F(\Box)(\delta) = \delta, \quad F(Appl_\Box C_1\varphi)(\delta) = Appl F(C_1)(\delta)\varphi, \quad F(Appl_\Box \varphi C_1)(\delta) = Appl \varphi F(C_1)(\delta).$$

Apply Proposition 2.59 with

$$A = Fun(Pattern, Expr_\tau),\ \Box^A(\delta) = \delta,\ G_1(f,\varphi)(\delta) = Appl f(\delta)\varphi \text{ and } G_2(f,\varphi)(\delta) = Appl \varphi f(\delta)$$

for every $f : Pattern \to Expr_\tau$ and every patterns δ, φ. Then

(i) $F(\Box)(\delta) = \delta = \Box^A(\delta)$ for every pattern δ. It follows that $F(\Box) = \Box^A$.

(ii) $F(Appl_\Box C_1\varphi)(\delta) = Appl F(C_1)(\delta)\varphi = G_1(F(C_1), \varphi)(\delta)$ for every pattern δ. It follows that $F(Appl_\Box C_1\varphi) = G_1(F(C_1), \varphi)$.

(iii) $F(Appl_\Box \varphi C_1)(\delta) = Appl \varphi F(C_1)(\delta) = G_2(F(C_1), \varphi)(\delta)$ for every pattern δ. It follows that $F(Appl_\Box \varphi C_1) = G_2(F(C_1), \varphi)$.

Thus, F is the unique mapping given by Proposition 2.59. □

Proposition 2.62. $C[\delta]$ *is a pattern.*

Proof. The proof is by induction on the context C, using the alternative definition of $C[\delta]$.

(i) $C = \Box$. Then $C[\delta] = \delta$ is a pattern.

(ii) $C = Appl_\Box C_1\varphi$ and, by the induction hypothesis, $C_1[\delta]$ is a pattern. As $C[\delta] = Appl C_1[\delta]\varphi$, it follows that $C[\delta]$ is a pattern.

(iii) $C = Appl_\Box \varphi C_1$ and, by the induction hypothesis, $C_1[\delta]$ is a pattern. As $C[\delta] = Appl \varphi C_1[\delta]$, it follows that $C[\delta]$ is a pattern.

□

2.9.3 Free variables of contexts

Definition 2.63. *The set $FV(C)$ of* ***free variables*** *of C is defined by recursion on contexts as follows:*

$$
\begin{array}{lll}
FV(\Box) & = & \emptyset, \\
FV(Appl_{\Box} C_1 \varphi) & = & FV(C_1) \cup FV(\varphi), \\
FV(Appl_{\Box} \varphi C_1) & = & FV(C_1) \cup FV(\varphi).
\end{array}
$$

Proof. Apply Proposition 2.59 with

$$A = 2^{EVar \cup SVar}, \quad \Box^A = \emptyset, \quad G_1(V, \varphi) = G_2(V, \varphi) = V \cup FV(\varphi).$$

Then

(i) $FV(\Box) = \emptyset = \Box^A$.

(ii) $FV(Appl_{\Box} C_1 \varphi) = FV(C_1) \cup FV(\varphi) = G_1(FV(C_1), \varphi)$.

(iii) $FV(Appl_{\Box} \varphi C_1) = FV(C_1) \cup FV(\varphi) = G_2(FV(C_1), \varphi)$.

Thus, $FV : \mathcal{C} \to 2^{EVar \cup SVar}$ is the unique mapping given by Proposition 2.59. $\Box$

2.10 Derived connectives. Conventions

We shall write in the sequel $\varphi \cdot \psi$ instead of $Appl\varphi\psi$, $\varphi \to \psi$ instead of $\to \varphi\psi$, $\exists x.\varphi$ instead of $\exists x\varphi$, and $\mu X.\varphi$ instead of $\mu X\varphi$. We shall also use parentheses when it is necessary to make clear the structure of a pattern. For example, we write $(\varphi \to \psi) \cdot \chi$.
Furthermore, from now on, $\varphi_1, \varphi_2, \ldots$ will denote patterns
The pattern $\varphi \cdot \psi$ is called an **application**. As a convention, application is associative to the left. Thus, if $\varphi_1, \ldots, \varphi_n$ are patterns, we write $\varphi_1 \cdot \varphi_2 \cdot \varphi_3 \ldots \cdot \varphi_n$ instead of $(\ldots((\varphi_1 \cdot \varphi_2) \cdot \varphi_3) \ldots \cdot \varphi_n)$.
We introduce derived connectives by the following abbreviations:

$$
\begin{array}{lll}
\bot = \mu X.X & \neg\varphi = \varphi \to \bot, & \top = \neg\bot, \\
\varphi \vee \psi = \neg\varphi \to \psi, & \varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi), & \varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi), \\
\forall x.\varphi = \neg\exists x.\neg\varphi, & \nu X.\varphi = \neg\mu X.\neg Subf^{X}_{\neg X}\varphi. &
\end{array}
$$

The scope of $\exists$, $\forall$, μ, ν goes as far as possible to the right unless it is limited by parentheses. For example, $\forall x.\varphi \to \psi$ is $\forall x.(\varphi \to \psi)$, $\mu X.\varphi \wedge \psi$ is $\mu X.(\varphi \wedge \psi)$, and $\exists x.\psi \to ((\exists y.\varphi) \to z)$ is $\exists x.\big(\psi \to ((\exists y.\varphi) \to z)\big)$.
To reduce the use of parentheses, we assume that

(i) application has the highest precedence.

(ii) μ, ν have higher precedence than the quantifiers $\exists$, $\forall$.

(iii) quantifiers $\exists$, $\forall$ have higher precedence than the propositional connectives.

(iv) $\neg$ has higher precedence than $\to$, $\wedge$, $\vee$, $\leftrightarrow$;

(v) $\wedge, \vee$ have higher precedence than $\to$, $\leftrightarrow$.

We define in the sequel **finite conjunctions and disjunctions**. Let $\varphi_1, \ldots, \varphi_n (n \geq 1)$ be patterns. Then $\bigwedge_{i=1}^{n} \varphi_i$ and $\bigvee_{i=1}^{n} \varphi_i$ are defined inductively as follows:

$$\bigwedge_{i=1}^{1} \varphi_i = \varphi_1, \quad \bigwedge_{i=1}^{2} \varphi_i = \varphi_1 \wedge \varphi_2, \quad \bigwedge_{i=1}^{n+1} \varphi_i = \left(\bigwedge_{i=1}^{n} \varphi_i\right) \wedge \varphi_{n+1}, \tag{3}$$

$$\bigvee_{i=1}^{1} \varphi_i = \varphi_1, \quad \bigvee_{i=1}^{2} \varphi_i = \varphi_1 \vee \varphi_2, \quad \bigvee_{i=1}^{n+1} \varphi_i = \left(\bigvee_{i=1}^{n} \varphi_i\right) \vee \varphi_{n+1}. \tag{4}$$

We also write $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n$ instead of $\bigwedge_{i=1}^{n} \varphi_i$ and $\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_n$ instead of $\bigvee_{i=1}^{n} \varphi_i$.

3 Semantics

Let $\tau = (EVar, SVar, \Sigma)$ be a signature.

Definition 3.1. *A τ-structure is a pair*

$$\mathcal{A} = (A, \_\star\_, \Sigma^{\mathcal{A}}),$$

where

(i) A is a nonempty set.

*(ii) $\_\star\_ : A \times A \to 2^A$ is a binary **application** function;*

(iii) $\Sigma^{\mathcal{A}} = \{\sigma^{\mathcal{A}} \subseteq A \mid \sigma \in \Sigma\}$.

Let $\mathcal{A}$ be a τ-structure. A is called the **universe** or the **domain** of the structure $\mathcal{A}$. We use the notation $A = |\mathcal{A}|$. If $a \in A$, we also write $a \in \mathcal{A}$. Furthermore, $\sigma^{\mathcal{A}}$ is called the **interpretation** or the **denotation** of the constant σ in $\mathcal{A}$. We extend the application function $\_\star\_$ as follows:

$$\_\star\_ : 2^A \times 2^A \to 2^A, \quad B \star C = \bigcup_{b \in B, c \in C} b \star c.$$

Definition 3.2. *An **$\mathcal{A}$-valuation** is a function $e : EVar \cup SVar \to A \cup 2^A$ that satisfies:*

$$e(x) \in A \text{ for all } x \in EVar \text{ and } e(X) \subseteq A \text{ for all } X \in SVar.$$

$\mathcal{A}$-valuations are also called **$\mathcal{A}$-assignments**.

In the following, $\mathcal{A}$ is a τ-structure and e is an $\mathcal{A}$-valuation.

Notation 3.3. *For any variable $x \in EVar$ and any $a \in A$, we define a new $\mathcal{A}$-valuation $e_{x \mapsto a}$ as follows: for all $v \in EVar$ and all $V \in SVar$,*

$$e_{x \mapsto a}(V) = e(V) \text{ and } e_{x \mapsto a}(v) = \begin{cases} e(v) & \text{if } v \neq x \\ a & \text{if } v = x. \end{cases}$$

For any variable $X \in SVar$ and any $B \subseteq A$, we define a new $\mathcal{A}$-valuation $e_{X \mapsto B}$ as follows: for all $v \in EVar$ and all $V \in SVar$,

$$e_{X \mapsto B}(v) = e(v) \text{ and } e_{X \mapsto B}(V) = \begin{cases} e(V) & \text{if } V \neq X \\ B & \text{if } V = X. \end{cases}$$

Definition 3.4. *The **patterns $\mathcal{A}$-valuation***

$$e^+ : Pattern \to 2^A$$

is defined by recursion on patterns as follows:

(i) $e^+(x) = \{e(x)\}$ if $x \in EVar$.

(ii) $e^+(X) = e(X)$ if $X \in SVar$.

(iii) $e^+(\sigma) = \sigma^{\mathcal{A}}$ for every $\sigma \in \Sigma$.

(iv) $e^+(\varphi \cdot \psi) = e^+(\varphi) \star e^+(\psi)$.

(v) $e^+(\varphi \to \psi) = C_A(e^+(\varphi) \setminus e^+(\psi))$.

(vi) $e^+(\exists x.\varphi) = \bigcup_{a \in A} (e_{x \mapsto a})^+ (\varphi)$.

(vii) $e^+(\mu X.\varphi) = \bigcap \left\{ B \subseteq A \mid (e_{X \mapsto B})^+ (\varphi) \subseteq B \right\}$.

We say that φ **evaluates to** $B \subseteq A$ (with e) if $e^+(\varphi) = B$. If $a \in e^+(\varphi)$, we say that a **matches** φ (with witness e).

Proposition 3.5. *For all patterns φ, ψ,*

(i) $e^+(\varphi \to \psi) = C_A e^+(\varphi) \cup e^+(\psi)$.

(ii) $e^+(\bot) = \emptyset$.

(iii) $e^+(\neg\varphi) = C_A e^+(\varphi)$.

(iv) $e^+(\top) = A$.

(v) $e^+(\varphi \vee \psi) = e^+(\varphi) \cup e^+(\psi)$.

(vi) $e^+(\varphi \wedge \psi) = e^+(\varphi) \cap e^+(\psi)$.

(vii) $e^+(\varphi \leftrightarrow \psi) = A \setminus (e^+(\varphi) \Delta e^+(\psi))$.

(viii) $e^+(\forall x.\varphi) = \bigcap_{a \in A} (e_{x \mapsto a})^+ (\varphi)$.

Proof. (i) We have that

$$e^+(\varphi \to \psi) = C_A(e^+(\varphi) \setminus e^+(\psi)) = C_A e^+(\varphi) \cup e^+(\psi).$$

(ii) We have that

$$\begin{aligned} e^+(\bot) = e^+(\mu X.X) &= \bigcap \left\{ B \subseteq A \mid (e_{X \mapsto B})^+ (X) \subseteq B \right\} = \bigcap \{ B \subseteq A \mid e_{X \mapsto B}(X) \subseteq B \} \\ &= \bigcap \{ B \subseteq A \mid B \subseteq B \} = \bigcap \{ B \subseteq A \} = \emptyset. \end{aligned}$$

(iii) $e^+(\neg\varphi) = e^+(\varphi \to \bot) = C_A e^+(\varphi) \cup e^+(\bot) = C_A e^+(\varphi) \cup \emptyset = C_A e^+(\varphi)$.

(iv) $e^+(\top) = e^+(\neg\bot) = C_A e^+(\bot) = C_A \emptyset = A$.

(v) We have that

$$\begin{aligned} e^+(\varphi \vee \psi) &= e^+(\neg\varphi \to \psi) = C_A e^+(\neg\varphi) \cup e^+(\psi) = C_A(C_A e^+(\varphi)) \cup e^+(\psi) \\ &= e^+(\varphi) \cup e^+(\psi). \end{aligned}$$

(vi) We have that

$$\begin{aligned} e^+(\varphi \wedge \psi) &= e^+(\neg(\neg\varphi \vee \neg\psi)) = C_A e^+(\neg\varphi \vee \neg\psi) = C_A(e^+(\neg\varphi) \cup e^+(\neg\psi)) \\ &= C_A(C_A e^+(\varphi) \cup C_A e^+(\psi)) = e^+(\varphi) \cap e^+(\psi). \end{aligned}$$

(vii) We have that

$$\begin{aligned} e^+(\varphi \leftrightarrow \psi) &= e^+\big((\varphi \to \psi) \wedge (\psi \to \varphi)\big) = e^+(\varphi \to \psi) \cap e^+(\psi \to \varphi) \\ &= C_A(e^+(\varphi) \setminus e^+(\psi)) \cap C_A(e^+(\psi) \setminus e^+(\varphi)) = C_A(e^+(\varphi) \Delta e^+(\psi)). \end{aligned}$$

(viii) We have that

$$\begin{aligned} e^+(\forall x.\varphi) &= e^+(\neg\exists x.\neg\varphi) = C_A e^+(\exists x.\neg\varphi) = C_A \left(\bigcup_{a \in A} (e_{x \mapsto a})^+ (\neg\varphi) \right) \\ &= C_A \left(\bigcup_{a \in A} \left(C_A (e_{x \mapsto a})^+ (\varphi) \right) \right) = \bigcap_{a \in A} C_A \left(C_A (e_{x \mapsto a})^+ (\varphi) \right) = \bigcap_{a \in A} (e_{x \mapsto a})^+ (\varphi). \end{aligned}$$

□

Definition 3.6. *Let φ be a pattern. We say that*

*(i) e **satisfies** φ in $\mathcal{A}$ if $e^+(\varphi) = A$.*
Notation: $\mathcal{A} \vDash \varphi[e]$.

*(ii) e **does not satisfy** φ in $\mathcal{A}$ if $e^+(\varphi) \neq A$.*
Notation: $\mathcal{A} \nvDash \varphi[e]$.

Proposition 3.7. *(i) For every element variable x, $\mathcal{A} \vDash x[e]$ iff $|A| = 1$.*

(ii) For every set variable X, $\mathcal{A} \vDash X[e]$ iff $e(X) = A$.

(iii) For every constant σ, $\mathcal{A} \vDash \sigma[e]$ iff $\sigma^{\mathcal{A}} = A$.

(iv) $\mathcal{A} \nvDash \bot[e]$ and $\mathcal{A} \vDash \top[e]$.

(v) For every pattern φ, $\mathcal{A} \vDash \neg\varphi[e]$ iff $e^+(\varphi) = \emptyset$.

(vi) For every patterns φ, ψ, $\mathcal{A} \vDash (\varphi \cdot \psi)[e]$ iff $\bigcup_{b \in e^+(\varphi), c \in e^+(\psi)} b \star c = A$.

(vii) For every patterns φ, ψ, $\mathcal{A} \vDash (\varphi \wedge \psi)[e]$ iff ($\mathcal{A} \vDash \varphi[e]$ and $\mathcal{A} \vDash \psi[e]$).

(viii) For every patterns φ, ψ, $\mathcal{A} \vDash (\varphi \vee \psi)[e]$ iff $e^+(\varphi) \cup e^+(\psi) = A$.

(ix) For every patterns φ, ψ, $\mathcal{A} \vDash (\varphi \to \psi)[e]$ iff $e^+(\varphi) \subseteq e^+(\psi)$.

(x) For every patterns φ, ψ, $\mathcal{A} \vDash (\varphi \leftrightarrow \psi)[e]$ iff $e^+(\varphi) = e^+(\psi)$.

(xi) For every patterns φ, ψ, $\mathcal{A} \vDash (\varphi \leftrightarrow \psi)[e]$ iff ($\mathcal{A} \vDash (\varphi \to \psi)[e]$ and $\mathcal{A} \vDash (\psi \to \varphi)[e]$).

(xii) For every pattern φ and every element variable x, $\mathcal{A} \vDash (\exists x.\varphi)[e]$ iff $\bigcup_{a \in A} (e_{x \mapsto a})^+(\varphi) = A$.

(xiii) For every pattern φ and every element variable x, if there exists $b \in A$ such that $\mathcal{A} \vDash \varphi[e_{x \mapsto b}]$, then $\mathcal{A} \vDash (\exists x.\varphi)[e]$.

(xiv) For every pattern φ and every element variable x, $\mathcal{A} \vDash (\forall x.\varphi)[e]$ iff $\mathcal{A} \vDash \varphi[e_{x \mapsto a}]$ for every $a \in A$.

(xv) For every pattern φ and every set variable X, if $\mathcal{A} \vDash \varphi[e_{X \mapsto B}]$ for every $B \subseteq A$, then $\mathcal{A} \vDash (\mu X.\varphi)[e]$.

Proof. (i) $\mathcal{A} \vDash x[e]$ iff $e^+(x) = A$ iff $\{e(x)\} = A$ iff $|A| = 1$.

(ii) $\mathcal{A} \vDash X[e]$ iff $e^+(X) = A$ iff $e(X) = A$.

(iii) $\mathcal{A} \vDash \sigma[e]$ iff $e^+(\sigma) = A$ iff $\sigma^{\mathcal{A}} = A$.

(iv) Obviously, since $e^+(\bot) = \emptyset$ and $e^+(\top) = A$.

(v) $\mathcal{A} \vDash \neg\varphi[e]$ iff $e^+(\neg\varphi) = A$ iff $A \setminus e^+(\varphi) = A$ iff $e^+(\varphi) = \emptyset$.

(vi) $\mathcal{A} \vDash (\varphi \cdot \psi)[e]$ iff $e^+(\varphi \cdot \psi) = A$ iff $e^+(\varphi) \star e^+(\psi) = A$ iff $\bigcup_{b \in e^+(\varphi), c \in e^+(\psi)} b \star c = A$.

(vii) $\mathcal{A} \vDash (\varphi \wedge \psi)[e]$ iff $e^+(\varphi \wedge \psi) = A$ iff $e^+(\varphi) \cap e^+(\psi) = A$ iff $e^+(\varphi) = e^+(\psi) = A$ (since $e^+(\varphi), e^+(\psi) \subseteq A$) iff ($\mathcal{A} \vDash \varphi[e]$ and $\mathcal{A} \vDash \psi[e]$).

(viii) $\mathcal{A} \vDash (\varphi \vee \psi)[e]$ iff $e^+(\varphi \vee \psi) = A$ iff $e^+(\varphi) \cup e^+(\psi) = A$.

(ix) $\mathcal{A} \vDash (\varphi \to \psi)[e]$ iff $e^+(\varphi \to \psi) = A$ iff $C_A(e^+(\varphi) \setminus e^+(\psi)) = A$ iff $e^+(\varphi) \setminus e^+(\psi) = \emptyset$ iff $e^+(\varphi) \subseteq e^+(\psi)$.

(x) $\mathcal{A} \vDash (\varphi \leftrightarrow \psi)[e]$ iff $e^+(\varphi \leftrightarrow \psi) = A$ iff $A \setminus (e^+(\varphi) \Delta e^+(\psi)) = A$ iff $e^+(\varphi) \Delta e^+(\psi) = \emptyset$ iff $e^+(\varphi) = e^+(\psi)$.

(xi) Apply (vii).

(xii) $\mathcal{A} \vDash (\exists x.\varphi)[e]$ iff $e^+(\exists x.\varphi) = A$ iff $\bigcup_{a\in A} (e_{x\mapsto a})^+(\varphi) = A$.

(xiii) Assume that $b \in A$ is such that $\mathcal{A} \vDash \varphi[e_{x\mapsto b}]$, so $(e_{x\mapsto b})^+(\varphi) = A$. Since $(e_{x\mapsto b})^+(\varphi) \subseteq \bigcup_{a\in A} (e_{x\mapsto a})^+(\varphi) = e^+(\exists x.\varphi)$, we must have $e^+(\exists x.\varphi) = A$, hence $\mathcal{A} \vDash (\exists x.\varphi)[e]$.

(xiv) $\mathcal{A} \vDash (\forall x\varphi)[e]$ iff $e^+(\forall x.\varphi) = A$ iff $\bigcap_{a\in A} (e_{x\mapsto a})^+(\varphi) = A$ iff for every $a \in A$, $(e_{x\mapsto a})^+(\varphi) = A$ iff for every $a \in A$, $\mathcal{A} \vDash \varphi[e_{x\mapsto a}]$.

(xv) Let us denote $\mathcal{B} = \left\{B \subseteq A \mid (e_{X\mapsto B})^+(\varphi) \subseteq B\right\}$. Obviously, $A \in \mathcal{B}$. Assume that for every $B \subseteq A$ we have that $\mathcal{A} \vDash \varphi[e_{X\mapsto B}]$, hence $(e_{X\mapsto B})^+(\varphi) = A$. It follows that $\mathcal{B} = \{A\}$, since if $B \subseteq A, B \neq A$, then $(e_{X\mapsto B})^+(\varphi) = A \not\subseteq B$. Thus, $e^+(\mu X.\varphi) = \bigcap \mathcal{B} = A$, so $\mathcal{A} \vDash (\mu X.\varphi)[e]$.

□

Proposition 3.8. *Let $\mathcal{A}$ be a τ-structure. Then for every pattern φ, the following holds:*

$(\star)$ *for every $\mathcal{A}$-valuations e_1, e_2, if $e_1|_{FV(\varphi)} = e_2|_{FV(\varphi)}$, then $e_1^+(\varphi) = e_2^+(\varphi)$.*

Proof. We prove $(\star)$ by induction on φ.

(i) $\varphi = x \in EVar$. Then $FV(\varphi) = \{x\}$. Let e_1, e_2 be two $\mathcal{A}$-valuations such that $e_1(x) = e_2(x)$. We get that $e_1^+(\varphi) = \{e_1(x)\} = \{e_2(x)\} = e_2^+(\varphi)$.

(ii) $\varphi = X \in SVar$. Then $FV(\varphi) = \{X\}$. Let e_1, e_2 be two $\mathcal{A}$-valuations such that $e_1(X) = e_2(X)$. We get that $e_1^+(\varphi) = e_1(X) = e_2(X) = e_2^+(\varphi)$.

(iii) $\varphi = \sigma \in \Sigma$. Then $FV(\varphi) = \emptyset$. Let e_1, e_2 be two $\mathcal{A}$-valuations. Then $e_1^+(\varphi) = \sigma^{\mathcal{A}} = e_2^+(\varphi)$.

(iv) $\varphi = \psi \cdot \chi$. Then $FV(\varphi) = FV(\psi) \cup FV(\chi)$. Let e_1, e_2 be two $\mathcal{A}$-valuations such that $e_1|_{FV(\varphi)} = e_2|_{FV(\varphi)}$. Then $e_1|_{FV(\psi)} = e_2|_{FV(\psi)}$ and $e_1|_{FV(\chi)} = e_2|_{FV(\chi)}$. Applying the induction hypothesis for ψ, χ, it follows that $e_1^+(\psi) = e_2^+(\psi)$ and $e_1^+(\chi) = e_2^+(\chi)$. We get that $e_1^+(\varphi) = e_1^+(\psi) \star e_1^+(\chi) = e_2^+(\psi) \star e_2^+(\chi) = e_2^+(\varphi.)$

(v) $\varphi = \psi \to \chi$. Then $FV(\varphi) = FV(\psi) \cup FV(\chi)$. Let e_1, e_2 be two $\mathcal{A}$-valuations such that $e_1|_{FV(\varphi)} = e_2|_{FV(\varphi)}$. Then $e_1|_{FV(\psi)} = e_2|_{FV(\psi)}$ and $e_1|_{FV(\chi)} = e_2|_{FV(\chi)}$. Applying the induction hypothesis for ψ, χ, it follows that $e_1^+(\psi) = e_2^+(\psi)$ and $e_1^+(\chi) = e_2^+(\chi)$. We get that $e_1^+(\varphi) = C_A(e_1^+(\psi) \setminus e_1^+(\chi)) = C_A(e_2^+(\psi) \setminus e_2^+(\chi)) = e_2^+(\varphi)$.

(vi) $\varphi = \exists x.\psi$. Then $FV(\varphi) = FV(\psi) \setminus \{x\}$. Let e_1, e_2 be two $\mathcal{A}$-valuations such that $e_1|_{FV(\varphi)} = e_2|_{FV(\varphi)}$. For every $a \in A$, we have that $e_{1x\mapsto a}|_{FV(\psi)} = e_{2x\mapsto a}|_{FV(\psi)}$, hence, applying the induction hypothesis for ψ, that $(e_{1x\mapsto a})^+(\psi) = (e_{2x\mapsto a})^+(\psi)$. We get that $e_1^+(\varphi) = \bigcup_{a\in A} (e_{1x\mapsto a})^+(\psi) = \bigcup_{a\in A} (e_{2x\mapsto a})^+(\psi) = e_2^+(\varphi)$.

(vii) $\varphi = \mu X.\psi$. Then $FV(\varphi) = FV(\psi) \setminus \{X\}$. Let e_1, e_2 be two $\mathcal{A}$-valuations such that $e_1|_{FV(\varphi)} = e_2|_{FV(\varphi)}$. For every $B \subseteq A$, we have that $e_{1X\mapsto B}|_{FV(\psi)} = e_{2X\mapsto B}|_{FV(\psi)}$, hence, applying the induction hypothesis for ψ, that $(e_{1X\mapsto B})^+(\psi) = (e_{2X\mapsto B})^+(\psi)$. We get that $e_1^+(\varphi) = \bigcap\{B \subseteq A \mid (e_{1X\mapsto B})^+(\psi) \subseteq B\} = \bigcap\{B \subseteq A \mid (e_{2X\mapsto B})^+(\psi) \subseteq B\} = e_2^+(\varphi)$.

□

Hence, the $\mathcal{A}$-valuation of a pattern depends only on the $\mathcal{A}$-valuations of the free variables of the pattern. As an immediate application of Proposition 3.8, we get

Corollary 3.9. *Let $\mathcal{A}$ be a τ-structure, φ be a pattern and e_1, e_2 be two $\mathcal{A}$-valuations such that $e_1|_{FV(\varphi)} = e_2|_{FV(\varphi)}$. Then $\mathcal{A} \vDash \varphi[e_1]$ iff $\mathcal{A} \vDash \varphi[e_2]$.*

3.1 Model, validity, satisfiability

Definition 3.10. *Let φ be a pattern and $\mathcal{A}$ be a τ-structure. We say that $\mathcal{A}$* ***satisfies*** *φ or that $\mathcal{A}$ is a* ***model*** *of φ if $\mathcal{A} \vDash \varphi[e]$ for every $\mathcal{A}$-valuation e.*
Notation: $\mathcal{A} \vDash \varphi$.

Proposition 3.11. *Let $\mathcal{A}$ be a τ-structure.*

(i) For every element variable x, $\mathcal{A} \vDash x$ iff $|A| = 1$.

(ii) For every constant symbol σ, $\mathcal{A} \vDash \sigma$ iff $\sigma^{\mathcal{A}} = A$.

(iii) $\mathcal{A} \nvDash \bot$ and $\mathcal{A} \vDash \top$.

(iv) For every patterns φ, ψ, $\mathcal{A} \vDash \varphi \wedge \psi$ iff ($\mathcal{A} \vDash \varphi$ and $\mathcal{A} \vDash \psi$).

(v) For every patterns φ, ψ, φ, $\mathcal{A} \vDash \varphi \leftrightarrow \psi$ iff ($\mathcal{A} \vDash \varphi \to \psi$ and $\mathcal{A} \vDash \psi \to \varphi$).

(vi) For every patterns φ, ψ, if $\mathcal{A} \vDash \varphi \to \psi$ and $\mathcal{A} \vDash \varphi$, then $\mathcal{A} \vDash \psi$.

(vii) For every patterns φ, ψ, if $\mathcal{A} \vDash \varphi \leftrightarrow \psi$, then ($\mathcal{A} \vDash \varphi$ iff $\mathcal{A} \vDash \psi$).

(viii) For every pattern φ and every element variable x, $\mathcal{A} \vDash \forall x.\varphi$ iff $\mathcal{A} \vDash \varphi$.

Proof. (i)-(vii) are obtained as an immediate consequence of Proposition 3.7.
(viii) $\Rightarrow$ Let e be an arbitrary $\mathcal{A}$-valuation. Since $\mathcal{A} \vDash \forall x.\varphi$, we have that $\mathcal{A} \vDash \varphi[e_{x \mapsto a}]$ for all $a \in A$. By letting $a := e(x)$, we get that $e = e_{x \mapsto a}$. Thus, $\mathcal{A} \vDash \varphi[e]$.
$\Leftarrow$ Let e be an arbitrary $\mathcal{A}$-valuation and $a \in A$. We have to prove that $\mathcal{A} \vDash \varphi[e_{x \mapsto a}]$. This follows immediately from the fact that $\mathcal{A} \vDash \varphi$. $\square$

Definition 3.12. *Let φ be a pattern. We say that φ is* ***valid*** *if $\mathcal{A} \vDash \varphi$ for every τ-structure $\mathcal{A}$.*
Notation: $\vDash \varphi$.

As an application of Propositions 3.11, we get

Remark 3.13. *(i) $\top$ is valid.*

(ii) For every patterns φ, ψ, $\vDash \varphi \wedge \psi$ iff ($\vDash \varphi$ and $\vDash \psi$).

(iii) For every patterns φ, ψ, $\vDash \varphi \leftrightarrow \psi$ iff ($\vDash \varphi \to \psi$ and $\vDash \psi \to \varphi$).

(iv) For every patterns φ, ψ, if $\vDash \varphi \to \psi$ and $\vDash \varphi$, then $\vDash \psi$.

(v) For every patterns φ, ψ, if $\vDash \varphi \leftrightarrow \psi$, then ($\vDash \varphi$ iff $\vDash \psi$).

In the sequel, we write "Let $(\mathcal{A}, e)$" instead of "Let $\mathcal{A}$ be a τ-structure and e be an $\mathcal{A}$-valuation", "there exists $(\mathcal{A}, e)$" instead of "there exists a τ-structure $\mathcal{A}$ and an $\mathcal{A}$-valuation e" and "for all $(\mathcal{A}, e)$" instead of "for every τ-structure $\mathcal{A}$ and every $\mathcal{A}$-valuation e".

3.1.1 Finite disjunctions and conjunctions

Let $\varphi_1, \ldots, \varphi_n (n \geq 1)$ be patterns.

Proposition 3.14. *Let $(\mathcal{A}, e)$. Then*

$$e^+(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) = \bigcap_{i=1}^{n} e^+(\varphi_i) \quad \textit{and} \quad e^+(\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_n) = \bigcup_{i=1}^{n} e^+(\varphi_i) \tag{5}$$

Proof. By induction on n. The case $n=1$ is obvious. For $n=2$ apply Proposition 3.5.(vi), (v).
$n \Rightarrow n+1$: We have that

$$\begin{aligned}
e^+(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n \wedge \varphi_{n+1}) &= e^+((\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) \wedge \varphi_{n+1}) \\
&= e^+(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n) \cap e^+(\varphi_{n+1}) = \bigcap_{i=1}^{n} e^+(\varphi_i) \cap e^+(\varphi_{n+1}) \\
&= \bigcap_{i=1}^{n+1} e^+(\varphi_i), \\
e^+(\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_n \vee \varphi_{n+1}) &= e^+((\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_n) \vee \varphi_{n+1}) \\
&= e^+(\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_n) \cup e^+(\varphi_{n+1}) = \bigcup_{i=1}^{n} e^+(\varphi_i) \cup e^+(\varphi_{n+1}) \\
&= \bigcup_{i=1}^{n+1} e^+(\varphi_i).
\end{aligned}$$

$\square$

It follows immediately that

Proposition 3.15. *The following hold:*

(i) For every $(\mathcal{A}, e)$,

- *(a) $\mathcal{A} \vDash (\varphi_1 \wedge \ldots \wedge \varphi_n)[e]$ iff $\mathcal{A} \vDash \varphi_i[e]$ for every $i = 1, \ldots, n$.*
- *(b) $\mathcal{A} \vDash (\varphi_1 \vee \ldots \vee \varphi_n)[e]$ iff $\bigcup_{i=1}^{n} e^+(\varphi_i) = A$.*

(ii) For every $\mathcal{A}$, $\mathcal{A} \vDash \varphi_1 \wedge \ldots \wedge \varphi_n$ iff $\mathcal{A} \vDash \varphi_i$ for every $i = 1, \ldots, n$.

(iii) $\vDash \varphi_1 \wedge \ldots \wedge \varphi_n$ iff $\vDash \varphi_i$ for every $i = 1, \ldots, n$.

3.2 Semantic consequence and equivalence

We define in the sequel three notions of semantic consequence and logic equivalence.

3.2.1 Notion 1

Definition 3.16. *Let φ, ψ be patterns. We say that*

*(i) ψ is a **global semantic consequence** of φ if for every $\mathcal{A}$,*

$$\mathcal{A} \vDash \varphi \text{ implies } \mathcal{A} \vDash \psi.$$

Notation: $\varphi \vDash_g \psi$.

*(ii) φ and ψ are **globally logically equivalent** if for every $\mathcal{A}$, $\mathcal{A} \vDash \varphi$ iff $\mathcal{A} \vDash \psi$.*
Notation: $\varphi \vDash\!\!\dashv_g \psi$.

Remark 3.17. *$\varphi \vDash\!\!\dashv_g \psi$ iff ($\varphi \vDash_g \psi$ and $\psi \vDash_g \varphi$).*

Proof. Obviously. $\square$

Remark 3.18. *(i) If $\varphi \vDash_g \psi$, then $\vDash \varphi$ implies $\vDash \psi$.*

(ii) If $\varphi \vDash\!\!\dashv_g \psi$, then $\vDash \varphi$ iff $\vDash \psi$.

Proof. (i) Assume that $\varphi \vDash_g \psi$ and that $\vDash \varphi$. We have to prove that $\vDash \psi$, hence that for every $\mathcal{A}$, $\mathcal{A} \vDash \psi$. Let $\mathcal{A}$ be arbitrary. Since $\vDash \varphi$, we have that $\mathcal{A} \vDash \varphi$. Since $\varphi \vDash_g \psi$, it follows that $\mathcal{A} \vDash \psi$.

(ii) By (i).

$\square$

3.2.2 Notion 2

Definition 3.19. *Let φ, ψ be patterns. We say that*

*(i) ψ is a **local semantic consequence** of φ if for every $(\mathcal{A}, e)$,*

$$\mathcal{A} \vDash \varphi[e] \text{ implies } \mathcal{A} \vDash \psi[e].$$

Notation: $\varphi \vDash_l \psi$.

*(ii) φ and ψ are **locally logically equivalent** if for every $(\mathcal{A}, e)$, $\mathcal{A} \vDash \varphi[e]$ iff $\mathcal{A} \vDash \psi[e]$.*
Notation: $\varphi \mathrel{\text{⫤}\!\vDash}_l \psi$.

Remark 3.20. *$\varphi \mathrel{\text{⫤}\!\vDash}_l \psi$ iff ($\varphi \vDash_l \psi$ and $\psi \vDash_l \varphi$).*

Remark 3.21. *(i) If $\varphi \vDash_l \psi$, then $\vDash \varphi$ implies $\vDash \psi$.*

(ii) If $\varphi \mathrel{\text{⫤}\!\vDash}_l \psi$, then $\vDash \varphi$ iff $\vDash \psi$.

Proof. (i) Assume that $\varphi \vDash_l \psi$ and that $\vDash \varphi$. We have to prove that $\vDash \psi$, hence that for every $(\mathcal{A}, e)$, $\mathcal{A} \vDash \psi[e]$. Let $(\mathcal{A}, e)$ be arbitrary. Since $\vDash \varphi$, we have that $\mathcal{A} \vDash \varphi[e]$. Since $\varphi \vDash_l \psi$, it follows that $\mathcal{A} \vDash \psi[e]$.

(ii) By (i).

□

3.2.3 Notion 3

Definition 3.22. *Let φ, ψ be patterns. We say that*

*(i) ψ is a **strong semantic consequence** of φ if for every $(\mathcal{A}, e)$, $e^+(\varphi) \subseteq e^+(\psi)$.*
Notation: $\varphi \vDash_s \psi$.

*(ii) φ and ψ are **strongly logically equivalent** if for every $(\mathcal{A}, e)$, $e^+(\varphi) = e^+(\psi)$.*
Notation: $\varphi \mathrel{\text{⫤}\!\vDash}_s \psi$.

Remark 3.23. *$\varphi \mathrel{\text{⫤}\!\vDash}_s \psi$ iff ($\varphi \vDash_s \psi$ and $\psi \vDash_s \varphi$).*

Remark 3.24. *(i) $\varphi \vDash_s \psi$ iff $\vDash \varphi \to \psi$.*

(ii) $\varphi \mathrel{\text{⫤}\!\vDash}_s \psi$ iff $\vDash \varphi \leftrightarrow \psi$.

Proof. (i) $\varphi \vDash_s \psi$ iff for every e, $e^+(\varphi) \subseteq e^+(\psi)$ iff for every e, $\vDash (\varphi \to \psi)[e]$ (by Proposition 3.7.(ix)) iff $\vDash \varphi \to \psi$.

(ii) $\varphi \mathrel{\text{⫤}\!\vDash}_s \psi$ iff for every e, $e^+(\varphi) = e^+(\psi)$ iff for every e, $\vDash (\varphi \leftrightarrow \psi)[e]$ (by Proposition 3.7.(x)) iff $\vDash \varphi \leftrightarrow \psi$.

□

Remark 3.25. *(i) If $\varphi \vDash_s \psi$, then $\vDash \varphi$ implies $\vDash \psi$.*

(ii) If $\varphi \mathrel{\text{⫤}\!\vDash}_s \psi$, then $\vDash \varphi$ iff $\vDash \psi$.

Proof. By Remark 3.24 and Remark 3.13,(iv),(v). □

3.2.4 Relations between the three notions

Proposition 3.26. *(i) $\varphi \vDash_s \psi$ implies $\varphi \vDash_l \psi$ implies $\varphi \vDash_g \psi$.*

(ii) $\varphi \mathrel{\text{⫤}\!\vDash}_s \psi$ implies $\varphi \mathrel{\text{⫤}\!\vDash}_l \psi$ implies $\varphi \mathrel{\text{⫤}\!\vDash}_g \psi$.

Proof. (i) $\varphi \vDash_s \psi$ implies $\varphi \vDash_l \psi$: Let $(\mathcal{A}, e)$ be such that $\mathcal{A} \vDash \varphi[e]$. We have to prove that $\mathcal{A} \vDash \psi[e]$. As $\mathcal{A} \vDash \varphi[e]$, we have that $e^+(\varphi) = A$. Since $e^+(\varphi) \subseteq e^+(\psi)$, it follows that $e^+(\psi) = A$, hence $\mathcal{A} \vDash \psi[e]$.

$\varphi \vDash_l \psi$ implies $\varphi \vDash_g \psi$: Let $\mathcal{A} \vDash \varphi$ and e be an arbitrary $\mathcal{A}$-valuation. We have to prove that $\mathcal{A} \vDash \psi[e]$. Since $\mathcal{A} \vDash \varphi$, we have that $\mathcal{A} \vDash \varphi[e]$. As $\varphi \vDash_l \psi$, it follows that $\mathcal{A} \vDash \psi[e]$.

(ii) Follows immediately from (i).

□

Remark 3.27. *Let x, y be element variables. Then $x \mathrel{\text{⫤}\!\vDash}_l y$, hence $x \mathrel{\text{⫤}\!\vDash}_g y$.*

Proof. Let (A, e) be arbitrary. Applying Proposition 3.7.(i), we get that (A, e) is a model of x iff $\mathcal{A} \vDash x[e]$ iff $|A| = 1$ iff $\mathcal{A} \vDash y[e]$ iff (A, e) is a model of y. Thus, $x \mathrel{\text{⫤}\!\vDash}_l y$. By Proposition 3.26.(ii), it follows that $x \mathrel{\text{⫤}\!\vDash}_g y$. □

Remark 3.28. *(i) $\varphi \vDash_g \psi$ does not imply $\varphi \vDash_l \psi$ and $\varphi \mathrel{\text{⫤}\!\vDash}_g \psi$ does not imply $\varphi \mathrel{\text{⫤}\!\vDash}_l \psi$.*

(ii) $\varphi \vDash_l \psi$ does not imply $\varphi \vDash_s \psi$ and $\varphi \mathrel{\text{⫤}\!\vDash}_l \psi$ does not imply $\varphi \mathrel{\text{⫤}\!\vDash}_s \psi$.

Proof. Let x and y be two different element variables. We give the following counterexamples.

(i) Let us prove that $x \vee y \mathrel{\text{⫤}\!\vDash}_g x \wedge y$. Let $\mathcal{A}$ be arbitrary. Then $\mathcal{A} \vDash x \vee y$ iff (for every e, $\mathcal{A} \vDash (x \vee y)[e]$) iff (for every e, $e^+(x \vee y) = A$) iff (for every e, $\{e(x)\} \cup \{e(y)\} = A$) iff $|A| = 1$, as we can take e such that $e(x) = e(y)$. Furthermore, $\mathcal{A} \vDash x \wedge y$ iff (for every e, $\mathcal{A} \vDash (x \wedge y)[e]$) iff (for every e, $e^+(x \wedge y) = A$) iff (for every e, $\{e(x)\} \cap \{e(y)\} = A$) iff $|A| = 1$.

Thus, $\mathcal{A} \vDash x \vee y$ iff $\mathcal{A} \vDash x \wedge y$ iff $|A| = 1$. Thus, $x \vee y \mathrel{\text{⫤}\!\vDash}_g x \wedge y$.

Let us prove now that $x \vee y \nvDash_l x \wedge y$. Let $A = \{1, 2\}$ and e be such that $e(x) = 1$ and $e(y) = 2$. Then $e^+(x \vee y) = \{e(x)\} \cup \{e(y)\} = A$. Thus, $\mathcal{A} \vDash (x \vee y)[e]$. On the other hand, $e^+(x \wedge y) = \{e(x)\} \cap \{e(y)\} = \emptyset$, hence $\mathcal{A} \nvDash (x \wedge y)[e]$.

(ii) By Proposition 3.27, $x \mathrel{\text{⫤}\!\vDash}_l y$. On the other hand, $x \nvDash_s y$: take an (A, e) such that $|A| \geq 2$ and $e(x) \neq e(y)$. Then $e^+(x) = \{e(x)\} \not\subseteq \{e(y)\} = e^+(y)$.

□

3.3 Sets of patterns

Let Γ be a set of patterns.

Definition 3.29. *Let $(\mathcal{A}, e)$. We say that e* ***satisfies*** *Γ in $\mathcal{A}$ if $\mathcal{A} \vDash \gamma[e]$ for every $\gamma \in \Gamma$.*
Notation: $\mathcal{A} \vDash \Gamma[e]$.

Definition 3.30. *Let $\mathcal{A}$ be a τ-structure. We say that $\mathcal{A}$* ***satisfies*** *Γ or that $\mathcal{A}$ is a* ***model*** *of Γ if every $\mathcal{A}$-valuation satisfies Γ in $\mathcal{A}$.*
Notation: $\mathcal{A} \vDash \Gamma$.

3.4 Semantic consequence and equivalence for sets of patterns

We extend to sets of patterns the notions introduced in Subsection 3.2.

3.4.1 Notion 1

Definition 3.31. *Let φ be a pattern. We say that φ is a **global semantic consequence** of Γ if for every $\mathcal{A}$,*

$$\mathcal{A} \vDash \Gamma \quad \textit{implies} \quad \mathcal{A} \vDash \varphi.$$

Notation: $\Gamma \vDash_g \varphi$.

This is the definition of semantic consequence from [9].
By letting $\Gamma = \{\psi\}$, we get the notion introduced in Definition 3.16.(i). Hence, we write $\psi \vDash_g \varphi$ instead of $\{\psi\} \vDash_g \varphi$.

Remark 3.32. *(i) $\emptyset \vDash_g \varphi$ iff $\vDash \varphi$.*

(ii) If φ is valid, then $\Gamma \vDash_g \varphi$.

Proof. (i) $\emptyset \vDash_g \varphi$ iff (for every $\mathcal{A}$, we have that $\mathcal{A} \vDash \varphi$) iff $\vDash \varphi$.

(ii) Obviously, as $\mathcal{A} \vDash \varphi$ for every $\mathcal{A}$.

□

Lemma 3.33. *For all patterns φ, ψ,*

(i) If $\varphi \vDash_g \psi$, then $\Gamma \vDash_g \varphi$ implies $\Gamma \vDash_g \psi$.

(ii) If $\varphi \dashv\vDash_g \psi$, then $\Gamma \vDash_g \varphi$ iff $\Gamma \vDash_g \psi$.

(iii) $\Gamma \vDash_g \varphi \wedge \psi$ iff ($\Gamma \vDash_g \varphi$ and $\Gamma \vDash_g \psi$).

(iv) $\Gamma \vDash_g \varphi \leftrightarrow \psi$ iff ($\Gamma \vDash_g \varphi \to \psi$ and $\Gamma \vDash_g \psi \to \varphi$).

Proof. (i) Assume that $\varphi \vDash_g \psi$ and $\Gamma \vDash_g \varphi$. Let $\mathcal{A}$ be a model of Γ. Then $\mathcal{A} \vDash \varphi$, as $\Gamma \vDash_g \varphi$. Furthermore, $\mathcal{A} \vDash \psi$ as $\varphi \vDash_g \psi$.

(ii) It follows from (i) and Remark 3.17.

(iii) We have that

$\Gamma \vDash_g \varphi \wedge \psi$	iff	for every model $\mathcal{A}$ of Γ, $\mathcal{A} \vDash \varphi \wedge \psi$
	iff	for every model $\mathcal{A}$ of Γ and every $\mathcal{A}$-valuation e, $\mathcal{A} \vDash (\varphi \wedge \psi)[e]$
	iff	for every model $\mathcal{A}$ of Γ and every $\mathcal{A}$-valuation e, $\mathcal{A} \vDash \varphi[e]$ and $\mathcal{A} \vDash \psi[e]$ (by Proposition 3.7.(vii))
	iff	(for every model $\mathcal{A}$ of Γ and every $\mathcal{A}$-valuation e, $\mathcal{A} \vDash \varphi[e]$) and (for every model $\mathcal{A}$ of Γ and every $\mathcal{A}$-valuation e, $\mathcal{A} \vDash \psi[e]$)
	iff	(for every model $\mathcal{A}$ of Γ, $\mathcal{A} \vDash \varphi$) and (for every model $\mathcal{A}$ of Γ, $\mathcal{A} \vDash \psi$)
	iff	$\Gamma \vDash_g \varphi$ and $\Gamma \vDash_g \psi$.

(iv) Apply (iii).

□

Definition 3.34. *Let Δ be a set of patterns. We say that*

*(i) Δ is a **semantic consequence** of Γ if for every $\mathcal{A}$,*

$$\mathcal{A} \vDash \Gamma \quad \textit{implies} \quad \mathcal{A} \vDash \Delta.$$

Notation: $\Gamma \vDash_g \Delta$.

*(ii) Γ and Δ are **logic equivalent** if for every $\mathcal{A}$,*

$$\mathcal{A} \models \Gamma \quad \textit{iff} \quad \mathcal{A} \models \Delta.$$

Notation: $\Gamma \mathrel{=}\!\mathrel{|}_g \Delta$.

Obviously, by letting $\Delta = \{\varphi\}$, we obtain the notion introduced in Definition 3.31. Furthermore, for $\Gamma = \{\psi\}$ and $\Delta = \{\varphi\}$, we get the notions introduced in Definition 3.16.

Remark 3.35. $\Gamma \mathrel{=}\!\mathrel{|}_g \Delta$ *iff* ($\Gamma \models_g \Delta$ *and* $\Delta \models_g \Gamma$).

Lemma 3.36. *(i) Assume that* $\Gamma \models_g \Delta$ *and that* $\varphi \mathrel{=}\!\mathrel{|}_g \psi$*. Then* $\Delta \models_g \varphi$ *implies* $\Gamma \models_g \psi$*.*

(ii) Assume that $\Gamma \mathrel{=}\!\mathrel{|}_g \Delta$ *and that* $\varphi \mathrel{=}\!\mathrel{|}_g \psi$*. Then* $\Gamma \models_g \varphi$ *iff* $\Delta \models_g \psi$*.*

Proof. (i) We have that

$\Delta \models_g \varphi$	iff	for every model $\mathcal{A}$ of Δ, $\mathcal{A} \models \varphi$
	iff	for every model $\mathcal{A}$ of Δ, $\mathcal{A} \models \psi$ (as $\varphi \mathrel{=}\!\mathrel{|}_g \psi$)
	implies	for every model $\mathcal{A}$ of Γ, $\mathcal{A} \models \psi$ (as $\Gamma \models_g \Delta$)
	iff	$\Gamma \models_g \psi$.

(ii) Apply (i).

□

Proposition 3.37 (Finite sets)**.** *Let* $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ *be a finite set of patterns. Then* $\Gamma \mathrel{=}\!\mathrel{|}_g \{\varphi_1 \wedge \ldots \wedge \varphi_n\}$*.*

Proof. Applying Proposition 3.15.(ii), we get that for every $\mathcal{A}$, $\mathcal{A} \models \Gamma$ iff $\mathcal{A} \models \varphi_i$ for every $i = 1, \ldots, n$ iff $\mathcal{A} \models \varphi_1 \wedge \ldots \wedge \varphi_n$. □

3.4.2 Notion 2

Definition 3.38. *Let* φ *be a pattern. We say that* φ *is a* ***local semantic consequence*** *of* Γ *if for every* $(\mathcal{A}, e)$*,*

$$\mathcal{A} \models \Gamma[e] \quad \textit{implies} \quad \mathcal{A} \models \varphi[e].$$

Notation: $\Gamma \models_l \varphi$.

By letting $\Gamma = \{\psi\}$, we get the notion introduced in Definition 3.19. Hence, we write $\psi \models_l \varphi$ instead of $\{\psi\} \models_l \varphi$.

Remark 3.39. *(i)* $\emptyset \models_l \varphi$ *iff* $\models \varphi$*.*

(ii) If φ *is valid, then* $\Gamma \models_l \varphi$*.*

Proof. (i) $\emptyset \models_l \varphi$ iff (for every $(\mathcal{A}, e)$, we have that $\mathcal{A} \models \varphi[e]$) iff (for every $\mathcal{A}$, we have that $\mathcal{A} \models \varphi$) iff $\models \varphi$.

(ii) Obviously, as $\mathcal{A} \models \varphi[e]$ for every $(\mathcal{A}, e)$.

□

Lemma 3.40. *For all patterns* φ*,* ψ*,*

(i) If $\varphi \models_l \psi$*, then* $\Gamma \models_l \varphi$ *implies* $\Gamma \models_l \psi$*.*

(ii) If $\varphi \mathrel{=}\!\mathrel{|}_l \psi$*, then* $\Gamma \models_l \varphi$ *iff* $\Gamma \models_l \psi$*.*

(iii) $\Gamma \models_l \varphi \wedge \psi$ *iff* ($\Gamma \models_l \varphi$ *and* $\Gamma \models_l \psi$)*.*

(iv) $\Gamma \models_l \varphi \leftrightarrow \psi$ *iff* ($\Gamma \models_l \varphi \to \psi$ *and* $\Gamma \models_l \psi \to \varphi$)*.*

Proof. (i) Assume that $\varphi \models_l \psi$ and $\Gamma \models_l \varphi$. Let $(\mathcal{A}, e)$ be such that $\mathcal{A} \models \Gamma[e]$. Since $\Gamma \models_l \varphi$, we get that $\mathcal{A} \models \varphi[e]$. Since $\varphi \models_l \psi$, it follows that $\mathcal{A} \models \psi[e]$.

(ii) It follows from (i) and Remark 3.20.

(iii) We have that

$$
\begin{array}{lll}
\Gamma \vDash_l \varphi \wedge \psi & \text{iff} & \text{for every } (\mathcal{A}, e) \text{ such that } \mathcal{A} \vDash \Gamma[e] \text{ we have that } \mathcal{A} \vDash (\varphi \wedge \psi)[e] \\
& \text{iff} & \text{for every } (\mathcal{A}, e) \text{ such that } \mathcal{A} \vDash \Gamma[e] \text{ we have that } \mathcal{A} \vDash \varphi[e] \text{ and } \mathcal{A} \vDash \psi[e] \\
& & \text{(by Proposition 3.7.(vii))} \\
& \text{iff} & \text{(for every } (\mathcal{A}, e) \text{ such that } \mathcal{A} \vDash \Gamma[e] \text{ we have that } \mathcal{A} \vDash \varphi[e]) \text{ and} \\
& & \text{(for every } (\mathcal{A}, e) \text{ such that } \mathcal{A} \vDash \Gamma[e] \text{ we have that } \mathcal{A} \vDash \psi[e]) \\
& \text{iff} & \Gamma \vDash_l \varphi \text{ and } \Gamma \vDash_l \psi.
\end{array}
$$

(iv) Apply (iii).

□

Definition 3.41. *Let Δ be a set of patterns. We say that*

(i) Δ is a ***local semantic consequence*** *of Γ if for every $(\mathcal{A}, e)$,*

$$\mathcal{A} \vDash \Gamma[e] \quad \textit{implies} \quad \mathcal{A} \vDash \Delta[e].$$

Notation: $\Gamma \vDash_l \Delta$.

(ii) Γ and Δ are ***locally logically equivalent*** *if for every $(\mathcal{A}, e)$,*

$$\mathcal{A} \vDash \Gamma[e] \quad \textit{iff} \quad \mathcal{A} \vDash \Delta[e].$$

Notation: $\Gamma \mathrel{\text{⫤}\!\vDash}_l \Delta$.

Obviously, by letting $\Delta = \{\varphi\}$, we obtain the notion introduced in Definition 3.38. Furthermore, for $\Gamma = \{\psi\}$ and $\Delta = \{\varphi\}$, we get the notions introduced in Definition 3.19.

Remark 3.42. *$\Gamma \mathrel{\text{⫤}\!\vDash}_l \Delta$ iff ($\Gamma \vDash_l \Delta$ and $\Delta \vDash_l \Gamma$).*

Lemma 3.43. *(i) Assume that $\Gamma \vDash_l \Delta$ and that $\varphi \mathrel{\text{⫤}\!\vDash}_l \psi$. Then $\Delta \vDash_l \varphi$ implies $\Gamma \vDash_l \psi$.*

(ii) Assume that $\Gamma \mathrel{\text{⫤}\!\vDash}_l \Delta$ and that $\varphi \mathrel{\text{⫤}\!\vDash}_l \psi$. Then $\Gamma \vDash_l \varphi$ iff $\Delta \vDash_l \psi$.

Proof. (i) We have that

$$
\begin{array}{lll}
\Delta \vDash_l \varphi & \text{iff} & \text{for every } (\mathcal{A}, e) \text{ such that } \mathcal{A} \vDash \Delta[e], \mathcal{A} \vDash \varphi[e] \\
& \text{iff} & \text{for every } (\mathcal{A}, e) \text{ such that } \mathcal{A} \vDash \Delta[e], \mathcal{A} \vDash \psi[e] \text{ (as } \varphi \mathrel{\text{⫤}\!\vDash}_l \psi) \\
& \text{iff} & \text{for every } (\mathcal{A}, e) \text{ such that } \mathcal{A} \vDash \Gamma[e], \mathcal{A} \vDash \psi[e] \text{ (as } \Gamma \vDash_l \Delta) \\
& \text{iff} & \Gamma \vDash_l \psi.
\end{array}
$$

(ii) Apply (i).

□

Proposition 3.44 (Finite sets)**.** *Let $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ be a set of patterns. Then $\Gamma \mathrel{\text{⫤}\!\vDash}_l \{\varphi_1 \wedge \ldots \wedge \varphi_n\}$.*

Proof. Applying Proposition 3.15.(i), we get that for every $(\mathcal{A}, e)$, $\mathcal{A} \vDash \Gamma[e]$ iff $\mathcal{A} \vDash \varphi_i[e]$ for every $i = 1, \ldots, n$ iff $\mathcal{A} \vDash (\varphi_1 \wedge \ldots \wedge \varphi_n)[e]$. □

3.4.3 Notion 3

Definition 3.45. *Let φ be a pattern. We say that φ is a **strong semantic consequence** of Γ if for every $(\mathcal{A}, e)$,*

$$\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi).$$

Notation: $\Gamma \vDash_s \varphi$.

By letting $\Gamma = \{\psi\}$, we get the notion introduced in Definition 3.22.

Remark 3.46. *(i) $\emptyset \vDash_s \varphi$ iff $\vDash \varphi$.*

(ii) If φ is valid, then $\Gamma \vDash_s \varphi$.

Proof. (i) $\emptyset \vDash_s \varphi$ iff (for every $(\mathcal{A}, e)$, we have that $\bigcap_{\gamma\in\emptyset} e^+(\gamma) \subseteq e^+(\varphi)$) iff (for every $(\mathcal{A}, e)$, we have that $A \subseteq e^+(\varphi)$) iff (for every $(\mathcal{A}, e)$, we have that $e^+(\varphi) = A$) iff $\vDash \varphi$.

(ii) Obviously, as $e^+(\varphi) = A$ for every $(\mathcal{A}, e)$.

□

Remark 3.47. *For all patterns φ, ψ,*

(i) If $\varphi \vDash_s \psi$, then $\Gamma \vDash_s \varphi$ implies $\Gamma \vDash_s \psi$.

(ii) If $\varphi \mathrel{=}\!\!\!\mid\vDash_s \psi$, then $\Gamma \vDash_s \varphi$ iff $\Gamma \vDash_s \psi$.

(iii) $\Gamma \vDash_s \varphi \wedge \psi$ iff ($\Gamma \vDash_s \varphi$ and $\Gamma \vDash_s \psi$).

(iv) $\Gamma \vDash_s \varphi \leftrightarrow \psi$ iff ($\Gamma \vDash_s \varphi \to \psi$ and $\Gamma \vDash_s \psi \to \varphi$).

Proof. (i) Assume that $\varphi \vDash_s \psi$ and that $\Gamma \vDash_s \varphi$. Let $(\mathcal{A}, e)$. Then $\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi) \subseteq e^+(\psi)$, hence $\Gamma \vDash_s \psi$.

(ii) It follows from (i) and Remark 3.23.

(iii)

$$\begin{array}{lll}
\Gamma \vDash_s \varphi \wedge \psi & \text{iff} & \text{for every } (\mathcal{A}, e),\ \bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi \wedge \psi) \\
& \text{iff} & \text{for every } (\mathcal{A}, e),\ \bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi) \cap e^+(\psi) \\
& \text{iff} & \text{for every } (\mathcal{A}, e),\ \bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi) \text{ and } \bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\psi) \\
& \text{iff} & \text{for every } (\mathcal{A}, e),\ \bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi) \text{ and} \\
& & \text{for every } (\mathcal{A}, e),\ \bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\psi) \\
& \text{iff} & \Gamma \vDash_s \varphi \text{ and } \Gamma \vDash_s \psi.
\end{array}$$

(iv) Apply (iii).

□

Definition 3.48. *Let Δ be a set of patterns. We say that*

*(i) Δ is a **strong semantic consequence** of Γ if for every $(\mathcal{A}, e)$,*

$$\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq \bigcap_{\delta\in\Delta} e^+(\delta).$$

Notation: $\Gamma \vDash_s \Delta$.

*(ii) Γ and Δ are **strongly logically equivalent** if for every $(\mathcal{A}, e)$,*

$$\bigcap_{\gamma\in\Gamma} e^+(\gamma) = \bigcap_{\delta\in\Delta} e^+(\delta).$$

Notation: $\Gamma \mathrel{=}\!\!\!\mid\vDash_s \Delta$.

Obviously, by letting $\Delta = \{\varphi\}$, we obtain the notion introduced in Definition 3.45. Furthermore, for $\Gamma = \{\psi\}$ and $\Delta = \{\varphi\}$, we get the notions introduced in Definition 3.22.

Remark 3.49. *$\Gamma \,\exists\!\models_s \Delta$ iff ($\Gamma \models_s \Delta$ and $\Delta \models_s \Gamma$).*

Lemma 3.50. *(i) Assume that $\Gamma \models_s \Delta$ and that $\varphi \,\exists\!\models_s \psi$. Then $\Delta \models_s \varphi$ implies $\Gamma \models_s \psi$.*

(ii) Assume that $\Gamma \,\exists\!\models_s \Delta$ and that $\varphi \,\exists\!\models_s \psi$. Then $\Gamma \models_s \varphi$ iff $\Delta \models_s \psi$.

Proof. (i) We have that

$$\begin{array}{lll} \Delta \models_s \varphi & \text{iff} & \text{for every } (\mathcal{A}, e) \bigcap_{\gamma\in\Delta} e^+(\gamma) \subseteq e^+(\psi) \\ & \text{iff} & \text{for every } (\mathcal{A}, e) \bigcap_{\gamma\in\Delta} e^+(\gamma) \subseteq e^+(\psi) \text{ (as } \varphi \,\exists\!\models_s \psi) \\ & \text{iff} & \text{for every } (\mathcal{A}, e) \bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\psi) \text{ (as } \Gamma \models_s \Delta) \\ & \text{iff} & \Gamma \models_s \psi. \end{array}$$

(ii) Apply (i).

□

Proposition 3.51. *Let $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ be a finite set of patterns. Then $\Gamma \,\exists\!\models_s \{\varphi_1 \wedge \ldots \wedge \varphi_n\}$.*

Proof. For every $(\mathcal{A}, e)$,

$$\bigcap_{\gamma\in\Gamma} e^+(\gamma) = \bigcap_{i=1}^{n} e^+(\varphi_i) = e^+(\varphi_1 \wedge \ldots \wedge \varphi_n).$$

□

3.4.4 Relations between the three notions

Proposition 3.52. *$\Gamma \models_s \varphi$ implies $\Gamma \models_l \varphi$ implies $\Gamma \models_g \varphi$.*

Proof. $\Gamma \models_s \varphi$ implies $\Gamma \models_l \varphi$: Let $(\mathcal{A}, e)$ be such that $\mathcal{A} \models \Gamma[e]$, so $e^+(\gamma) = A$ for every $\gamma \in \Gamma$. Hence, $\bigcap_{\gamma\in\Gamma} e^+(\gamma) = A$. As $\Gamma \models_s \varphi$, it follows that $e^+(\varphi) = A$, that is $\mathcal{A} \models \varphi[e]$.
$\Gamma \models_l \varphi$ implies $\Gamma \models_g \varphi$: Assume that $\mathcal{A} \models \Gamma$ and let e be arbitrary. Then $\mathcal{A} \models \Gamma[e]$. Applying now the fact that $\Gamma \models_l \varphi$, we get that $\mathcal{A} \models \varphi[e]$.

□

3.5 Closed patterns

Definition 3.53. *A pattern φ is said to be a* ***closed pattern*** *if $FV(\varphi) = \emptyset$, that is φ does not have free variables.*

Let $\mathcal{A}$ be a τ-structure.

Proposition 3.54. *Let φ be a closed pattern. For every $\mathcal{A}$-valuations e_1, e_2, we have that $e_1^+(\varphi) = e_2^+(\varphi)$. In particular, $\mathcal{A} \models \varphi[e_1]$ iff $\mathcal{A} \models \varphi[e_2]$.*

Proof. Apply Proposition 3.8 and the fact that $FV(\varphi) = \emptyset$. □

Corollary 3.55. *Let φ be a closed pattern. Then either $\mathcal{A} \models \varphi[e]$ for all e or $\mathcal{A} \not\models \varphi[e]$ for all e. Hence, $\mathcal{A} \models \varphi$ iff $\mathcal{A} \models \varphi[e]$ for one $\mathcal{A}$-valuation e.*

Proposition 3.56. *Let Γ be a set of closed patterns. Then for every pattern φ, $\Gamma \models_l \varphi$ iff $\Gamma \models_g \varphi$.*

Proof. $\Rightarrow$ By Proposition 3.52.
$\Leftarrow$ Let $(\mathcal{A}, e)$ be such that $\mathcal{A} \models \Gamma[e]$, that is $\mathcal{A} \models \gamma[e]$ for all $\gamma \in \Gamma$. By Corollary 3.55, we get that $\mathcal{A} \models \gamma$ for all $\gamma \in \Gamma$, that is $\mathcal{A} \models \Gamma$. As $\Gamma \models_g \varphi$, it follows that $\mathcal{A} \models \varphi$. In particular, $\mathcal{A} \models \varphi[e]$. □

3.6 Predicates

Definition 3.57. *Let φ be a pattern.*

*(i) If $\mathcal{A}$ is a τ-structure, we say that φ is an **$\mathcal{A}$-predicate** or a **predicate in** A if for any $\mathcal{A}$-valuation e, $e^+(\varphi) \in \{\emptyset, A\}$.*

*(ii) φ is a **predicate** iff it is an $\mathcal{A}$-predicate for every τ-structure $\mathcal{A}$.*

We shall denote by **$\mathcal{A}$-Predicates** the set of $\mathcal{A}$-predicates and by **Predicates** the set of predicates.

Proposition 3.58. *(i) For every τ-structure $\mathcal{A}$, the set $\mathcal{A}$-Predicates contains $\bot$, $\top$ and is closed to $\neg, \to, \vee, \wedge, \leftrightarrow$ and to $\exists x$, $\forall x$ (for any element variable x). Furthermore, the set $\mathcal{A}$-Predicates is closed to finite conjunctions and disjunctions.*

(ii) The set Predicates contains $\bot$, $\top$ and is closed to $\neg, \to, \vee, \wedge, \leftrightarrow$ and to $\exists x$, $\forall x$ (for any variable x). Furthermore, the set Predicates is closed to finite conjunctions and disjunctions.

Proof.

(i) Let e be an arbitrary $\mathcal{A}$-valuation.

(a) Since $e^+(\bot) = \emptyset$ and $e^+(\top) = A$, we have that $\bot, \top \in \mathcal{A}$-Predicates.

(b) If φ is an $\mathcal{A}$-predicate, then $e^+(\varphi) \in \{\emptyset, A\}$, hence $e^+(\neg\varphi) = A \setminus e^+(\varphi) \in \{\emptyset, A\}$, that is $\neg\varphi$ is an $\mathcal{A}$-predicate.

(c) Assume that $\varphi_1, \ldots, \varphi_n (n \geq 1)$ are $\mathcal{A}$-predicates, so $e^+(\varphi_i) \in \{\emptyset, A\}$ for every $i = 1, \ldots, n$. Then

$$e^+(\varphi_1 \wedge \ldots \wedge \varphi_n) = \bigcap_{i=1}^{n} e^+(\varphi_i) = \begin{cases} \emptyset & \text{if there exists } i = 1, \ldots, n \text{ such that } e^+(\varphi_i) = \emptyset \\ A & \text{otherwise,} \end{cases}$$
$$e^+(\varphi_1 \vee \ldots \vee \varphi_n) = \bigcup_{i=1}^{n} e^+(\varphi_i) = \begin{cases} A & \text{if there exists } i = 1, \ldots, n \text{ such that } e^+(\varphi_i) = A, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, $\varphi_1 \wedge \ldots \wedge \varphi_n$ and $\varphi_1 \vee \ldots \vee \varphi_n$ are $\mathcal{A}$-predicates.

(d) Assume that φ, ψ are $\mathcal{A}$-predicates, so $e^+(\varphi), e^+(\psi) \in \{\emptyset, A\}$. Then $\varphi \wedge \psi$ and $\varphi \vee \psi$. Furthermore,

$$e^+(\varphi \to \psi) = \begin{cases} \emptyset & \text{if } e^+(\varphi) = A \text{ and } e^+(\psi) = \emptyset \\ A & \text{otherwise} \end{cases},$$
$$e^+(\varphi \leftrightarrow \psi) = \begin{cases} A & \text{if } e^+(\varphi) = e^+(\psi) \\ & \emptyset\text{otherwise.} \end{cases}$$

Thus, $\varphi \to \psi, \varphi \leftrightarrow \psi$ are $\mathcal{A}$-predicates.

(e) Assume that φ is an $\mathcal{A}$-predicate. Then, for every $a \in A$, $(e_{x\mapsto a})^+(\varphi) \in \{\emptyset, A\}$, hence $e^+(\exists x.\varphi) = \bigcup_{a\in A} (e_{x\mapsto a})^+(\varphi) \in \{\emptyset, A\}$ and $e^+(\forall x.\varphi) = \bigcap_{a\in A} (e_{x\mapsto a})^+(\varphi) \in \{\emptyset, A\}$. Thus, $\exists x.\varphi$ and $\forall x.\varphi$ are $\mathcal{A}$-predicates.

(ii) Obviously, by (i).

□

Proposition 3.59. *Let Γ be a set of closed predicates (that is, closed patterns that are predicates). Then for every pattern φ, $\Gamma \vDash_g \varphi$ iff $\Gamma \vDash_l \varphi$ iff $\Gamma \vDash_s \varphi$.*

Proof. $\Gamma \vDash_g \varphi$ iff $\Gamma \vDash_l \varphi$: By Proposition 3.56.
$\Gamma \vDash_l \varphi$ iff $\Gamma \vDash_s \varphi$: $\Leftarrow$ By Proposition 3.52.
$\Rightarrow$ Let $(\mathcal{A}, e)$. We have the following cases:

(i) There exists $\gamma_0 \in \Gamma$ such that $e^+(\gamma_0) \neq A$. Then $e^+(\gamma_0) = \emptyset$, so $\bigcap_{\gamma\in\Gamma} e^+(\gamma) = \emptyset \subseteq e^+(\varphi)$. Thus, $\Gamma \vDash_s \varphi$.

(ii) $e^+(\gamma) = A$ for all $\gamma \in \Gamma$, hence $\mathcal{A} \vDash \Gamma[e]$. Since $\Gamma \vDash_l \varphi$, we get that $\mathcal{A} \vDash \varphi[e]$, that is $e^+(\varphi) = A$. We have got that $\bigcap_{\gamma\in\Gamma} e^+(\gamma) = A = e^+(\varphi)$. Thus, $\Gamma \vDash_s \varphi$.

□

Proposition 3.60. *Let Γ be a set of patterns, σ be a predicate pattern and φ, ψ be patterns such that*

$$\varphi \vDash_l \psi.$$

Then

(i) $\sigma \to \varphi \vDash_l \sigma \to \psi$*;*

(ii) $\sigma \to \varphi \vDash_g \sigma \to \psi$*;*

(iii) $\Gamma \vDash_l \sigma \to \varphi$ *implies* $\Gamma \vDash_l \sigma \to \psi$*;*

(iv) $\Gamma \vDash_g \sigma \to \varphi$ *implies* $\Gamma \vDash_g \sigma \to \psi$*.*

Proof. (i) Let $(\mathcal{A}, e)$ be such that $\mathcal{A} \vDash (\sigma \to \varphi)[e]$, hence

$$(*) \qquad e^+(\sigma) \subseteq e^+(\varphi).$$

We have to prove that $\mathcal{A} \vDash (\sigma \to \psi)[e]$, hence that

$$(**) \qquad e^+(\sigma) \subseteq e^+(\psi).$$

We have two cases:

(a) $e^+(\sigma) = \emptyset$. Then obviously, $(**)$ holds.

(b) $e^+(\sigma) = A$. Then, by $(*)$, $e^+(\varphi) = A$, so $\mathcal{A} \vDash \varphi[e]$. As $\varphi \vDash_l \psi$, we must have that $\mathcal{A} \vDash \psi[e]$, that is $e^+(\psi) = A$. Thus, $(**)$ holds.

(ii) Apply (i) and the fact that $\sigma \to \varphi \vDash_l \sigma \to \psi$ implies $\sigma \to \varphi \vDash_g \sigma \to \psi$, by Proposition 3.26.(i).

(iii) Apply (i) and Lemma 3.40.(i).

(iv) Apply (ii) and Lemma 3.33.(i).

□

3.7 Tautologies

Let $\mathcal{A}$ be a τ-structure.

Definition 3.61. *A propositional $\mathcal{A}$-evaluation is a mapping $F : Pattern \to 2^A$ that satisfies for any patterns φ, ψ,*

(i) $F(\bot) = \emptyset$.

(ii) $F(\varphi \to \psi) = C_A(F(\varphi) \setminus F(\psi))$.

Lemma 3.62. *Let F be a propositional $\mathcal{A}$-evaluation. For any patterns φ,ψ,*

(i) $F(\varphi \to \psi) = C_A F(\varphi) \cup F(\psi)$.

(ii) $F(\top) = A$.

(iii) $F(\neg\varphi) = C_A F(\varphi)$.

(iv) $F(\varphi \vee \psi) = F(\varphi) \cup F(\psi)$.

(v) $F(\varphi \wedge \psi) = F(\varphi) \cap F(\psi)$.

(vi) $F(\varphi \leftrightarrow \psi) = C_A(F(\varphi) \Delta F(\psi))$.

Proof. Replace e^+ with F in the proof of Proposition 3.5. $\square$

Lemma 3.63. *Let F be a propositional $\mathcal{A}$-evaluation. For any patterns φ,ψ,*

(i) $F(\varphi \to \psi) = A$ *iff* $F(\varphi) \subseteq F(\psi)$.

(ii) $F(\varphi \leftrightarrow \psi) = A$ *iff* $F(\varphi) = F(\psi)$.

Proof. Replace e^+ with F in the proof of Proposition 3.7.(ix), (x). $\square$

Proposition 3.64. *Let e be an $\mathcal{A}$-valuation. The mapping*

$$V_{e,\mathcal{A}} : Pattern \to 2^A, \quad V_{e,\mathcal{A}}(\varphi) = e^+(\varphi)$$

is a propositional $\mathcal{A}$-evaluation.

Proof. We have that

(i) $V_{e,\mathcal{A}}(\bot) = e^+(\bot) = \emptyset$

(ii) $V_{e,\mathcal{A}}(\varphi \to \psi) = e^+(\varphi \to \psi) = C_A(e^+(\varphi) \setminus e^+(\psi)) = C_A(V_{e,\mathcal{A}}(\varphi) \setminus V_{e,\mathcal{A}}(\psi))$.

$\square$

Definition 3.65. *A pattern φ is a* ***tautology*** *if $F(\varphi) = A$ for any τ-structure $\mathcal{A}$ and any propositional $\mathcal{A}$-evaluation F.*
Notation: $\models^t \varphi$.

Proposition 3.66. *Any tautology is valid.*

Proof. Assume that φ is a tautology and let $(\mathcal{A}, e)$. Since, by Proposition 3.64, $V_{e,\mathcal{A}}$ is a propositional $\mathcal{A}$-evaluation, we have that $V_{e,\mathcal{A}}(\varphi) = A$, that is $e^+(\varphi) = A$, so $\mathcal{A} \models \varphi[e]$. $\square$

3.7.1 Examples of tautologies and tautological equivalences

Proposition 3.67. *The following are tautologies, hence valid, for all patterns φ, ψ, χ,*

$$\varphi \leftrightarrow \varphi, \tag{6}$$
$$\varphi \vee \varphi \leftrightarrow \varphi, \tag{7}$$
$$\varphi \leftrightarrow \varphi \wedge \varphi, \tag{8}$$
$$\varphi \vee \psi \leftrightarrow \psi \vee \varphi, \tag{9}$$
$$\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi, \tag{10}$$
$$\varphi \to \varphi \vee \psi, \tag{11}$$
$$\varphi \wedge \psi \to \varphi, \tag{12}$$
$$\bot \to \varphi, \tag{13}$$
$$\varphi \vee \neg\varphi, \tag{14}$$
$$(\varphi \to (\psi \to \chi)) \leftrightarrow (\varphi \wedge \psi \to \chi), \tag{15}$$
$$(\varphi \to (\psi \to \chi)) \leftrightarrow (\psi \to (\varphi \to \chi)), \tag{16}$$

Proof. Let $\mathcal{A}$ be a τ-structure and F be an propositional $\mathcal{A}$-evaluation.
(6): We have that $e^+(\varphi) = e^+(\varphi)$, hence, by Lemma 3.63, $F(\varphi \leftrightarrow \varphi) = A$.
(7): We have that $F(\varphi \vee \varphi) = F(\varphi) \cup F(\varphi) = F(\varphi)$, hence, by Lemma 3.63, $F(\varphi \vee \varphi \leftrightarrow \varphi) = A$.
(8): We have that $F(\varphi \wedge \varphi) = F(\varphi) \cap F(\varphi) = F(\varphi)$, hence, by Lemma 3.63, $F(\varphi \leftrightarrow \varphi \wedge \varphi) = A$.
(9): We have that $F(\varphi \vee \psi) = F(\varphi) \cup F(\psi) = F(\psi) \cup F(\varphi) = F(\psi \vee \varphi)$, hence, by Lemma 3.63, $F(\varphi \vee \psi \leftrightarrow \psi \vee \varphi) = A$.
(10): We have that $F(\varphi \wedge \psi) = F(\varphi) \cap F(\psi) = F(\psi) \cap F(\varphi) = F(\psi \wedge \varphi)$, hence, by Lemma 3.63, $F(\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi) = A$.
(11): We have that $F(\varphi) \subseteq F(\varphi) \cup F(\psi) = F(\varphi \vee \psi)$, hence, by Lemma 3.63, $F(\varphi \to \varphi \vee \psi) = A$.
(12): We have that $F(\varphi \wedge \psi) = F(\varphi) \cap F(\psi) \subseteq F(\varphi)$, hence, by Lemma 3.63, $F(\varphi \wedge \psi \to \varphi) = A$.
(13): We have that $F(\bot \to \varphi) = C_A(F(\bot) \setminus F(\varphi)) = C_A(\emptyset \setminus F(\varphi)) = C_A\emptyset = A$.
(14): We have that $F(\varphi \vee \neg\varphi) = F(\varphi) \cup F(\neg\varphi) = F(\varphi) \cup C_A F(\varphi) = A$.
(15): We have that

$$\begin{aligned}
F(\varphi \to (\psi \to \chi)) &= C_A F(\varphi) \cup F(\psi \to \chi) = C_A F(\varphi) \cup (C_A F(\psi) \cup F(\chi)) \\
&= (C_A F(\varphi) \cup C_A F(\psi)) \cup F(\chi) \overset{(148)}{=} C_A(F(\varphi) \cap F(\psi)) \cup F(\chi) \\
&= C_A F(\varphi \wedge \psi) \cup F(\chi) = F(\varphi \wedge \psi \to \chi),
\end{aligned}$$

hence, by Lemma 3.63, $F((\varphi \to (\psi \to \chi)) \leftrightarrow (\varphi \wedge \psi \to \chi)) = A$.
(16): We have that

$$\begin{aligned}
F(\varphi \to (\psi \to \chi)) &\overset{(15)}{=} F(\varphi \wedge \psi \to \chi) = C_A F(\varphi \wedge \psi) \cup F(\chi) \\
&= C_A F(\psi \wedge \varphi) \cup F(\chi) \quad \text{as } F(\varphi \wedge \psi) = F(\psi \wedge \varphi) \text{ by (10)} \\
&= F(\psi \wedge \varphi \to \chi) \overset{(15)}{=} F(\psi \to (\varphi \to \chi)).
\end{aligned}$$

□

3.8 Change of $\mathcal{A}$-valuations

Notation 3.68. *For any distinct element variables x, z and any $a, b \in A$, we define a new $\mathcal{A}$-valuation $e_{x\mapsto a, z\mapsto b}$ as follows: for all $v \in EVar$ and all $V \in SVar$,*

$$e_{x\mapsto a,z\mapsto b}(V) = e(V), \qquad e_{x\mapsto a,z\mapsto b}(v) = \begin{cases} e(v) & \text{if } v \neq x, z \\ a & \text{if } v = x \\ b & \text{if } v = z \end{cases} \tag{17}$$

It is obvious that $e_{x\mapsto a,z\mapsto b} = (e_{x\mapsto a})_{z\mapsto b} = (e_{z\mapsto b})_{x\mapsto a}$.

Notation 3.69. *For any distinct set variables X, Z and any $B, C \subseteq A$, we define a new $\mathcal{A}$-valuation $e_{X\mapsto B, Z\mapsto C}$ as follows: for all $v \in EVar$ and all $V \in SVar$,*

$$e_{X\mapsto B,Z\mapsto C}(v) = e(v), \qquad e_{X\mapsto B,Z\mapsto C}(V) = \begin{cases} e(V) & \text{if } V \neq X, Z \\ B & \text{if } V = X \\ C & \text{if } V = Z \end{cases} \tag{18}$$

It is obvious that $e_{X\mapsto B,Z\mapsto C} = (e_{X\mapsto B})_{Z\mapsto C} = (e_{Z\mapsto C})_{X\mapsto B}$.

Notation 3.70. *For any element variable x and set variable X and any $a \in A, B \subseteq A$, we define a new $\mathcal{A}$-valuation $e_{x\mapsto a, X\mapsto B}$ as follows: for all $v \in EVar$ and all $V \in SVar$,*

$$e_{x\mapsto a,X\mapsto B}(v) = \begin{cases} e(v) & \text{if } v \neq x \\ a & \text{if } v = x \end{cases} \qquad e_{x\mapsto a,X\mapsto B}(V) = \begin{cases} e(V) & \text{if } V \neq X \\ B & \text{if } V = X \end{cases} \tag{19}$$

It is obvious that $e_{x\mapsto a,X\mapsto b} = (e_{x\mapsto a})_{X\mapsto B} = (e_{X\mapsto B})_{x\mapsto a}$.

3.9 Substitution

3.9.1 Element variables

Proposition 3.71. *Let φ, δ be patterns and x be an element variable such that x is free for δ in φ. Assume that*

(*) $\mathcal{A}$ is a τ-structure such that for every $\mathcal{A}$-valuation e, there exists $c_e \in A$ such that $e^+(\delta) = \{c_e\}$.

Then for every $\mathcal{A}$-valuation e,

$$e^+(Subf_\delta^x \varphi) = (e_{x\mapsto c_e})^+(\varphi).$$

Proof. If $x \notin FV(\varphi)$, then $Subf_\delta^x \varphi = \varphi$ and $e = e_{x\mapsto c_e}$, by Proposition 3.8. The conclusion is obvious.
We prove by induction on patterns that for all patterns φ such that $x \in FV(\varphi)$, the following holds:

$$\text{for every } \mathcal{A}\text{-valuation } e,\ e^+(Subf_\delta^x \varphi) = (e_{x\mapsto c_e})^+(\varphi).$$

We use the definition by recursion on paterns of $Subf_\delta^x \varphi$, given by Remark 2.31.

(i) φ is an atomic pattern. Since $x \in FV(\varphi)$, we have that $\varphi = x \in EVar$. Then $Subf_\delta^x \varphi = \delta$. For every $\mathcal{A}$-valuation e, we get that $e^+(Subf_\delta^x \varphi) = e^+(\delta) = \{c_e\}$. Furthermore, $(e_{x\mapsto c_e})^+(\varphi) = (e_{x\mapsto c_e})^+(x) = \{c_e\}$.

(ii) $\varphi = \psi \cdot \chi$. Then $x \in FV(\psi)$, $x \in FV(\chi)$ and x is free for δ in ψ, χ. Let e be an $\mathcal{A}$-valuation. We can apply the induction hypothesis for ψ and χ to get that $e^+(Subf_\delta^x \psi) = (e_{x\mapsto c_e})^+(\psi)$ and $e^+(Subf_\delta^x \chi) = (e_{x\mapsto c_e})^+(\chi)$. It follows that

$$\begin{aligned} e^+(Subf_\delta^x \varphi) &= e^+(Subf_\delta^x \psi \cdot Subf_\delta^x \chi) = e^+(Subf_\delta^x \psi) \star e^+(Subf_\delta^x \chi) \\ &= (e_{x\mapsto c_e})^+(\psi) \star (e_{x\mapsto c_e})^+(\chi) = (e_{x\mapsto c_e})^+(\psi \cdot \chi) = (e_{x\mapsto c_e})^+(\varphi). \end{aligned}$$

(iii) $\varphi = \psi \to \chi$. Then $x \in FV(\psi)$, $x \in FV(\chi)$ and x is free for δ in ψ, χ. Let e be an $\mathcal{A}$-valuation. We can apply the induction hypothesis for ψ and χ to get that $e^+(Subf_\delta^x \psi) = (e_{x\mapsto c_e})^+(\psi)$ and $e^+(Subf_\delta^x \chi) = (e_{x\mapsto c_e})^+(\chi)$. It follows that

$$\begin{aligned} e^+(Subf_\delta^x \varphi) &= e^+(Subf_\delta^x \psi \to Subf_\delta^x \chi) = C_A\left(e^+(Subf_\delta^x \psi) \setminus e^+(Subf_\delta^x \chi)\right) \\ &= C_A\left((e_{x\mapsto c_e})^+(\psi) \setminus (e_{x\mapsto c_e})^+(\chi)\right) = (e_{x\mapsto c_e})^+(\psi \to \chi) \\ &= (e_{x\mapsto c_e})^+(\varphi). \end{aligned}$$

(iv) $\varphi = \exists z.\psi$. Since $x \in FV(\varphi)$, we have that $x \neq z$. Then $x \in FV(\psi)$ and x is free for δ in ψ. Let e be an $\mathcal{A}$-valuation. We get that

$$\begin{aligned} e^+(Subf_\delta^x \varphi) &= e^+(\exists z. Subf_\delta^x \psi) = \bigcup_{a\in A} (e_{z\mapsto a})^+(Subf_\delta^x \psi) \\ &= \left((e_{z\mapsto a})_{x\mapsto c_{e_{z\mapsto a}}}\right)^+(\psi) \text{ by the induction hypothesis for } \psi \text{ applied to } e_{z\mapsto a}. \end{aligned}$$

As x is free for δ in φ and $x \in FV(\psi)$, it follows that $z \notin FV(\delta)$. Hence, by Proposition 3.8, $e^+(\delta) = (e_{z\mapsto a})^+(\delta)$, so $c_e = c_{e_{z\mapsto a}}$. It follows that

$$\begin{aligned} e^+(Subf_\delta^x \varphi) &= \left((e_{z\mapsto a})_{x\mapsto c_e}\right)^+(\psi) = \bigcup_{a\in A} (e_{x\mapsto c_e, z\mapsto a})^+(\psi) = \bigcup_{a\in A} \left((e_{x\mapsto c_e})_{z\mapsto a}\right)^+(\psi) \\ &= (e_{x\mapsto c_e})^+(\varphi), \end{aligned}$$

since $x \neq z$, hence $(e_{x\mapsto c_e})_{z\mapsto a} = (e_{z\mapsto c_e})_{x\mapsto a} = e_{x\mapsto c_e, z\mapsto a}$.

(v) $\varphi = \mu X.\psi$. Then $x \in FV(\psi)$ and x is free for δ in ψ. Let e be an $\mathcal{A}$-valuation.

$$\begin{aligned} e^+(Subf^x_\delta\varphi) = e^+(\mu X.Subf^x_\delta\psi) &= \bigcap\left\{B \subseteq A \mid (e_{X\mapsto B})^+ (Subf^x_\delta\psi) \subseteq B\right\} \\ &= \bigcap\left\{B \subseteq A \mid \left((e_{X\mapsto B})_{x\mapsto c_{e_{X\mapsto B}}}\right)^+ (\psi) \subseteq B\right\} \\ &\quad \text{by the induction hypothesis for } \psi \text{ applied to } e_{X\mapsto B}. \end{aligned}$$

As x is free for δ in φ and $x \in FV(\psi)$, it follows that $X \notin FV(\delta)$. Hence, by Proposition 3.8, $e^+(\delta) = (e_{X\mapsto B})^+(\delta)$, so $c_e = c_{e_{X\mapsto B}}$. Thus, $\left((e_{X\mapsto B})_{x\mapsto c_{e_{X\mapsto B}}}\right)^+ (\psi) = \left((e_{X\mapsto B})_{x\mapsto c_e}\right)^+ (\psi)$ It follows that

$$\begin{aligned} e^+(Subf^x_\delta\varphi) &= \bigcap\left\{B \subseteq A \mid \left((e_{X\mapsto B})_{x\mapsto c_e}\right)^+ (\psi) \subseteq B\right\} \\ &= \bigcap\left\{B \subseteq A \mid ((e_{x\mapsto c_e})_{X\mapsto B})^+ (\psi) \subseteq B\right\} \\ &= (e_{x\mapsto c_e})^+ (\varphi). \end{aligned}$$

□

As an immediate consequence of Proposition 3.71, we get the following.

Corollary 3.72. *Let φ be a pattern and x, y be variables such that x is free for y in φ. Then for every τ-structure $\mathcal{A}$ and every $\mathcal{A}$-valuation e,*

$$e^+(Subf^x_y\varphi) = \left(e_{x\mapsto e(y)}\right)^+ (\varphi).$$

Proposition 3.73. *Let φ be a pattern and x, y be variables such that x is free for y in φ. Then*

$$\vDash Subf^x_y\varphi \to \exists x.\varphi, \tag{20}$$

$$\vDash \forall x.\varphi \to Subf^x_y\varphi. \tag{21}$$

Proof. Let $(\mathcal{A}, e)$. Apply Corollary 3.72 to get that $e^+(Subf^x_y\varphi) = \left(e_{x\mapsto e(y)}\right)^+ (\varphi)$. It follows that
(22): $e^+(\varphi_x(y)) = \left(e_{x\mapsto e(y)}\right)^+ (\varphi) \subseteq \bigcup_{a\in A} (e_{x\mapsto a})^+ (\varphi) = e^+(\exists x.\varphi)$.
(23): $e^+(\varphi_x(y)) = \left(e_{x\mapsto e(y)}\right)^+ (\varphi) \supseteq \bigcap_{a\in A} (e_{x\mapsto a})^+ (\varphi) = e^+(\forall x.\varphi)$. □

Bounded substitution

Proposition 3.74. *Let φ be a pattern and x, y variables such that y does not occur in φ. Then*

$$\vDash \varphi \leftrightarrow Subb^x_y\varphi.$$

Proof. If $x = y$, then obviously $Subb^x_y\varphi = \varphi$, hence $\vDash \varphi \leftrightarrow Subb^x_y\varphi$. Assume that $x \neq y$.
Let $\mathcal{A}$ be a τ-structure. We prove by induction on φ that

$$\text{for every } \mathcal{A}\text{-valuation } e,\ e^+(\varphi) = e^+(Subb^x_y\varphi).$$

We use the definition by recursion on paterns of $Subb^x_y\varphi$, given by Remark 2.38.

(i) φ is an atomic pattern. Then $Subb^x_y\varphi = \varphi$, so $e^+(\varphi) = e^+(Subb^x_y\varphi)$ for every $\mathcal{A}$-valuation e.

(ii) $\varphi = \psi \cdot \chi$. Let e be an $\mathcal{A}$-valuation. By the induction hypothesis for ψ, χ, we have that $e^+(\psi) = e^+(Subb^x_y\psi)$ and $e^+(\chi) = e^+(Subb^x_y\chi)$. It follows that

$$\begin{aligned} e^+(Subb^x_y\varphi) &= e^+(Subb^x_y(\psi) \cdot Subb^x_y(\chi)) = e^+(Subb^x_y\psi) \star e^+(Subb^x_y\chi) = e^+(\psi) \star e^+(\chi) \\ &= e^+(\psi \cdot \chi) = e^+(\varphi). \end{aligned}$$

(iii) $\varphi = \psi \to \chi$. Let e be an $\mathcal{A}$-valuation. By the induction hypothesis for ψ, χ, we have that $e^+(\psi) = e^+(Subb_y^x\psi)$ and $e^+(\chi) = e^+(Subb_y^x\chi)$. It follows that

$$\begin{aligned} e^+(Subb_y^x\varphi) &= e^+(Subb_y^x\psi \to Subb_y^x\chi) = C_A e^+(Subb_y^x\psi) \cup e^+(Subb_y^x\chi) = C_A e^+(\psi) \cup e^+(\chi) \\ &= e^+(\psi \to \chi) = e^+(\varphi). \end{aligned}$$

(iv) $\varphi = \exists z.\psi$. We have two cases:

1. $x \neq z$. Then

$$\begin{aligned} e^+(Subb_y^x\varphi) = e^+(\exists z.Subb_y^x\psi) &= \bigcup_{a\in A} (e_{z\mapsto a})^+ (Subb_y^x\psi) \\ &= \bigcup_{a\in A} (e_{z\mapsto a})^+ (\psi) \text{ by the induction hypothesis for } \psi \text{ applied to } e_{z\mapsto a} \\ &= e^+(\varphi). \end{aligned}$$

2. $x = z$. Then

$$e^+(Subb_y^x\varphi) = e^+\left(\exists y.Subf_y^x\left(Subb_y^x\psi\right)\right) = \bigcup_{a\in A} (e_{y\mapsto a})^+ \left(Subf_y^x\left(Subb_y^x\psi\right)\right).$$

As $x \neq y$ and y does not occur in ψ, we have that x is free for y in $Subb_y^x\psi$, by Proposition 2.40. We can apply then Corollary 3.72 to get that

$$\begin{aligned} (e_{y\mapsto a})^+ \left(Subf_y^x\left(Subb_y^x\psi\right)\right) &= \left((e_{y\mapsto a})_{x\mapsto e_{y\mapsto a}(y)}\right)^+ \left(Subb_y^x\psi\right) = \left((e_{y\mapsto a})_{x\mapsto a}\right)^+ \left(Subb_y^x\psi\right) \\ &= \left((e_{x\mapsto a})_{y\mapsto a}\right)^+ \left(Subb_y^x\psi\right) = (e_{x\mapsto a})^+ \left(Subb_y^x\psi\right), \end{aligned}$$

since y does not occur in ψ, so $y \notin FV\left(Subb_y^x\psi\right)$, hence we can apply Proposition 3.8. It follows that

$$\begin{aligned} e^+(Subb_y^x\varphi) &= \bigcup_{a\in A} (e_{x\mapsto a})^+ \left(Subb_y^x\psi\right) \\ &= \bigcup_{a\in A} (e_{x\mapsto a})^+ (\psi) \text{ by the induction hypothesis for } \psi \text{ applied to } e_{x\mapsto a} \\ &= e^+(\varphi). \end{aligned}$$

(v) $\varphi = \mu X.\psi$. Then

$$\begin{aligned} e^+(Subb_y^x\varphi) = e^+(\mu X.Subb_y^x\psi) &= \bigcap \left\{B \subseteq A \mid (e_{X\mapsto B})^+ (Subb_y^x\psi) \subseteq B\right\} \\ &= \bigcap \left\{B \subseteq A \mid (e_{X\mapsto B})^+ (\psi) \subseteq B\right\} \\ &\quad \text{by the induction hypothesis for } \psi \text{ applied to } e_{X\mapsto B} \\ &= e^+(\varphi). \end{aligned}$$

□

Free substitution

Proposition 3.75. *Let φ be a pattern and x,y be variables. Then*

$$\models \varphi_x(y) \to \exists x.\varphi, \tag{22}$$
$$\models \forall x.\varphi \to \varphi_x(y). \tag{23}$$

Proof. We have the following two cases:

(i) x is free for y in φ. Then $\varphi_x(y) = Subf_y^x\varphi$, so we apply Corollary 3.73 to get the conclusion.

(ii) x is not free for y in φ. Then $\varphi_x(y) = Subf_y^x\theta$, where $\theta = Subb_z^y\varphi$, with z a new variable, $z \notin Var(\varphi) \cup \{y\}$. Since x is free for y in θ, we can apply Corollary 3.73 (with $\varphi = \theta$) to get that

$$\vDash \varphi_x(y) \to \exists x.\theta \text{ and } \vDash \forall x.\theta \to \varphi_x(y). \tag{24}$$

Furthermore, by Proposition 3.74, we have that

$$\vDash \theta \leftrightarrow \varphi. \tag{25}$$

Let $(\mathcal{A}, e)$. Then

$$e^+(\varphi_x(y)) \overset{(24)}{\subseteq} e^+(\exists x.\theta) = \bigcup_{a\in A} (e_{x\mapsto a})^+(\theta) \overset{(25)}{=} \bigcup_{a\in A} (e_{x\mapsto a})^+(\varphi) = e^+(\exists x.\varphi),$$
$$e^+(\varphi_x(y)) \overset{(24)}{\supseteq} e^+(\forall x.\theta) = \bigcap_{a\in A} (e_{x\mapsto a})^+(\theta) \overset{(25)}{=} \bigcap_{a\in A} (e_{x\mapsto a})^+(\varphi) = e^+(\forall x.\varphi),$$

Hence, $\mathcal{A} \vDash (\varphi_x(y) \to \exists x.\varphi)[e]$ and $\mathcal{A} \vDash (\forall x.\varphi \to \varphi_x(y))[e]$.

□

3.9.2 Set variables

Proposition 3.76. *Let φ, δ be patterns and X be a set variable such that X is free for δ in φ. Then for every τ-structure $\mathcal{A}$ and every $\mathcal{A}$-valuation e,*

$$e^+(Subf_\delta^X\varphi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\varphi).$$

Proof. If $X \notin FV(\varphi)$, then $Subf_\delta^X\varphi = \varphi$ and $e = e_{X\mapsto e^+(\delta)}$, so the conclusion is obvious.
Let $\mathcal{A}$ be a τ-structure. We prove by induction on patterns that for all patterns φ such that $X \in FV(\varphi)$, the following holds:

$$\text{for every } \mathcal{A}\text{-valuation } e,\ e^+(Subf_\delta^X\varphi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\varphi).$$

We use the definition by recursion on paterns of $Subf_\delta^X\varphi$, given by Remark 2.42.
Assume in the sequel that $X \in FV(\varphi)$. The proof is by induction on φ.

(i) φ is an atomic pattern. Since $X \in FV(\varphi)$, we have that $\varphi = X$. Then $Subf_\delta^X\varphi = \delta$. For every $\mathcal{A}$-valuation e, we get that $e^+(Subf_\delta^X\varphi) = e^+(\delta)$. Furthermore, $\left(e_{X\mapsto e^+(\delta)}\right)^+(\varphi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(X) = e^+(\delta)$.

(ii) $\varphi = \psi \cdot \chi$. Then $X \in FV(\psi)$, $X \in FV(\chi)$ and X is free for δ in ψ, χ. Let e be an $\mathcal{A}$-valuation. We can apply the induction hypothesis for ψ and χ to get that $e^+(Subf_\delta^X\psi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\psi)$ and $e^+(Subf_\delta^X\chi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\chi)$. It follows that

$$\begin{aligned} e^+(Subf_\delta^X\varphi) &= e^+(Subf_\delta^X\psi \cdot Subf_\delta^X\chi) = e^+(Subf_\delta^X\psi) \star e^+(Subf_\delta^X\chi) \\ &= \left(e_{X\mapsto e^+(\delta)}\right)^+(\psi) \star \left(e_{X\mapsto e^+(\delta)}\right)^+(\chi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\psi\cdot\chi) \\ &= \left(e_{X\mapsto e^+(\delta)}\right)^+(\varphi). \end{aligned}$$

(iii) $\varphi = \psi \to \chi$. Then $X \in FV(\psi)$, $X \in FV(\chi)$ and X is free for δ in ψ, χ. Let e be an $\mathcal{A}$-valuation. We can apply the induction hypothesis for ψ and χ to get that $e^+(Subf_\delta^X \psi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\psi)$ and $e^+(Subf_\delta^X \chi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\chi)$. It follows that

$$\begin{aligned}
e^+(Subf_\delta^X \varphi) &= e^+(Subf_\delta^X \psi \to Subf_\delta^X \chi) = C_A e^+ Subf_\delta^X \psi \cup e^+(Subf_\delta^X \chi) \\
&= C_A \left(e_{X\mapsto e^+(\delta)}\right)^+(\psi) \cup \left(e_{X\mapsto e^+(\delta)}\right)^+(\chi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\psi \to \chi) \\
&= \left(e_{X\mapsto e^+(\delta)}\right)^+(\varphi).
\end{aligned}$$

(iv) $\varphi = \exists x.\psi$. Then $X \in FV(\psi)$ and X is free for δ in ψ. Let e be an $\mathcal{A}$-valuation.

$$\begin{aligned}
e^+(Subf_\delta^X \varphi) = e^+(\exists x. Subf_\delta^X \psi) &= \bigcup_{a\in A} (e_{x\mapsto a})^+ (Subf_\delta^X \psi) \\
&= \bigcup_{a\in A} \left((e_{x\mapsto a})_{X\mapsto (e_{x\mapsto a})^+(\delta)}\right)^+(\psi)
\end{aligned}$$

by the induction hypothesis for ψ applied to $e_{x\mapsto a}$.

As X is free for δ in φ and $X \in FV(\psi)$, it follows that $x \notin FV(\delta)$. Hence, by Proposition 3.8, $e^+(\delta) = (e_{x\mapsto a})^+(\delta)$.

It follows that

$$\begin{aligned}
e^+(Subf_\delta^X \varphi) &= \bigcup_{a\in A} \left((e_{x\mapsto a})_{X\mapsto e^+(\delta)}\right)^+(\psi) = \bigcup_{a\in A} \left(\left(e_{X\mapsto e^+(\delta)}\right)_{x\mapsto a}\right)^+(\psi) \\
&= \left(e_{X\mapsto e^+(\delta)}\right)^+(\exists x.\psi)
\end{aligned}$$

(v) $\varphi = \mu Z.\psi$. Since $X \in FV(\varphi)$, we have that $X \neq Z$. Then $X \in FV(\psi)$ and X is free for δ in ψ. Let e be an $\mathcal{A}$-valuation. We get that

$$\begin{aligned}
e^+(Subf_\delta^X \varphi) = e^+(\mu Z. Subf_\delta^X \psi) &= \bigcap \left\{ B \subseteq A \mid (e_{Z\mapsto B})^+ (Subf_\delta^X \psi) \subseteq B \right\} \\
&= \bigcap \left\{ B \subseteq A \mid \left((e_{Z\mapsto B})_{X\mapsto (e_{Z\mapsto B})^+(\delta)}\right)^+(\psi) \subseteq B \right\}
\end{aligned}$$

by the induction hypothesis for ψ applied to $e_{Z\mapsto B}$.

As X is free for δ in φ and $X \in FV(\varphi)$, it follows that $Z \notin FV(\delta)$. Hence, by Proposition 3.8, $e^+(\delta) = (e_{Z\mapsto B})^+(\delta)$. It follows that

$$\begin{aligned}
e^+(Subf_\delta^X \varphi) &= \bigcap \left\{ B \subseteq A \mid \left((e_{Z\mapsto B})_{X\mapsto e^+(\delta)}\right)^+(\psi) \subseteq B \right\} \\
&= \bigcap \left\{ B \subseteq A \mid \left(\left(e_{X\mapsto e^+(\delta)}\right)_{Z\mapsto B}\right)^+(\psi) \subseteq B \right\} \\
&= \left(e_{X\mapsto e^+(\delta)}\right)^+(\mu Z.\psi) = \left(e_{X\mapsto e^+(\delta)}\right)^+(\varphi).
\end{aligned}$$

□

As an immediate consequence of Proposition 3.76, we get the following.

Corollary 3.77. *Let φ be a pattern and X, Y be variables such that X is free for Y in φ. Then for every τ-structure $\mathcal{A}$ and every $\mathcal{A}$-valuation e,*

$$e^+(Subf_Y^X \varphi) = \left(e_{X\mapsto e(Y)}\right)^+(\varphi).$$

Bounded substitution

Proposition 3.78. *Let φ be a pattern and X, Y variables such that Y does not occur in φ. Then*

$$\vDash \varphi \leftrightarrow Subb_Y^X \varphi.$$

Proof. If $X = Y$, then obviously $Subb_Y^X \varphi = \varphi$, hence $\vDash \varphi \leftrightarrow Subb_Y^X \varphi$. Assume that $X \neq Y$. Let $\mathcal{A}$ be a τ-structure. We prove by induction on φ that

$$\text{for any } \mathcal{A}\text{-valuation } e,\ e^+(\varphi) = e^+(Subb_Y^X \varphi).$$

We use the definition by recursion on paterns of $Subb_Y^X \varphi$ given by Remark 2.47.

(i) φ is an atomic pattern. Then $Subb_Y^X \varphi = \varphi$, hence $e^+(\varphi) = e^+(Subb_Y^X \varphi)$ for every $\mathcal{A}$-valuation e.

(ii) $\varphi = \psi \cdot \chi$. Let e be an $\mathcal{A}$-valuation. By the induction hypothesis for ψ, χ, we have that $e^+(\psi) = e^+(Subb_Y^X \psi)$ and $e^+(\chi) = e^+(Subb_Y^X \chi)$. It follows that

$$\begin{aligned} e^+(Subb_Y^X \varphi) &= e^+(Subb_Y^X(\psi) \cdot Subb_Y^X(\chi)) = e^+(Subb_Y^X \psi) \star e^+(Subb_Y^X \psi) = e^+(\psi) \star e^+(\chi) \\ &= e^+(\psi \cdot \chi) = e^+(\varphi). \end{aligned}$$

(iii) $\varphi = \psi \to \chi$. Let e be an $\mathcal{A}$-valuation. By the induction hypothesis for ψ, χ, we have that $e^+(\psi) = e^+(Subb_Y^X \psi)$ and $e^+(\chi) = e^+(Subb_Y^X \chi)$. It follows that

$$\begin{aligned} e^+(Subb_Y^X \varphi) &= e^+(Subb_Y^X \psi \to Subb_Y^X \chi) = C_A e^+(Subb_Y^X \psi) \cup e^+(Subb_Y^X \chi) \\ &= C_A e^+(\psi) \cup e^+(\chi) = e^+(\psi \to \chi) = e^+(\varphi). \end{aligned}$$

(iv) $\varphi = \exists x.\psi$. Then

$$\begin{aligned} e^+(Subb_Y^X \varphi) = e^+(\exists x.Subb_Y^X \psi) &= \bigcap_{a \in A} (e_{x \mapsto a})^+ (Subb_Y^X \psi) \\ &= \bigcap_{a \in A} (e_{x \mapsto a})^+ (\varphi) \quad \text{by the induction hypothesis for } \psi \text{ applied to } e_{x \mapsto a} \\ &= e^+(\varphi). \end{aligned}$$

(v) $\varphi = \mu Z.\psi$. We have two cases:

1. $X \neq Z$, then

$$\begin{aligned} e^+(Subb_Y^X \varphi) = e^+(\mu Z.Subb_Y^X \psi) &= \bigcap \left\{ B \subseteq A \mid (e_{Z \mapsto B})^+ (Subb_Y^X \psi) \subseteq B \right\} \\ &= \bigcap \left\{ B \subseteq A \mid (e_{Z \mapsto B})^+ (\psi) \subseteq B \right\} \\ &\quad \text{by the induction hypothesis for } \psi \text{ applied to } e_{Z \mapsto B} \\ &= e^+(\varphi). \end{aligned}$$

2. $X = Z$. Then

$$\begin{aligned} e^+(Subb_Y^X \varphi) &= e^+ \left(\mu Y.Subf_Y^X \left(Subb_Y^X \psi \right) \right) \\ &= \bigcap \left\{ B \subseteq A \mid (e_{Y \mapsto B})^+ \left(Subf_Y^X \left(Subb_Y^X \psi \right) \right) \subseteq B \right\}. \end{aligned}$$

As Y does not occur in ψ, we have that that X is free for Y in $Subb_Y^X \psi$, by Proposition 2.49. We can apply then Corollary 3.77 to get that

$$\begin{aligned} (e_{Y \mapsto B})^+ \left(Subf_Y^X \left(Subb_Y^X \psi \right) \right) &= \left((e_{Y \mapsto B})_{X \mapsto e_{Y \mapsto B}(Y)} \right)^+ \left(Subb_Y^X \psi \right) \\ &= ((e_{Y \mapsto B})_{X \mapsto B})^+ \left(Subb_Y^X \psi \right) = (e_{X \mapsto B})^+ \left(Subb_Y^X \psi \right) \end{aligned}$$

since Y does not occur in ψ, so $Y \notin FV\left(Subb_Y^X \psi\right)$, hence we can apply Proposition 3.8. It follows that

$$\begin{aligned}
e^+(Subb_Y^X \varphi) &= \bigcap \left\{ B \subseteq A \mid (e_{X \mapsto B})^+ \left(Subb_Y^X \psi\right) \subseteq B \right\} \\
&= \bigcap \left\{ B \subseteq A \mid (e_{X \mapsto B})^+ (\psi) \subseteq B \right\} \\
&\quad \text{by the induction hypothesis for } \psi \text{ applied to } e_{X \mapsto B} \\
&= e^+(\varphi).
\end{aligned}$$

□

3.10 Semantics of ν

Proposition 3.79. *For every pattern φ and set variable X,*

$$e^+(\nu X.\varphi) = \bigcup \left\{ D \subseteq A \mid (e_{X \mapsto D})^+ (\varphi) \supseteq D \right\}.$$

Proof. As $\nu X.\varphi = \neg \mu X.\neg Subf_{\neg X}^X \varphi$, we get that

$$\begin{aligned}
e^+(\nu X.\varphi) &= e^+(\neg \mu X.\neg Subf_{\neg X}^X \varphi) = C_A e^+(\mu X.\neg Subf_{\neg X}^X \varphi) \\
&= C_A \bigcap \left\{ B \subseteq A \mid (e_{X \mapsto B})^+ (\neg Subf_{\neg X}^X \varphi)) \subseteq B \right\} \\
&= C_A \bigcap \left\{ B \subseteq A \mid C_A \left(e_{X \mapsto B}\right)^+ (Subf_{\neg X}^X \varphi)) \subseteq B \right\} \\
&= \bigcup \left\{ C_A B \mid B \subseteq A \text{ such that } C_A \left(e_{X \mapsto B}\right)^+ (Subf_{\neg X}^X \varphi)) \subseteq B \right\} \\
&= \bigcup \left\{ C_A B \mid B \subseteq A \text{ such that } \left(e_{X \mapsto B}\right)^+ (Subf_{\neg X}^X \varphi)) \supseteq C_A B \right\}.
\end{aligned}$$

As, obviously, X is free for $\neg X$ in φ, we have by Proposition 3.76 that

$$\begin{aligned}
(e_{X \mapsto B})^+ (Subf_{\neg X}^X \varphi) &= \left((e_{X \mapsto B})_{X \mapsto (e_{X \mapsto B})^+(\neg X)} \right)^+ (\varphi) = \left((e_{X \mapsto B})_{X \mapsto C_A B} \right)^+ (\varphi) \\
&= (e_{X \mapsto C_A B})^+ (\varphi).
\end{aligned}$$

It follows that

$$\begin{aligned}
e^+(\nu X.\varphi) &= \bigcup \left\{ C_A B \mid B \subseteq A \text{ such that } \left(e_{X \mapsto C_A B}\right)^+ (\varphi) \supseteq C_A B \right\} \\
&= \bigcup \left\{ D \subseteq A \mid (e_{X \mapsto D})^+ (\varphi) \supseteq D \right\}.
\end{aligned}$$

□

3.11 Least and greatest fixpoints

Let φ be a pattern and X be a set variable. Define, for every $(\mathcal{A}, e)$, the functions

$$\mathcal{F}_{\varphi,X}^{\mathcal{A},e} : 2^A \to 2^A, \quad \mathcal{F}_{\varphi,X}^{\mathcal{A},e}(B) = (e_{X \mapsto B})^+ (\varphi), \tag{26}$$

$$\mathcal{G}_{\varphi,X}^{\mathcal{A},e} : 2^A \to 2^A, \quad \mathcal{G}_{\varphi,X}^{\mathcal{A},e}(B) = C_A \mathcal{F}_{\varphi,X}^{\mathcal{A},e}(B) = C_A \left(e_{X \mapsto B}\right)^+ (\varphi). \tag{27}$$

Proposition 3.80. *(i) If φ is positive in X, then for all $(\mathcal{A}, e)$, $\mathcal{F}_{\varphi,X}^{\mathcal{A},e}$ is monotone.*

(ii) If φ is negative in X, then for all $(\mathcal{A}, e)$, $\mathcal{G}_{\varphi,X}^{\mathcal{A},e}$ is monotone.

Proof. We prove (i) and (ii) simultaneously by induction on φ. We use Remarks 2.28, 2.29. Let $B, C \subseteq A$ be such that $B \subseteq C$.

(a) φ is atomic and $\varphi \neq X$. Then φ is both positive and negative in X. As $X \notin FV(\varphi)$, we have that $(e_{X\mapsto B})^+(\varphi) = (e_{X\mapsto C})^+(\varphi) = e^+(\varphi)$. It follows that $\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) = \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C) = e^+(\varphi)$ and $\mathcal{G}^{\mathcal{A},e}_{\varphi,X}(B) = \mathcal{G}^{\mathcal{A},e}_{\varphi,X}(C) = C_A e^+(\varphi)$.

(b) $\varphi = X$. Then φ is positive in X, hence $\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) = (e_{X\mapsto B})^+(X) = B \subseteq C = (e_{X\mapsto C})^+(X) = \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C)$.

(c) $\varphi = \psi \cdot \chi$.

(i) Assume that φ is positive in X. We have that both ψ, χ are positive in X. We can apply the induction hypothesis to get that $\mathcal{F}^{\mathcal{A},e}_{\psi,X}(B) \subseteq \mathcal{F}^{\mathcal{A},e}_{\psi,X}(C)$ and $\mathcal{F}^{\mathcal{A},e}_{\chi,X}(B) \subseteq \mathcal{F}^{\mathcal{A},e}_{\chi,X}(C)$. It follows that

$$\begin{aligned}\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &= (e_{X\mapsto B})^+(\psi\cdot\chi) = (e_{X\mapsto B})^+(\psi) \star (e_{X\mapsto B})^+(\chi) = \mathcal{F}^{\mathcal{A},e}_{\psi,X}(B) \star \mathcal{F}^{\mathcal{A},e}_{\chi,X}(B) \\ &\overset{(169)}{\subseteq} \mathcal{F}^{\mathcal{A},e}_{\psi,X}(C) \star \mathcal{F}^{\mathcal{A},e}_{\chi,X}(C) = (e_{X\mapsto C})^+(\psi) \star (e_{X\mapsto C})^+(\chi) = (e_{X\mapsto C})^+(\psi\cdot\chi) \\ &= \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C).\end{aligned}$$

(ii) Assume that φ is negative in X. We have that both ψ, χ are negative in X. We can apply the induction hypothesis to get that $\mathcal{G}^{\mathcal{A},e}_{\psi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e}_{\psi,X}(C)$ and $\mathcal{G}^{\mathcal{A},e}_{\chi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e}_{\chi,X}(C)$, hence $\mathcal{F}^{\mathcal{A},e}_{\psi,X}(B) \supseteq \mathcal{F}^{\mathcal{A},e}_{\psi,X}(C)$ and $\mathcal{F}^{\mathcal{A},e}_{\chi,X}(B) \supseteq \mathcal{F}^{\mathcal{A},e}_{\chi,X}(C)$. It follows that

$$\begin{aligned}\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &= (e_{X\mapsto B})^+(\psi\cdot\chi) = (e_{X\mapsto B})^+(\psi) \star (e_{X\mapsto B})^+(\chi) = \mathcal{F}^{\mathcal{A},e}_{\psi,X}(B) \star \mathcal{F}^{\mathcal{A},e}_{\chi,X}(B) \\ &\overset{(169)}{\supseteq} \mathcal{F}^{\mathcal{A},e}_{\psi,X}(C) \star \mathcal{F}^{\mathcal{A},e}_{\chi,X}(C) = (e_{X\mapsto C})^+(\psi) \star (e_{X\mapsto C})^+(\chi) = (e_{X\mapsto C})^+(\psi\cdot\chi) \\ &= \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C).\end{aligned}$$

Thus, $\mathcal{G}^{\mathcal{A},e}_{\varphi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e}_{\varphi,X}(C)$.

(d) $\varphi = \psi \to \chi$.

(i) Assume that φ is positive in X. We have that ψ is negative in X and χ is positive in X. We can apply the induction hypothesis for ψ to get that $\mathcal{G}^{\mathcal{A},e}_{\psi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e}_{\psi,X}(C)$ and the induction hypothesis for χ to get that $\mathcal{F}^{\mathcal{A},e}_{\chi,X}(B) \subseteq \mathcal{F}^{\mathcal{A},e}_{\chi,X}(C)$. It follows that

$$\begin{aligned}\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &= (e_{X\mapsto B})^+(\psi\to\chi) = C_A\,(e_{X\mapsto B})^+(\psi) \cup (e_{X\mapsto B})^+(\chi) \\ &= \mathcal{G}^{\mathcal{A},e}_{\psi,X}(B) \cup \mathcal{F}^{\mathcal{A},e}_{\chi,X}(B) \\ &\subseteq \mathcal{G}^{\mathcal{A},e}_{\psi,X}(C) \cup \mathcal{F}^{\mathcal{A},e}_{\chi,X}(C) \\ &= C_A\,(e_{X\mapsto C})^+(\psi) \cup (e_{X\mapsto C})^+(\chi) = (e_{X\mapsto C})^+(\psi\to\chi) \\ &= \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C).\end{aligned}$$

(ii) Assume that φ is negative in X. We have that ψ is positive in X and χ is negative in X. We can apply the induction hypothesis for ψ to get that $\mathcal{F}^{\mathcal{A},e}_{\psi,X}(B) \subseteq \mathcal{F}^{\mathcal{A},e}_{\psi,X}(C)$ and the induction hypothesis for χ to get that $\mathcal{G}^{\mathcal{A},e}_{\chi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e}_{\chi,X}(C)$. It follows that

$$\begin{aligned}\mathcal{G}^{\mathcal{A},e}_{\varphi,X}(B) &= C_A\,(e_{X\mapsto B})^+(\psi\to\chi) = C_A\Big(C_A\,(e_{X\mapsto B})^+(\psi) \cup (e_{X\mapsto B})^+(\chi)\Big) \\ &= (e_{X\mapsto B})^+(\psi) \cap C_A\,(e_{X\mapsto B})^+(\chi) = \mathcal{F}^{\mathcal{A},e}_{\psi,X}(B) \cap \mathcal{G}^{\mathcal{A},e}_{\chi,X}(B) \\ &\subseteq \mathcal{F}^{\mathcal{A},e}_{\psi,X}(C) \cap \mathcal{G}^{\mathcal{A},e}_{\chi,X}(C) = (e_{X\mapsto C})^+(\psi) \cap C_A\,(e_{X\mapsto C})^+(\chi) \\ &= C_A\Big(C_A\,(e_{X\mapsto C})^+(\psi) \cup (e_{X\mapsto C})^+(\chi)\Big) = C_A\,(e_{X\mapsto C})^+(\psi\to\chi) \\ &= \mathcal{G}^{\mathcal{A},e}_{\varphi,X}(C).\end{aligned}$$

(e) $\varphi = \exists x.\psi$.

(i) Assume that φ is positive in X. We have that ψ is positive in X. It follows that

$$\begin{aligned}
\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &= (e_{X\mapsto B})^+(\exists x.\psi) = \bigcup_{a\in A}((e_{X\mapsto B})_{x\mapsto a})^+(\psi) = \bigcup_{a\in A}((e_{x\mapsto a})_{X\mapsto B})^+(\psi)\\
&= \bigcup_{a\in A}\mathcal{F}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(B) \subseteq \bigcup_{a\in A}\mathcal{F}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(C)\\
&\quad \text{by the induction hypothesis for } \psi \text{ applied to } e_{x\mapsto a}\\
&= \bigcup_{a\in A}((e_{x\mapsto a})_{X\mapsto C})^+(\psi) = \bigcup_{a\in A}((e_{X\mapsto C})_{x\mapsto a})^+(\psi) = (e_{X\mapsto C})^+(\exists x.\psi)\\
&= \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C).
\end{aligned}$$

(ii) Assume that φ is negative in X. We have that ψ is negative in X. It follows that

$$\begin{aligned}
\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &= (e_{X\mapsto B})^+(\exists x.\psi) = \bigcup_{a\in A}((e_{X\mapsto B})_{x\mapsto a})^+(\psi) = \bigcup_{a\in A}((e_{x\mapsto a})_{X\mapsto B})^+(\psi)\\
&= \bigcup_{a\in A}\mathcal{F}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(B) \supseteq \bigcup_{a\in A}\mathcal{F}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(C)
\end{aligned}$$

as, by the induction hypothesis for ψ applied to $e_{x\mapsto a}$, we have that $\mathcal{G}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(C)$, hence $\mathcal{F}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(B) \supseteq \mathcal{F}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(C)$. Thus,

$$\begin{aligned}
\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &\supseteq \bigcup_{a\in A}\mathcal{F}^{\mathcal{A},e_{x\mapsto a}}_{\psi,X}(C) = \bigcup_{a\in A}((e_{x\mapsto a})_{X\mapsto C})^+(\psi) = \bigcup_{a\in A}((e_{X\mapsto C})_{x\mapsto a})^+(\psi)\\
&= (e_{X\mapsto C})^+(\exists x.\psi) = \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C).
\end{aligned}$$

Hence, $\mathcal{G}^{\mathcal{A},e}_{\varphi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e}_{\varphi,X}(C)$.

(f) $\varphi = \mu Z.\psi$. If $Z = X$, then φ is both positive and negative in X. As $X \notin FV(\varphi)$, we have that $(e_{X\mapsto B})^+(\varphi) = (e_{X\mapsto C})^+(\varphi) = e^+(\varphi)$. It follows that $\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) = \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C) = e^+(\varphi)$ and $\mathcal{G}^{\mathcal{A},e}_{\varphi,X}(B) = \mathcal{G}^{\mathcal{A},e}_{\varphi,X}(C) = C_A e^+(\varphi)$.

Suppose that $X \neq Z$.

(i) Assume that φ is positive in X. We have that ψ is positive in X. Then

$$\begin{aligned}
\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &= (e_{X\mapsto B})^+(\mu Z.\psi) = \bigcap\left\{D \subseteq A \mid ((e_{X\mapsto B})_{Z\mapsto D})^+(\psi) \subseteq D\right\}\\
&= \bigcap\left\{D \subseteq A \mid ((e_{Z\mapsto D})_{X\mapsto B})^+(\psi) \subseteq D\right\} = \bigcap\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq D\right\}.
\end{aligned}$$

By the induction hypothesis for ψ applied to $e_{Z\mapsto D}$, we have that $\mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C)$, so $\mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C) \subseteq D$ implies $\mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq D$, hence

$$\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C) \subseteq D\right\} \subseteq \left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq D\right\}.$$

Thus,

$$\bigcap\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq D\right\} \subseteq \bigcap\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C) \subseteq D\right\}.$$

It follows that

$$\begin{aligned}
\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &\subseteq \bigcap\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C) \subseteq D\right\} = \bigcap\left\{D \subseteq A \mid ((e_{Z\mapsto D})_{X\mapsto C})^+(\psi) \subseteq D\right\}\\
&= \bigcap\left\{D \subseteq A \mid ((e_{X\mapsto C})_{Z\mapsto D})^+(\psi) \subseteq D\right\}\\
&= (e_{X\mapsto C})^+(\mu Z.\psi) = \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C).
\end{aligned}$$

(ii) Assume that φ is negative in X. We have that ψ is negative in X. Then

$$\begin{aligned}\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &= (e_{X\mapsto B})^+(\mu Z.\psi) = \bigcap\left\{D \subseteq A \mid ((e_{X\mapsto B})_{Z\mapsto D})^+(\psi) \subseteq D\right\} \\ &= \bigcap\left\{D \subseteq A \mid ((e_{Z\mapsto D})_{X\mapsto B})^+(\psi) \subseteq D\right\} = \bigcap\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq D\right\}.\end{aligned}$$

By the induction hypothesis for ψ applied to $e_{Z\mapsto D}$, we have that $\mathcal{G}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C)$, so $\mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \supseteq \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C)$. It follows that $\mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq D$ implies $\mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C) \subseteq D$, hence

$$\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq D\right\} \subseteq \left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C) \subseteq D\right\}.$$

Thus,

$$\bigcap\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C) \subseteq D\right\} \subseteq \bigcap\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(B) \subseteq D\right\}.$$

It follows that

$$\begin{aligned}\mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) &\supseteq \bigcap\left\{D \subseteq A \mid \mathcal{F}^{\mathcal{A},e_{Z\mapsto D}}_{\psi,X}(C) \subseteq D\right\} = \bigcap\left\{D \subseteq A \mid ((e_{Z\mapsto D})_{X\mapsto C})^+(\psi) \subseteq D\right\} \\ &= \bigcap\left\{D \subseteq A \mid ((e_{X\mapsto C})_{Z\mapsto D})^+(\psi) \subseteq D\right\} \\ &= (e_{X\mapsto C})^+(\mu Z.\psi) = \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(C).\end{aligned}$$

Hence, $\mathcal{G}^{\mathcal{A},e}_{\varphi,X}(B) \subseteq \mathcal{G}^{\mathcal{A},e}_{\varphi,X}(C)$.

□

As a consequence of Knaster-Tarski Theorem A.9, it follows that if φ is positive in X, then $\mathcal{F}^{\mathcal{A},e}_{\varphi,X}$ has a least fixpoint $\mu\mathcal{F}^{\mathcal{A},e}_{\varphi,X}$ and a greatest fixpoint $\nu\mathcal{F}^{\mathcal{A},e}_{\varphi,X}$, given by:

$$\begin{aligned}\mu\mathcal{F}^{\mathcal{A},e}_{\varphi,X} &= \bigcap\{B \subseteq A \mid \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B) \subseteq B\} = \bigcap\{B \subseteq A \mid (e_{X\mapsto B})^+(\varphi) \subseteq B\} \\ &= \bigcap\{B \subseteq A \mid (e_{X\mapsto B})^+(\varphi) = B\}, \\ \nu\mathcal{F}^{\mathcal{A},e}_{\varphi,X} &= \bigcup\{B \subseteq A \mid B \subseteq \mathcal{F}^{\mathcal{A},e}_{\varphi,X}(B)\} = \bigcup\{B \subseteq A \mid B \subseteq (e_{X\mapsto B})^+(\varphi)\} \\ &= \bigcup\{B \subseteq A \mid (e_{X\mapsto B})^+(\varphi) = B\}.\end{aligned}$$

Furthermore, by Definition 3.4.(vii) and Proposition 3.79, we get that

$$\mu\mathcal{F}^{\mathcal{A},e}_{\varphi,X} = e^+(\mu X.\varphi) \quad \text{and} \quad \nu\mathcal{F}^{\mathcal{A},e}_{\varphi,X} = e^+(\nu X.\varphi). \tag{28}$$

Thus,

Proposition 3.81. *Let φ be a pattern and X be a set variable such that φ is positive in X. Then, for every $(\mathcal{A}, e)$,*

(i) $\left(e_{X\mapsto e^+(\mu X.\varphi)}\right)^+(\varphi) = e^+(\mu X.\varphi)$ and $\left(e_{X\mapsto e^+(\nu X.\varphi)}\right)^+(\varphi) = e^+(\nu X.\varphi)$.

(ii) For every $B \subseteq A$ such that $(e_{X\mapsto B})^+(\varphi) = B$,

$$e^+(\mu X.\varphi) \subseteq B \subseteq e^+(\nu X.\varphi).$$

3.12 Equivalences, logical consequences, validities

Proposition 3.82. *For all patterns φ, ψ, χ,*

$$\vDash \varphi \leftrightarrow \varphi, \tag{29}$$
$$\vDash \varphi \leftrightarrow \psi \quad \textit{iff} \quad \vDash \psi \leftrightarrow \varphi, \tag{30}$$
$$\vDash \varphi \leftrightarrow \psi \textit{ and } \vDash \psi \leftrightarrow \chi \quad \textit{implies} \quad \vDash \varphi \leftrightarrow \chi, \tag{31}$$
$$\vDash \varphi \to \psi \textit{ and } \vDash \psi \to \chi \quad \textit{implies} \quad \vDash \varphi \to \chi. \tag{32}$$

Proof. (29): It follows by (6) and Proposition 3.66.
(30): We have that $\vDash \varphi \leftrightarrow \psi$ iff $\mathcal{A} \vDash (\varphi \leftrightarrow \psi)[e]$ for every $(\mathcal{A}, e)$ iff $e^+(\varphi) = e^+(\psi)$ for every $(\mathcal{A}, e)$ iff $e^+(\psi) = e^+(\varphi)$ for every $(\mathcal{A}, e)$ iff $\mathcal{A} \vDash (\psi \leftrightarrow \varphi)[e]$ for every $(\mathcal{A}, e)$ iff $\vDash \psi \leftrightarrow \varphi$.
(31): Let $(\mathcal{A}, e)$. Then $\mathcal{A} \vDash (\varphi \leftrightarrow \psi)[e]$, hence $e^+(\varphi) = e^+(\psi)$. Furthermore, $\mathcal{A} \vDash (\psi \leftrightarrow \chi)[e]$, hence $e^+(\psi) = e^+(\chi)$. It follows that $e^+(\varphi) = e^+(\chi)$. Thus, $\mathcal{A} \vDash (\varphi \leftrightarrow \chi)[e]$.
(32): Assume that $\vDash \varphi \to \psi$ and $\vDash \psi \to \chi$. Let $(\mathcal{A}, e)$. Then $\mathcal{A} \vDash (\varphi \to \psi)[e]$, hence $e^+(\varphi) \subseteq e^+(\psi)$. Furthermore, $\mathcal{A} \vDash (\psi \to \chi)[e]$, hence $e^+(\psi) \subseteq e^+(\chi)$. It follows that $e^+(\varphi) \subseteq e^+(\chi)$. Thus, $\mathcal{A} \vDash (\varphi \to \chi)[e]$. □

Proposition 3.83. *For any pattern φ and any variable x,*

$$\vDash \forall x.\varphi \to \exists x.\varphi, \tag{33}$$
$$\vDash \forall x.\varphi \to \varphi, \tag{34}$$
$$\vDash \varphi \to \exists x.\varphi, \tag{35}$$
$$\vDash \exists x.x, \tag{36}$$
$$\vDash \forall x.\varphi \textit{ iff } \vDash \varphi \tag{37}$$

Proof. We apply Proposition 3.7.(ix).
(33): Let $(\mathcal{A}, e)$. Then $e^+(\forall x.\varphi) = \bigcap_{a\in A} (e_{x\mapsto a})^+(\varphi) \subseteq \bigcup_{a\in A} (e_{x\mapsto a})^+(\varphi) = e^+(\exists x.\varphi)$.
(34): Let $(\mathcal{A}, e)$. Then $e^+(\forall x.\varphi) = \bigcap_{a\in A} (e_{x\mapsto a})^+(\varphi) \subseteq \left(e_{x\mapsto e(x)}\right)^+(\varphi) = e^+(\varphi)$.
(35): Let $(\mathcal{A}, e)$. Then $e^+(\varphi) = \left(e_{x\mapsto e(x)}\right)^+(\varphi) \subseteq \bigcup_{a\in A} (e_{x\mapsto a})^+(\varphi) = e^+(\exists x.\varphi)$.
(36): Let $(\mathcal{A}, e)$. Then $e^+(\exists x.x) = \bigcup_{a\in A} (e_{x\mapsto a})^+(x) = \bigcup_{a\in A}\{a\} = A$. Thus, $\mathcal{A} \vDash (\exists x.x)[e]$.
(37): By Proposition 3.11.(viii). □

Proposition 3.84. *For any pattern φ and any variable x,*

$$\vDash \exists x.\varphi \leftrightarrow \varphi \quad \textit{if } x \notin FV(\varphi), \tag{38}$$
$$\vDash \forall x.\varphi \leftrightarrow \varphi \quad \textit{if } x \notin FV(\varphi). \tag{39}$$

Proof. We apply Proposition 3.7.(x). Let $(\mathcal{A}, e)$. As $x \notin FV(\varphi)$, we can apply Proposition 3.8 to get that $(e_{x\mapsto a})^+(\varphi) = e^+(\varphi)$ for any $a \in A$.
(38): We have that $e^+(\exists x.\varphi) = \bigcup_{a\in A} (e_{x\mapsto a})^+(\varphi) = \bigcup_{a\in A} e^+(\varphi) = e^+(\varphi)$.
(39): We have that $e^+(\forall x.\varphi) = \bigcap_{a\in A} (e_{x\mapsto a})^+(\varphi) = \bigcap_{a\in A} e^+(\varphi) = e^+(\varphi)$. □

Proposition 3.85. *For any pattern φ and any variable x,*

$$\vDash \forall x.(\varphi \to \psi) \to (\forall x.\varphi \to \forall x.\psi). \tag{40}$$

Proof. Let $(\mathcal{A}, e)$. We have that

$$
\begin{aligned}
e^+(\forall x.\varphi \to \forall x.\psi) &= C_A\left(e_{x\mapsto a}\right)^+(\forall x\varphi) \cup \left(e_{x\mapsto a}\right)^+(\forall x\psi) \quad \text{by Lemma 3.5.(i)}\\
&= C_A\left(\bigcap_{a\in A}\left(e_{x\mapsto a}\right)^+(\varphi)\right) \cup \left(e_{x\mapsto a}\right)^+(\psi)\\
&= \bigcup_{a\in A} C_A\left(e_{x\mapsto a}\right)^+(\varphi) \cup \bigcap_{a\in A}\left(e_{x\mapsto a}\right)^+(\psi) \quad \text{by (157)}\\
&= \bigcap_{a\in A}\left(\bigcup_{a\in A} C_A\left(e_{x\mapsto a}\right)^+(\varphi) \cup \left(e_{x\mapsto a}\right)^+(\psi)\right) \quad \text{by (158)}\\
&\supseteq \bigcap_{a\in A}\left(C_A\left(e_{x\mapsto a}\right)^+(\varphi) \cup \left(e_{x\mapsto a}\right)^+(\psi)\right)\\
&= \bigcap_{a\in A}\left(e_{x\mapsto a}\right)^+(\varphi \to \psi) \quad \text{by Lemma 3.5.(i)}\\
&= e^+(\forall x.(\varphi \to \psi)) \quad \text{by Lemma 3.5.(viii)}
\end{aligned}
$$

We have got that $e^+(\forall x.(\varphi \to \psi)) \subseteq e^+(\forall x.\varphi \to \forall x.\psi)$. Apply Proposition 3.7.(ix) to conclude that $\mathcal{A} \vDash \big(\forall x.(\varphi \to \psi) \to (\forall x.\varphi \to \forall x.\psi)\big)[e]$. ☐

3.13 Rules

Let Γ be a set of patterns.

3.13.1 $\vDash_s$

Proposition 3.86. *For all patterns φ, ψ, χ,*

$$
\begin{aligned}
\Gamma \vDash_s \varphi \text{ and } \Gamma \vDash_s \varphi \to \psi &\quad\Rightarrow\quad \Gamma \vDash_s \psi && (41)\\
\Gamma \vDash_s \varphi \to \psi \text{ and } \Gamma \vDash_s \psi \to \chi &\quad\Rightarrow\quad \Gamma \vDash_s \varphi \to \chi && (42)\\
\Gamma \vDash_s \varphi \wedge \psi \to \chi &\quad\Rightarrow\quad \Gamma \vDash_s \varphi \to (\psi \to \chi) && (43)\\
\Gamma \vDash_s \varphi \to (\psi \to \chi) &\quad\Rightarrow\quad \Gamma \vDash_s \varphi \wedge \psi \to \chi && (44)
\end{aligned}
$$

Proof. Let $(\mathcal{A}, e)$.
(41): Assume that $\Gamma \vDash_s \varphi$ and $\Gamma \vDash_s \varphi \to \psi$. Then $\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi)$ and $\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi \to \psi)$, hence $\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi) \cap e^+(\varphi \to \psi)$. Remark that

$$
\begin{aligned}
e^+(\varphi) \cap e^+(\varphi \to \psi) &= e^+(\varphi) \cap \left(C_A e^+(\varphi) \cup e^+(\psi)\right) = \left(e^+(\varphi) \cap C_A e^+(\varphi)\right) \cup \left(e^+(\varphi) \cap e^+(\psi)\right)\\
&= \emptyset \cup \left(e^+(\varphi) \cap e^+(\psi)\right) = e^+(\varphi) \cap e^+(\psi) \subseteq e^+(\psi).
\end{aligned}
$$

Hence, $\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\psi)$, that is $\Gamma \vDash_s \psi$.
(42): Assume that $\Gamma \vDash_s \varphi \to \psi$ and $\Gamma \vDash_s \psi \to \chi$. Then $\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi \to \psi)$ and $\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\psi \to \chi)$, hence $\bigcap_{\gamma\in\Gamma} e^+(\gamma) \subseteq e^+(\varphi \to \psi) \cap e^+(\psi \to \chi)$. Remark now that

$$
\begin{aligned}
e^+(\varphi \to \psi) \cap e^+(\psi \to \chi) &= C_A\big(e^+(\varphi) \setminus e^+(\psi)\big) \cap C_A\big(e^+(\psi) \setminus e^+(\chi)\big)\\
&\overset{(147)}{=} C_A\Big(\left(e^+(\varphi) \setminus e^+(\psi)\right) \cup \left(e^+(\psi) \setminus e^+(\chi)\right)\Big)\\
&\subseteq C_A\big(e^+(\varphi) \setminus e^+(\chi)\big) \text{ by (144) and (152)}\\
&= e^+(\varphi \to \chi).
\end{aligned}
$$

(43), (44): By (15) and Remark 3.24.(ii), we have that $\varphi \to (\psi \to \chi) \mathrel{\text{=}\!\!|}\!\vDash_s \varphi \wedge \psi \to \chi$. Apply now Remark 3.47.(ii). ☐

3.13.2 $\vDash_l$

Proposition 3.87. *For all patterns φ, ψ, χ,*

$$\Gamma \vDash_l \varphi \text{ and } \Gamma \vDash_l \varphi \to \psi \quad \Rightarrow \quad \Gamma \vDash_l \psi \tag{45}$$

$$\Gamma \vDash_l \varphi \to \psi \text{ and } \Gamma \vDash_l \psi \to \chi \quad \Rightarrow \quad \Gamma \vDash_l \varphi \to \chi \tag{46}$$

$$\Gamma \vDash_l \varphi \wedge \psi \to \chi \quad \Rightarrow \quad \Gamma \vDash_l \varphi \to (\psi \to \chi) \tag{47}$$

$$\Gamma \vDash_l \varphi \to (\psi \to \chi) \quad \Rightarrow \quad \Gamma \vDash_l \varphi \wedge \psi \to \chi \tag{48}$$

Proof. Let $(\mathcal{A}, e)$ be such that $\mathcal{A} \vDash \Gamma[e]$.
(45): Assume that $\Gamma \vDash_l \varphi$ and $\Gamma \vDash_l \varphi \to \psi$. Then $\mathcal{A} \vDash \varphi[e]$ and $\mathcal{A} \vDash (\varphi \to \psi)[e]$, hence $e^+(\varphi) \subseteq e^+(\psi)$. It follows that $e^+(\psi) = A$, hence $\mathcal{A} \vDash \psi[e]$.
(46): Assume that $\Gamma \vDash_l \varphi \to \psi$ and $\Gamma \vDash_l \psi \to \chi$. Then $\mathcal{A} \vDash (\varphi \to \psi)[e]$ and $\mathcal{A} \vDash (\psi \to \chi)[e]$, hence $e^+(\varphi) \subseteq e^+(\psi)$ and $e^+(\psi) \subseteq e^+(\chi)$. It follows that $e^+(\varphi) \subseteq e^+(\chi)$, so $\mathcal{A} \vDash (\varphi \to \chi)[e]$.
(47), (48): By (15) and Remark 3.24.(ii), we have that $\varphi \to (\psi \to \chi) \vDash\!\!\!\dashv_s \varphi \wedge \psi \to \chi$, hence, by Remark 3.26.(ii), $\varphi \to (\psi \to \chi) \vDash\!\!\!\dashv_l \varphi \wedge \psi \to \chi$. Apply now Remark 3.40.(ii) □

3.13.3 $\vDash_g$

Proposition 3.88. *For all patterns φ, ψ, χ,*

$$\Gamma \vDash_g \varphi \text{ and } \Gamma \vDash_g \varphi \to \psi \quad \Rightarrow \quad \Gamma \vDash_g \psi, \tag{49}$$

$$\Gamma \vDash_g \varphi \to \psi \text{ and } \Gamma \vDash_g \psi \to \chi \quad \Rightarrow \quad \Gamma \vDash_g \varphi \to \chi, \tag{50}$$

$$\Gamma \vDash_g \varphi \wedge \psi \to \chi \quad \Rightarrow \quad \Gamma \vDash_g \varphi \to (\psi \to \chi), \tag{51}$$

$$\Gamma \vDash_g \varphi \to (\psi \to \chi), \quad \Rightarrow \quad \Gamma \vDash_g \varphi \wedge \psi \to \chi, \tag{52}$$

$$\Gamma \vDash_g \varphi \to \psi \quad \Rightarrow \quad \Gamma \vDash_g \chi \vee \varphi \to \chi \vee \psi. \tag{53}$$

Proof. Let $\mathcal{A}$ be such that $\mathcal{A} \vDash \Gamma$.
(49): Assume that $\Gamma \vDash_g \varphi$ and $\Gamma \vDash_g \varphi \to \psi$. Let e be an $\mathcal{A}$-evaluation. Then $\mathcal{A} \vDash \varphi[e]$ and $\mathcal{A} \vDash (\varphi \to \psi)[e]$, so $A = e^+(\varphi) \subseteq e^+(\psi)$. It follows that $e^+(\psi) = A$, hence $\mathcal{A} \vDash \psi[e]$.
(50): Assume that $\Gamma \vDash_g \varphi \to \psi$ and $\Gamma \vDash_g \psi \to \chi$. Let e be an $\mathcal{A}$-evaluation. Then $\mathcal{A} \vDash (\varphi \to \psi)[e]$ and $\mathcal{A} \vDash (\psi \to \chi)[e]$, hence $e^+(\varphi) \subseteq e^+(\psi)$ and $e^+(\psi) \subseteq e^+(\chi)$. It follows that $e^+(\varphi) \subseteq e^+(\chi)$, so $\mathcal{A} \vDash (\varphi \to \chi)[e]$.
(51), (52): By (15) and Remark 3.24.(ii), we have that $\varphi \to (\psi \to \chi) \vDash\!\!\!\dashv_s \varphi \wedge \psi \to \chi$, hence, by Remark 3.26.(ii), $\varphi \to (\psi \to \chi) \vDash\!\!\!\dashv_g \varphi \wedge \psi \to \chi$. Apply now Remark 3.33.(ii).
(53): Assume that $\Gamma \vDash_g \varphi \to \psi$. Let e be an $\mathcal{A}$-evaluation. Then $\mathcal{A} \vDash (\varphi \to \psi)[e]$, hence $e^+(\varphi) \subseteq e^+(\psi)$. We get that $e^+(\chi \vee \varphi) = e^+(\chi) \cup e^+(\varphi) \subseteq e^+(\chi) \cup e^+(\psi) = e^+(\chi \vee \psi)$. Thus, $\mathcal{A} \vDash (\chi \vee \varphi \to \chi \vee \psi)[e]$. □

Proposition 3.89. *For all patterns φ, ψ and variables x,*

$$\Gamma \vDash_g \varphi \to \psi \quad \Rightarrow \quad \Gamma \vDash_g \exists x.\varphi \to \psi \quad \text{if } x \notin FV(\psi)., \tag{54}$$

$$\Gamma \vDash_g \varphi \quad \Rightarrow \quad \Gamma \vDash_g \forall x.\varphi. \tag{55}$$

Proof. Let $\mathcal{A}$ be such that $\mathcal{A} \vDash \Gamma$.
(54) Assume that $\Gamma \vDash_g \varphi \to \psi$. Let e be an $\mathcal{A}$-evaluation. Then for every $a \in A$, $\mathcal{A} \vDash (\varphi \to \psi)[e_{x \mapsto a}]$, so $(e_{x \mapsto a})^+(\varphi) \subseteq (e_{x \mapsto a})^+(\psi)$. Since $x \notin FV(\psi)$, we have that $(e_{x \mapsto a})^+(\psi) = e^+(\psi)$. Thus, we have got that for every $a \in A$, $(e_{x \mapsto a})^+(\varphi) \subseteq e^+(\psi)$. It follows that $e^+(\exists x.\varphi) = \bigcup_{a \in A} (e_{x \mapsto a})^+(\varphi) \subseteq e^+(\psi)$, that is $\mathcal{A} \vDash (\exists x.\varphi \to \psi)[e]$.
(55) Assume that $\Gamma \vDash_g \varphi$. Let e be an $\mathcal{A}$-evaluation. Then for every $a \in A$, $\mathcal{A} \vDash \varphi[e_{x \mapsto a}]$, as $\mathcal{A} \vDash \varphi$. Thus, $\mathcal{A} \vDash \forall x \varphi$. □

3.14 Application

Proposition 3.90. *For all patterns φ, ψ, χ,*

$$\begin{aligned}
&\vDash \varphi \cdot \bot \leftrightarrow \bot, && (56)\\
&\vDash \bot \cdot \varphi \leftrightarrow \bot, && (57)\\
&\vDash (\varphi \vee \psi) \cdot \chi \leftrightarrow \varphi \cdot \chi \vee \psi \cdot \chi, && (58)\\
&\vDash \chi \cdot (\varphi \vee \psi) \leftrightarrow \chi \cdot \varphi \vee \chi \cdot \psi, && (59)\\
&\vDash (\exists x.\varphi) \cdot \psi \leftrightarrow \exists x.\varphi \cdot \psi \quad \textit{if } x \notin FV(\psi), && (60)\\
&\vDash \psi \cdot (\exists x.\varphi) \leftrightarrow \exists x.\psi \cdot \varphi \quad \textit{if } x \notin FV(\psi). && (61)
\end{aligned}$$

Proof. Let $(\mathcal{A}, e)$.
(56), (57): Apply (159) to get that $e^+(\varphi \cdot \bot) = e^+(\varphi) \star e^+(\bot) = e^+(\varphi) \star \emptyset = \emptyset = e^+(\bot)$ and $e^+(\bot \cdot \varphi) = e^+(\bot) \star e^+(\varphi) = \emptyset \star e^+(\varphi) = \emptyset = e^+(\bot)$.

(58): We have that $e^+((\varphi \vee \psi) \cdot \chi) = e^+(\varphi \vee \psi) \star e^+(\chi) = (e^+(\varphi) \cup e^+(\psi)) \star e^+(\chi) \overset{(160)}{=} (e^+(\varphi) \star e^+(\chi)) \cup (e^+(\psi) \star e^+(\chi)) = e^+(\varphi \cdot \chi) \cup e^+(\psi \cdot \chi) = e^+(\varphi \cdot \chi \vee \psi \cdot \chi)$.

(59): We have that $e^+(\chi \cdot (\varphi \vee \psi)) = e^+(\chi) \star e^+(\varphi \vee \psi) = e^+(\chi) \star (e^+(\varphi) \cup e^+(\psi)) \overset{(161)}{=} (e^+(\chi) \star e^+(\varphi)) \cup (e^+(\chi) \star e^+(\psi)) = e^+(\chi \cdot \varphi) \cup e^+(\chi \cdot \psi) = e^+(\chi \cdot \varphi \vee \chi \cdot \psi)$.
(60): We have that

$$\begin{aligned}
e^+((\exists x.\varphi) \cdot \psi) = e^+(\exists x.\varphi) \star e^+(\psi) = \left(\bigcup_{a \in A} (e_{x \mapsto a})^+(\varphi)\right) \star e^+(\psi) &\overset{(164)}{=} \bigcup_{a \in A} ((e_{x \mapsto a})^+(\varphi) \star e^+(\psi))\\
&= \bigcup_{a \in A} \left((e_{x \mapsto a})^+(\varphi) \star (e_{x \mapsto a})^+(\psi)\right) \quad \text{as } x \notin FV(\psi), \text{ hence } (e_{x \mapsto a})^+(\psi) = e^+(\psi)\\
&= \bigcup_{a \in A} (e_{x \mapsto a})^+(\varphi \cdot \psi) = e^+(\exists x.\varphi \cdot \psi).
\end{aligned}$$

(61): We have that

$$\begin{aligned}
e^+(\psi \cdot (\exists x.\varphi)) = e^+(\psi) \star e^+(\exists x.\varphi) = e^+(\psi) \star \left(\bigcup_{a \in A} (e_{x \mapsto a})^+(\varphi)\right) &\overset{(165)}{=} \bigcup_{a \in A} (e^+(\psi) \star (e_{x \mapsto a})^+(\varphi))\\
&= \bigcup_{a \in A} \left((e_{x \mapsto a})^+(\psi) \star (e_{x \mapsto a})^+(\varphi)\right) \quad \text{as } x \notin FV(\psi), \text{ hence } (e_{x \mapsto a})^+(\psi) = e^+(\psi)\\
&= \bigcup_{a \in A} (e_{x \mapsto a})^+(\psi \cdot \varphi) = e^+(\exists x.\psi \cdot \varphi).
\end{aligned}$$

$\square$

Proposition 3.91. *For all patterns φ, ψ, χ,*

$$\begin{aligned}
&\vDash (\varphi \wedge \psi) \cdot \chi \to \varphi \cdot \chi \wedge \psi \cdot \chi, && (62)\\
&\vDash \chi \cdot (\varphi \wedge \psi) \to \chi \cdot \varphi \wedge \chi \cdot \psi, && (63)\\
&\vDash (\forall x.\varphi) \cdot \psi \to \forall x.\varphi \cdot \psi \quad \textit{if } x \notin FV(\psi), && (64)\\
&\vDash \psi \cdot (\forall x.\varphi) \to \forall x.\psi \cdot \varphi \quad \textit{if } x \notin FV(\psi). && (65)
\end{aligned}$$

Proof. Let $(\mathcal{A}, e)$.

(62): We have that $e^+((\varphi \wedge \psi) \cdot \chi) = e^+(\varphi \wedge \psi) \star e^+(\chi) = (e^+(\varphi) \cap e^+(\psi)) \star e^+(\chi) \overset{(162)}{\subseteq} (e^+(\varphi) \star e^+(\chi)) \cap (e^+(\psi) \star e^+(\chi)) = e^+(\varphi \cdot \chi) \cap e^+(\psi \cdot \chi) = e^+(\varphi \cdot \chi \wedge \psi \cdot \chi)$.

(63): We have that $e^+(\chi \cdot (\varphi \wedge \psi)) = e^+(\chi) \star e^+(\varphi \wedge \psi) = e^+(\chi) \star (e^+(\varphi) \cap e^+(\psi)) \overset{(163)}{\subseteq} (e^+(\chi) \star e^+(\varphi)) \cap (e^+(\chi) \star e^+(\psi)) = e^+(\chi \cdot \varphi) \cap e^+(\chi \cdot \psi) = e^+(\chi \cdot \varphi \wedge \chi \cdot \psi)$.

(64): We have that

$$\begin{aligned} e^{+}((\forall x.\varphi)\cdot\psi) &= e^{+}(\forall x.\varphi)\star e^{+}(\psi) = \left(\bigcap_{a\in A}(e_{x\mapsto a})^{+}(\varphi)\right)\star e^{+}(\psi) \overset{(166)}{\subseteq} \bigcap_{a\in A}((e_{x\mapsto a})^{+}(\varphi)\star e^{+}(\psi)) \\ &= \bigcap_{a\in A}\left((e_{x\mapsto a})^{+}(\varphi)\star(e_{x\mapsto a})^{+}(\psi)\right) \quad \text{as } x\notin FV(\psi), \text{ hence } (e_{x\mapsto a})^{+}(\psi)=e^{+}(\psi) \\ &= \bigcap_{a\in A}(e_{x\mapsto a})^{+}(\varphi\cdot\psi) = e^{+}(\forall x.\varphi\cdot\psi). \end{aligned}$$

(65):

$$\begin{aligned} e^{+}(\psi\cdot(\forall x.\varphi)) &= e^{+}(\psi)\star e^{+}(\forall x.\varphi) = e^{+}(\psi)\star\left(\bigcap_{a\in A}(e_{x\mapsto a})^{+}(\varphi)\right) \overset{(167)}{\subseteq} \bigcap_{a\in A}(e^{+}(\psi)\star(e_{x\mapsto a})^{+}(\varphi)) \\ &= \bigcap_{a\in A}\left((e_{x\mapsto a})^{+}(\psi)\star(e_{x\mapsto a})^{+}(\varphi)\right) \quad \text{as } x\notin FV(\psi), \text{ hence } (e_{x\mapsto a})^{+}(\psi)=e^{+}(\psi) \\ &= \bigcap_{a\in A}(e_{x\mapsto a})^{+}(\psi\cdot\varphi) = e^{+}(\forall x.\psi\cdot\varphi). \end{aligned}$$

□

3.14.1 $\vDash_l$

Proposition 3.92. *For all patterns φ, ψ, χ,*

$$\varphi\to\psi \ \vDash_l \chi\cdot\varphi\to\chi\cdot\psi, \tag{66}$$
$$\varphi\leftrightarrow\psi \ \vDash_l \chi\cdot\varphi\leftrightarrow\chi\cdot\psi, \tag{67}$$
$$\varphi\to\psi \ \vDash_l \varphi\cdot\chi\to\psi\cdot\chi, \tag{68}$$
$$\varphi\leftrightarrow\psi \ \vDash_l \varphi\cdot\chi\leftrightarrow\psi\cdot\chi, \tag{69}$$
$$\varphi\to\psi \ \vDash_l \chi\cdot\neg\psi\to\chi\cdot\neg\varphi, \tag{70}$$
$$\varphi\leftrightarrow\psi \ \vDash_l \chi\cdot\neg\psi\leftrightarrow\chi\cdot\neg\varphi, \tag{71}$$
$$\varphi\to\psi \ \vDash_l \neg\psi\cdot\chi\to\neg\varphi\cdot\chi, \tag{72}$$
$$\varphi\leftrightarrow\psi \ \vDash_l \neg\psi\cdot\chi\leftrightarrow\neg\varphi\cdot\chi. \tag{73}$$

Proof. Let $(\mathcal{A}, e)$.
(66): Assume that $\mathcal{A}\vDash(\varphi\to\psi)[e]$, hence $e^{+}(\varphi)\subseteq e^{+}(\psi)$. It follows that $e^{+}(\chi\cdot\varphi) = e^{+}(\chi)\star e^{+}(\varphi) \overset{(168)}{\subseteq} e^{+}(\chi)\star e^{+}(\psi) = e^{+}(\chi\cdot\psi)$, hence $\mathcal{A}\vDash(\chi\cdot\varphi\to\chi\cdot\psi)[e]$.
(67): Assume that $\mathcal{A}\vDash(\varphi\leftrightarrow\psi)[e]$. Then, by Proposition 3.7.(xi), $\mathcal{A}\vDash(\varphi\to\psi)[e]$ and $\mathcal{A}\vDash(\psi\to\varphi)[e]$. Apply now (66) to get that $\mathcal{A}\vDash(\chi\cdot\varphi\to\chi\cdot\psi)[e]$ and $\mathcal{A}\vDash(\chi\cdot\psi\to\chi\cdot\varphi)[e]$, hence, by applying again Proposition 3.7.(xi), $\mathcal{A}\vDash(\chi\cdot\varphi\leftrightarrow\chi\cdot\psi)[e]$
(68): Assume that $\mathcal{A}\vDash(\varphi\to\psi)[e]$, hence $e^{+}(\varphi)\subseteq e^{+}(\psi)$. It follows that $e^{+}(\varphi\cdot\chi) = e^{+}(\varphi)\star e^{+}(\chi) \overset{(168)}{\subseteq} e^{+}(\psi)\star e^{+}(\chi) = e^{+}(\psi\cdot\chi)$, hence $\mathcal{A}\vDash(\varphi\cdot\chi\to\psi\cdot\chi)[e]$.
(69): Assume that $\mathcal{A}\vDash(\varphi\leftrightarrow\psi)[e]$. Then, by Proposition 3.7.(xi), $\mathcal{A}\vDash(\varphi\to\psi)[e]$ and $\mathcal{A}\vDash(\psi\to\varphi)[e]$. Apply now (68) to get that $\mathcal{A}\vDash(\varphi\cdot\chi\to\psi\cdot\chi)[e]$ and $\mathcal{A}\vDash(\psi\cdot\chi\to\varphi\cdot\chi)[e]$, hence, by applying again Proposition 3.7.(xi), $\mathcal{A}\vDash(\varphi\cdot\chi\leftrightarrow\psi\cdot\chi)[e]$.
(70): Assume that $\mathcal{A}\vDash(\varphi\to\psi)[e]$, hence $e^{+}(\varphi)\subseteq e^{+}(\psi)$. It follows that $e^{+}(\chi\cdot\neg\psi) = e^{+}(\chi)\star e^{+}(\neg\psi) = e^{+}(\chi)\star C_A e^{+}(\psi) \overset{(170)}{\subseteq} e^{+}(\chi)\star C_A e^{+}(\varphi) = e^{+}(\chi\cdot\neg\varphi)$, hence $\mathcal{A}\vDash(\chi\cdot\neg\psi\to\chi\cdot\neg\varphi)[e]$.
(71): Assume that $\mathcal{A}\vDash(\varphi\leftrightarrow\psi)[e]$. Then, by Proposition 3.7.(xi), $\mathcal{A}\vDash(\varphi\to\psi)[e]$ and $\mathcal{A}\vDash(\psi\to\varphi)[e]$. Apply now (70) to get $\mathcal{A}\vDash(\chi\cdot\neg\psi\to\chi\cdot\neg\varphi)[e]$ and $\mathcal{A}\vDash(\chi\cdot\neg\varphi\to\chi\cdot\neg\psi)[e]$, hence, by applying again Proposition 3.7.(xi), $\mathcal{A}\vDash(\chi\cdot\neg\psi\leftrightarrow\chi\cdot\neg\varphi)[e]$.

(72): Assume that $\mathcal{A} \vDash (\varphi \to \psi)[e]$, hence $e^+(\varphi) \subseteq e^+(\psi)$. It follows that $e^+(\neg\psi \cdot \chi) = e^+(\neg\psi) \star e^+(\chi) = C_A e^+(\psi) \star e^+(\chi) \overset{(170)}{\subseteq} C_A e^+(\varphi) \star e^+(\chi) = e^+(\neg\varphi) \star e^+(\chi) = e^+(\neg\varphi \cdot \chi)$, hence $\mathcal{A} \vDash (\neg\psi \cdot \chi \to \neg\varphi \cdot \chi)[e]$.
(73): Assume that $\mathcal{A} \vDash (\varphi \leftrightarrow \psi)[e]$. Then, by Proposition 3.7.(xi), $\mathcal{A} \vDash (\varphi \to \psi)[e]$ and $\mathcal{A} \vDash (\psi \to \varphi)[e]$. Apply now (72) to get that $\mathcal{A} \vDash (\neg\psi \cdot \chi \to \neg\varphi \cdot \chi)[e]$ and $\mathcal{A} \vDash (\neg\varphi \cdot \chi \to \neg\psi \cdot \chi)[e]$, hence, by applying again Proposition 3.7.(xi), $\mathcal{A} \vDash (\neg\psi \cdot \chi \leftrightarrow \neg\varphi \cdot \chi)[e]$. □

Proposition 3.93. *Let Γ be a set of patterns and φ, ψ, χ be patterns. Then*

$$\Gamma \vDash_l \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_l \varphi \cdot \chi \to \psi \cdot \chi, \tag{74}$$
$$\Gamma \vDash_l \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_l \varphi \cdot \chi \leftrightarrow \psi \cdot \chi, \tag{75}$$
$$\Gamma \vDash_l \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_l \chi \cdot \varphi \to \chi \cdot \psi, \tag{76}$$
$$\Gamma \vDash_l \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_l \chi \cdot \varphi \leftrightarrow \chi \cdot \psi, \tag{77}$$
$$\Gamma \vDash_l \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_l \chi \cdot \neg\psi \to \chi \cdot \neg\varphi, \tag{78}$$
$$\Gamma \vDash_l \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_l \chi \cdot \neg\psi \leftrightarrow \chi \cdot \neg\varphi, \tag{79}$$
$$\Gamma \vDash_l \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_l \neg\psi \cdot \chi \to \neg\varphi \cdot \chi, \tag{80}$$
$$\Gamma \vDash_l \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_l \neg\psi \cdot \chi \leftrightarrow \neg\varphi \cdot \chi. \tag{81}$$

Proof. Apply Proposition 3.92 and Lemma 3.40.(i). □

Proposition 3.94. *Let χ be a predicate pattern. Then for all patterns φ, ψ,*

$$\sigma \to (\varphi \to \psi) \ \vDash_l \sigma \to (\chi \cdot \varphi \to \chi \cdot \psi), \tag{82}$$
$$\sigma \to (\varphi \to \psi) \ \vDash_l \sigma \to (\varphi \cdot \chi \to \psi \cdot \chi). \tag{83}$$

Proof. Apply (66), (68) and Proposition 3.60.(i).

□

Proposition 3.95. *Let χ be a predicate pattern and $\Gamma \cup \{\varphi, \psi, \chi\}$ be a set of patterns Then*

$$\Gamma \vDash_l \sigma \to (\varphi \to \psi) \quad \textit{implies} \quad \Gamma \vDash_l \sigma \to (\chi \cdot \varphi \to \chi \cdot \psi), \tag{84}$$
$$\Gamma \vDash_l \sigma \to (\varphi \to \psi) \quad \textit{implies} \quad \Gamma \vDash_l \sigma \to (\varphi \cdot \chi \to \psi \cdot \chi). \tag{85}$$

Proof. Apply (66), (68) and Proposition 3.60.(iii). □

3.14.2 $\vDash_g$

Proposition 3.96. *For all patterns φ, ψ, χ,*

$$\varphi \to \psi \ \vDash_g \chi \cdot \varphi \to \chi \cdot \psi, \tag{86}$$
$$\varphi \leftrightarrow \psi \ \vDash_g \chi \cdot \varphi \leftrightarrow \chi \cdot \psi, \tag{87}$$
$$\varphi \to \psi \ \vDash_g \varphi \cdot \chi \to \psi \cdot \chi, \tag{88}$$
$$\varphi \leftrightarrow \psi \ \vDash_g \varphi \cdot \chi \leftrightarrow \psi \cdot \chi, \tag{89}$$
$$\varphi \to \psi \ \vDash_g \chi \cdot \neg\psi \to \chi \cdot \neg\varphi, \tag{90}$$
$$\varphi \leftrightarrow \psi \ \vDash_g \chi \cdot \neg\psi \leftrightarrow \chi \cdot \neg\varphi, \tag{91}$$
$$\varphi \to \psi \ \vDash_g \neg\psi \cdot \chi \to \neg\varphi \cdot \chi, \tag{92}$$
$$\varphi \leftrightarrow \psi \ \vDash_g \neg\psi \cdot \chi \leftrightarrow \neg\varphi \cdot \chi. \tag{93}$$

Proof. Apply Proposition 3.92 and Proposition 3.26.(i). □

Proposition 3.97. *Let Γ be a set of patterns and φ, ψ, χ be patterns. Then*

$$\Gamma \vDash_g \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_g \varphi \cdot \chi \to \psi \cdot \chi, \tag{94}$$
$$\Gamma \vDash_g \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_g \varphi \cdot \chi \leftrightarrow \psi \cdot \chi, \tag{95}$$
$$\Gamma \vDash_g \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_g \chi \cdot \varphi \to \chi \cdot \psi, \tag{96}$$
$$\Gamma \vDash_g \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_g \chi \cdot \varphi \leftrightarrow \chi \cdot \psi, \tag{97}$$
$$\Gamma \vDash_g \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_g \chi \cdot \neg\psi \to \chi \cdot \neg\varphi, \tag{98}$$
$$\Gamma \vDash_g \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_g \chi \cdot \neg\psi \leftrightarrow \chi \cdot \neg\varphi, \tag{99}$$
$$\Gamma \vDash_g \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_g \neg\psi \cdot \chi \to \neg\varphi \cdot \chi, \tag{100}$$
$$\Gamma \vDash_g \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_g \neg\psi \cdot \chi \leftrightarrow \neg\varphi \cdot \chi. \tag{101}$$

Proof. Apply Proposition 3.96 and Lemma 3.33.(i). □

Proposition 3.98. *Let σ be a predicate pattern. Then for all patterns φ, ψ, χ,*

$$\sigma \to (\varphi \to \psi) \ \vDash_g \sigma \to (\chi \cdot \varphi \to \chi \cdot \psi), \tag{102}$$
$$\sigma \to (\varphi \to \psi) \ \vDash_g \sigma \to (\varphi \cdot \chi \to \psi \cdot \chi). \tag{103}$$

Proof. Apply (66), (68) and Proposition 3.60.(ii).

□

Proposition 3.99. *Let σ be a predicate pattern and $\Gamma \cup \{\varphi, \psi, \chi\}$ be a set of patterns Then*

$$\Gamma \vDash_g \sigma \to (\varphi \to \psi) \quad \textit{implies} \quad \Gamma \vDash_g \sigma \to (\chi \cdot \varphi \to \chi \cdot \psi), \tag{104}$$
$$\Gamma \vDash_g \sigma \to (\varphi \to \psi) \quad \textit{implies} \quad \Gamma \vDash_g \sigma \to (\varphi \cdot \chi \to \psi \cdot \chi). \tag{105}$$

Proof. Apply (66), (68) and Proposition 3.60.(iv). □

3.15 Contexts

Proposition 3.100. *Let C be a context, φ be a pattern, and $(\mathcal{A}, e)$.*

$$e^+(C[\varphi]) = \begin{cases} e^+(\varphi) & \textit{if } C = \square, \\ e^+(C_1[\varphi]) \star e^+(\psi) & \textit{if } C = \mathit{Appl}_\square C_1 \psi, \\ e^+(\psi) \star e^+(C_1[\varphi]) & \textit{if } \mathit{Appl}_\square \psi C_1. \end{cases} \tag{106}$$

Proof. Apply Proposition 2.58. We have the following cases:

(i) $C = \square$. Then $C[\varphi] = \varphi$, so $e^+(C[\varphi]) = e^+(\varphi)$.

(ii) $C = \mathit{Appl}_\square C_1 \psi$. Then $C[\varphi] = C_1[\varphi] \cdot \psi$, hence $e^+(C[\varphi]) = e^+(C_1[\varphi]) \star e^+(\psi)$.

(iii) $C = \mathit{Appl}_\square \psi C_1$. Then $C[\varphi] = \psi \cdot C_1[\varphi]$, hence $e^+(C[\varphi]) = e^+(\psi) \star e^+(C_1[\varphi])$.

□

Proposition 3.101. *Let C be a context and φ, ψ be patterns.*

$$\vDash C[\bot] \leftrightarrow \bot, \tag{107}$$
$$\vDash C[\varphi \vee \psi] \leftrightarrow C[\varphi] \vee C[\psi], \tag{108}$$
$$\vDash C[\exists x.\varphi] \leftrightarrow \exists x.C[\varphi] \qquad \textit{if } x \notin FV(C), \tag{109}$$
$$\vDash C[\varphi \wedge \psi] \to C[\varphi] \wedge C[\psi], \tag{110}$$
$$\vDash C[\forall x.\varphi] \to \forall x.C[\varphi] \qquad \textit{if } x \notin FV(C). \tag{111}$$

Proof. (107): The proof is by induction on contexts:

(i) $C = \Box$. Then $C[\bot] = \bot$. Apply (29).

(ii) $C = \mathit{Appl}_\Box C_1 \psi$ and, by the induction hypothesis for C_1, $\vDash C_1[\bot] \leftrightarrow \bot$. Since $C[\bot] = C_1[\bot] \cdot \psi$, we can apply (89) to get that $C_1[\bot] \leftrightarrow \bot \vDash_g C[\bot] \leftrightarrow \bot \cdot \psi$ and Remark 3.18.(i) to get that $\vDash C[\bot] \leftrightarrow \bot \cdot \psi$. Hence, by (57) and (31), it follows that $\vDash C[\bot] \leftrightarrow \bot$.

(iii) $C = \mathit{Appl}_\Box \psi C_1$ and, by the induction hypothesis for C_1, $\vDash C_1[\bot] \leftrightarrow \bot$. Since $C[\bot] = \psi \cdot C_1[\bot]$, we can apply (87) to get that $C_1[\bot] \leftrightarrow \bot \vDash_g C[\bot] \leftrightarrow \psi \cdot \bot$ and Remark 3.18.(i) to get that $\vDash C[\bot] \leftrightarrow \psi \cdot \bot$. Hence, by (56) and (31), it follows that $\vDash C[\bot] \leftrightarrow \bot$.

(108): The proof is by induction on contexts:

(i) $C = \Box$. Then $C[\varphi \vee \psi] = \varphi \vee \psi = C[\varphi] \vee C[\psi]$. Apply (29).

(ii) $C = \mathit{Appl}_\Box C_1 \chi$ and, by the induction hypothesis for C_1, $\vDash C_1[\varphi \vee \psi] \leftrightarrow C_1[\varphi] \vee C_1[\psi]$. Since $C[\varphi \vee \psi] = C_1[\varphi \vee \psi] \cdot \chi$, we can apply (89) to get that $C_1[\varphi \vee \psi] \leftrightarrow C_1[\varphi] \vee C_1[\psi] \vDash_g C[\varphi \vee \psi] \leftrightarrow (C_1[\varphi] \vee C_1[\psi]) \cdot \chi$ and Remark 3.18.(i) to get that $\vDash C[\varphi \vee \psi] \leftrightarrow (C_1[\varphi] \vee C_1[\psi]) \cdot \chi$. Hence, by (58) and (31), it follows that $\vDash C[\varphi \vee \psi] \leftrightarrow C_1[\varphi] \cdot \chi \vee C_2[\varphi] \cdot \chi = C[\varphi] \vee C[\psi]$.

(iii) $C = \mathit{Appl}_\Box \chi C_1$ and, by the induction hypothesis for C_1, $\vDash C_1[\varphi \vee \psi] \leftrightarrow C_1[\varphi] \vee C_1[\psi]$. Since $C[\varphi \vee \psi] = \chi \cdot C_1[\varphi \vee \psi]$, we can apply (87) to get that $C_1[\varphi \vee \psi] \leftrightarrow C_1[\varphi] \vee C_1[\psi] \vDash_g C[\varphi \vee \psi] \leftrightarrow \chi \cdot (C_1[\varphi] \vee C_1[\psi])$ and Remark 3.18.(i) to get that $\vDash C[\varphi \vee \psi] \leftrightarrow \chi \cdot (C_1[\varphi] \vee C_1[\psi])$. Hence, by (59) and (31), it follows that $\vDash C[\varphi \vee \psi] \leftrightarrow \chi \cdot C_1[\varphi] \vee \chi \cdot C_2[\varphi] = C[\varphi] \vee C[\psi]$.

(109): The proof is by induction on contexts:

(i) $C = \Box$. Then $C[\exists x.\varphi] = \exists x.\varphi = \exists x.C[\varphi]$. Apply (29).

(ii) $C = \mathit{Appl}_\Box C_1 \psi$. As $x \notin FV(C)$ and, by Definition 2.63, $FV(C) = FV(C_1) \cup FV(\varphi)$, we have that $x \notin FV(C_1)$, so we can apply the induction hypothesis for C_1 to get that $\vDash C_1[\exists x.\varphi] \leftrightarrow \exists x.C_1[\varphi]$. Since $C[\exists x.\varphi] = C_1[\exists x.\varphi] \cdot \psi$, we can apply (89) to get that $C_1[\exists x.\varphi] \leftrightarrow \exists x.C_1[\varphi] \vDash_g C[\exists x.\varphi] \leftrightarrow (\exists x.C_1[\varphi]) \cdot \psi$ and Remark 3.18.(i) to get that $\vDash C[\exists x.\varphi] \leftrightarrow (\exists x.C_1[\varphi]) \cdot \psi$. Hence, by (60) and (31), it follows that $\vDash C[\exists x.\varphi] \leftrightarrow \exists x.C_1[\varphi] \cdot \psi = \exists x.C[\varphi]$.

(iii) $C = \mathit{Appl}_\Box \psi C_1$. As $x \notin FV(C)$ and, by Definition 2.63, $FV(C) = FV(C_1) \cup FV(\varphi)$, we have that $x \notin FV(C_1)$, so we can apply the induction hypothesis for C_1 to get that $\vDash C_1[\exists x.\varphi] \leftrightarrow \exists x.C_1[\varphi]$. Since $C[\exists x.\varphi] = \psi \cdot C_1[\exists x.\varphi]$, we can apply (87) to get that $C_1[\exists x.\varphi] \leftrightarrow \exists x.C_1[\varphi] \vDash_g C[\exists x.\varphi] \leftrightarrow \psi \cdot (\exists x.C_1[\varphi])$ and Remark 3.18.(i) to get that $\vDash C[\exists x.\varphi] \leftrightarrow \psi \cdot (\exists x.C_1[\varphi])$. Hence, by (61) and (31), it follows that $\vDash C[\exists x.\varphi] \leftrightarrow \exists x.\psi \cdot C_1[\varphi] = \exists x.C[\varphi]$.

(110): The proof is by induction on contexts:

(i) $C = \Box$. Then $C[\varphi \wedge \psi] = \varphi \wedge \psi = C[\varphi] \wedge C[\psi]$. Apply (29) and Remark 3.13.(iii).

(ii) $C = \mathit{Appl}_\Box C_1 \chi$ and, by the induction hypothesis for C_1, $\vDash C_1[\varphi \wedge \psi] \to C_1[\varphi] \wedge C_1[\psi]$. Since $C[\varphi \wedge \psi] = C_1[\varphi \wedge \psi] \cdot \chi$, we can apply (88) to get that $C_1[\varphi \wedge \psi] \to C_1[\varphi] \wedge C_1[\psi] \vDash_g C[\varphi \wedge \psi] \to (C_1[\varphi] \wedge C_1[\psi]) \cdot \chi$ and Remark 3.18.(i) to get that $\vDash C[\varphi \wedge \psi] \to (C_1[\varphi] \wedge C_1[\psi]) \cdot \chi$. Hence, by (62) and (32), it follows that $\vDash C[\varphi \wedge \psi] \to C_1[\varphi] \cdot \chi \wedge C_2[\psi] \cdot \chi = C[\varphi] \wedge C[\psi]$.

(iii) $C = \mathit{Appl}_\Box \chi C_1$ and, by the induction hypothesis for C_1, $\vDash C_1[\varphi \wedge \psi] \to C_1[\varphi] \wedge C_1[\psi]$. Since $C[\varphi \wedge \psi] = \chi \cdot C_1[\varphi \wedge \psi]$, we can apply (86) to get that $C_1[\varphi \wedge \psi] \to C_1[\varphi] \wedge C_1[\psi] \vDash_g C[\varphi \wedge \psi] \to \chi \cdot (C_1[\varphi] \wedge C_1[\psi])$ and Remark 3.18.(i) to get that $\vDash C[\varphi \wedge \psi] \to \chi \cdot (C_1[\varphi] \wedge C_1[\psi])$. Hence, by (63) and (32), it follows that $\vDash C[\varphi \wedge \psi] \to \chi \cdot C_1[\varphi] \wedge \chi \cdot C_2[\psi] = C[\varphi] \wedge C[\psi]$.

(111): The proof is by induction on contexts:

(i) $C = \Box$. Then $C[\forall x.\varphi] = \forall x.\varphi = \forall x.C[\varphi]$. Apply (29) and Remark 3.13.(iii).

(ii) $C = Appl_{\Box} C_1 \psi$. As $x \notin FV(C)$ and, by Definition 2.63, $FV(C) = FV(C_1) \cup FV(\varphi)$, we have that $x \notin FV(C_1)$, so we can apply the induction hypothesis for C_1 to get that $\vDash C_1[\forall x.\varphi] \to \forall x.C_1[\varphi]$. Since $C[\forall x.\varphi] = C_1[\forall x.\varphi] \cdot \psi$, we can apply (88) to get that $C_1[\forall x.\varphi] \to \forall x.C_1[\varphi] \vDash_g C[\forall x.\varphi] \to (\forall x.C_1[\varphi]) \cdot \psi$ and Remark 3.18.(i) to get that $\vDash C[\forall x.\varphi] \to (\forall x.C_1[\varphi]) \cdot \psi$. Hence, by (64) and (32), it follows that $\vDash C[\forall x.\varphi] \to \forall x.C_1[\varphi] \cdot \psi = \forall x.C[\varphi]$.

(iii) $C = Appl_{\Box} \psi C_1$. As $x \notin FV(C)$ and, by Definition 2.63, $FV(C) = FV(C_1) \cup FV(\varphi)$, we have that $x \notin FV(C_1)$, so we can apply the induction hypothesis for C_1 to get that $\vDash C_1[\forall x.\varphi] \to \forall x.C_1[\varphi]$. Since $C[\forall x.\varphi] = \psi \cdot C_1[\forall x.\varphi]$, we can apply (86) to get that $C_1[\forall x.\varphi] \to \forall x.C_1[\varphi] \vDash_g C[\forall x.\varphi] \to \psi \cdot (\forall x.C_1[\varphi])$ and Remark 3.18.(i) to get that $\vDash C[\forall x.\varphi] \to \psi \cdot (\forall x.C_1[\varphi])$. Hence, by (65) and (32), it follows that $\vDash C[\forall x.\varphi] \to \forall x.\psi \cdot C_1[\varphi] = \forall x.C[\varphi]$.

$\Box$

3.15.1 $\vDash_l$

Proposition 3.102. *Let C be a context. Then for every patterns, φ, ψ,*

$$\varphi \to \psi \ \vDash_l C[\varphi] \to C[\psi], \tag{112}$$
$$\varphi \leftrightarrow \psi \ \vDash_l C[\varphi] \leftrightarrow C[\psi]. \tag{113}$$

Proof. (112): Let $(\mathcal{A}, e)$ be such that $\mathcal{A} \vDash (\varphi \to \psi)[e]$. We prove that $\mathcal{A} \vDash (C[\varphi] \to C[\psi])[e]$ by induction on C.

(i) $C = \Box$. Then $C[\varphi] \to C[\psi] = \varphi \to \psi$ and the conclusion is obvious.

(ii) $C = Appl_{\Box} C_1 \chi$. By the induction hypothesis, we have that $\mathcal{A} \vDash (C_1[\varphi] \to C_1[\psi])[e]$. Apply now (68) to get that $\mathcal{A} \vDash (C_1[\varphi] \cdot \chi \to C_1[\psi] \cdot \chi)[e]$, that is $\mathcal{A} \vDash (C[\varphi] \to C[\psi])[e]$.

(iii) $C = Appl_{\Box} \chi C_1$. By the induction hypothesis, we have that $\mathcal{A} \vDash (C_1[\varphi] \to C_1[\psi])[e]$. Apply now (66) to get that $\mathcal{A} \vDash (\chi \cdot C_1[\varphi] \to \chi \cdot C_1[\psi])[e]$, that is $\mathcal{A} \vDash (C[\varphi] \to C[\psi])[e]$.

(113): Apply (112) and Proposition 3.7.(xi). $\Box$

Proposition 3.103. *Let C be a context, Γ be a set of patterns and φ, ψ be patterns. Then*

$$\Gamma \vDash_l \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_l C[\varphi] \to C[\psi], \tag{114}$$
$$\Gamma \vDash_l \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_l C[\varphi] \leftrightarrow C[\psi]. \tag{115}$$

Proof. Apply Proposition 3.102 and Lemma 3.40.(i). $\Box$

3.15.2 $\vDash_g$

Proposition 3.104. *Let C be a context. Then for every patterns, φ, ψ,*

$$\varphi \to \psi \ \vDash_g C[\varphi] \to C[\psi], \tag{116}$$
$$\varphi \leftrightarrow \psi \ \vDash_g C[\varphi] \leftrightarrow C[\psi]. \tag{117}$$

Proof. Apply Proposition 3.102 and Proposition 3.26.(i). $\Box$

Proposition 3.105. *Let C be a context, Γ be a set of patterns and φ, ψ be patterns. Then*

$$\Gamma \vDash_g \varphi \to \psi \quad \textit{implies} \quad \Gamma \vDash_g C[\varphi] \to C[\psi], \tag{118}$$
$$\Gamma \vDash_g \varphi \leftrightarrow \psi \quad \textit{implies} \quad \Gamma \vDash_g C[\varphi] \leftrightarrow C[\psi]. \tag{119}$$

Proof. Apply Proposition 3.104 and Lemma 3.33.(i). $\Box$

3.15.3 (Singleton Variable)

Lemma 3.106. *Let C be a context and φ be a pattern. Then for every $(\mathcal{A}, e)$,*

$$e^+(\varphi) = \emptyset \quad \textit{implies} \quad e^+(C[\varphi]) = \emptyset.$$

Proof. Since $e^+(\varphi) = \emptyset = e^+(\bot)$, we have that $\mathcal{A} \vDash (\varphi \leftrightarrow \bot)[e]$. Apply (113) to get that $\mathcal{A} \vDash (C[\varphi] \leftrightarrow C[\bot])[e]$, so $e^+(C[\varphi]) = e^+(C[\bot])$. As, by (107), $\vDash C[\bot] \leftrightarrow \bot$, it follows that $e^+(C[\bot]) = e^+(\bot) = \emptyset$. Thus, we have got that $e^+(C[\varphi]) = \emptyset$. □

Proposition 3.107. *For every contexts C_1, C_2, pattern φ and element variable x,*

$$\vDash \neg(C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg\varphi]). \tag{120}$$

Proof. Denote $\theta = C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg\varphi]$. Let $(\mathcal{A}, e)$. We have to prove that $\mathcal{A} \vDash (\neg\theta)[e]$, that is $e^+(\neg\theta) = A$, hence that $e^+(\theta) = \emptyset$. We have two cases:

(i) $e(x) \in e^+(\varphi)$. Then $e^+(x \wedge \neg\varphi) = e^+(x) \cap e^+(\neg\varphi) = \{e(x)\} \cap C_A e^+(\varphi) = \emptyset$. Apply Lemma 3.106 to get that $e^+(C_2[x \wedge \neg\varphi]) = \emptyset$. It follows that $e^+(\theta) = e^+(C_1[x \wedge \varphi]) \cap e^+(C_2[x \wedge \neg\varphi]) = e^+(C_1[x \wedge \varphi]) \cap \emptyset = \emptyset$.

(ii) $e(x) \notin e^+(\varphi)$. Then $e^+(x \wedge \varphi) = \{e(x)\} \cap e^+(\varphi) = e^+(x) \cap e^+(\varphi) = \emptyset$. Apply Lemma 3.106 to get that $e^+(C_1[x \wedge \varphi]) = \emptyset$. It follows that $e^+(\theta) = e^+(C_1[x \wedge \varphi]) \cap e^+(C_2[x \wedge \neg\varphi]) = \emptyset \cap e^+(C_2[x \wedge \neg\varphi]) = \emptyset$.

□

3.16 Set variables

Let Γ be a set of patterns.

3.16.1 $\vDash_g$

Proposition 3.108. *Let φ, ψ be patterns and X be a set variable such that X is free for ψ in φ. Then*

$$\varphi \vDash_g Subf_{\psi}^{X} \varphi.$$

Proof. Let $\mathcal{A}$ be a model of φ and e be an arbitrary $\mathcal{A}$-evaluation. We have to prove that $\mathcal{A} \vDash Subf_{\psi}^{X} \varphi[e]$. By Proposition 3.76, we have that $e^+(Subf_{\psi}^{X} \varphi) = \left(e_{X \mapsto e^+(\psi)}\right)^+(\varphi)$. Since $\mathcal{A} \vDash \varphi$, we have in particular that $\mathcal{A} \vDash \varphi[e_{X \mapsto e^+(\psi)}]$. Thus, $\mathcal{A} \vDash Subf_{\psi}^{X} \varphi[e]$. □

Proposition 3.109. *Let $\Gamma \cup \{\varphi, \psi\}$ be a set of patterns and X be a set variable such that X is free for ψ in φ. Then*

$$\Gamma \vDash_g \varphi \quad \textit{implies} \quad \Gamma \vDash_g Subf_{\psi}^{X} \varphi.$$

Proof. Apply Proposition 3.108 and Lemma 3.33.(i). □

Proposition 3.110. *For all patterns φ, ψ and any set variable X,*

$$\varphi \vDash_g \varphi_X(\psi).$$

Proof. We have the following two cases:

(i) X is free for ψ in φ. Then $\varphi_X(\psi) = Subf_{\psi}^{X} \varphi$, so we apply Proposition 3.108 to get the conclusion.

(ii) X is not free for ψ in φ. Then $\varphi_X(\psi) = Subf_\psi^X \theta$ with

$$\theta = Subb_{Z_1}^{U_1} Subb_{Z_2}^{U_2} \dots Subb_{Z_p}^{U_p} Subb_{z_1}^{u_1} Subb_{z_2}^{u_2} \dots Subb_{z_k}^{u_k} \varphi,$$

where

(a) $u_1, \dots, u_k$ are the element variables and $U_1, \dots, U_p$ are the set variables that occur bound in φ and also occur in ψ;

(b) $z_1, \dots, z_k$ are new element variables and $Z_1, \dots, Z_p$ are new set variables, that do not occur in φ or ψ.

Applying repeatedly Propositions 3.74, 3.78, we get that

$$\models \theta \leftrightarrow \varphi. \tag{121}$$

Let $\mathcal{A}$ be a model of φ. By (121), we get that $\mathcal{A} \models \theta$. Apply Proposition 3.108 (with $\varphi = \theta$) to get that $\mathcal{A} \models Subf_\psi^X \theta = \varphi_X(\psi)$.

□

Proposition 3.111. *For all patterns φ, ψ,*

$$\Gamma \models_g \varphi \quad \textit{implies} \quad \Gamma \models_g \varphi_X(\psi).$$

Proof. Apply Proposition 3.110 and Lemma 3.33.(i). □

3.17 μ

Proposition 3.112. *Let φ be a pattern and X be a set variable such that φ is positive in X and X is free for $\mu X.\varphi$ in φ. Then*

$$\models Subf_{\mu X.\varphi}^X \varphi \leftrightarrow \mu X.\varphi. \tag{122}$$

Proof. Let $(\mathcal{A}, e)$. We have that

$$\begin{aligned} e^+\left(Subf_{\mu X.\varphi}^X \varphi\right) &= \left(e_{X \mapsto e^+(\mu X.\varphi)}\right)^+(\varphi) \quad \text{by Proposition 3.76} \\ &= e^+(\mu X.\varphi) \quad \text{by Proposition 3.81.(i)} \end{aligned}$$

□

3.17.1 $\models_l$

Proposition 3.113. *Let φ, ψ be patterns and X be a set variable such that X is free for ψ in φ. Then*

$$Subf_\psi^X \varphi \to \psi \models_l \mu X.\varphi \to \psi.$$

Proof. Let $(\mathcal{A}, e)$ be such that $\mathcal{A} \models (Subf_\psi^X \varphi \to \psi)[e]$, so $e^+\left(Subf_\psi^X \varphi\right) \subseteq e^+(\psi)$. By Propositions 3.76, we have that $e^+\left(Subf_\psi^X \varphi\right) = \left(e_{X \mapsto e^+(\psi)}\right)^+(\varphi)$. We get that $\left(e_{X \mapsto e^+(\psi)}\right)^+(\varphi) \subseteq e^+(\psi)$, hence $e^+(\psi) \in \{B \subseteq A \mid (e_{X \mapsto B})^+(\varphi) \subseteq B\}$. It follows that

$$e^+(\mu X.\varphi) = \bigcap \{B \subseteq A \mid (e_{X \mapsto B})^+(\varphi) \subseteq B\} \subseteq e^+(\psi),$$

so $\mathcal{A} \models (\mu X.\varphi \to \psi)[e]$. □

Proposition 3.114. *Let $\Gamma \cup \{\varphi, \psi\}$ be a set of patterns and X be a set variable such that X is free for ψ in φ. Then*

$$\Gamma \models_l Subf_\psi^X \varphi \to \psi \quad \textit{implies} \quad \Gamma \models_l \mu X.\varphi \to \psi.$$

Proof. Apply Proposition 3.113 and Lemma 3.40.(i). □

3.17.2 $\models_g$

Proposition 3.115. *Let φ, ψ be patterns and X be a set variable such that X is free for ψ in φ. Then*

$$Subf^X_\psi \varphi \to \psi \models_g \mu X.\varphi \to \psi.$$

Proof. Apply Proposition 3.113 and Proposition 3.26.(i). □

Proposition 3.116. *Let $\Gamma \cup \{\varphi, \psi\}$ be a set of patterns and X be a set variable such that X is free for ψ in φ. Then*

$$\Gamma \models_g Subf^X_\psi \varphi \to \psi \quad \text{implies} \quad \Gamma \models_g \mu X.\varphi \to \psi.$$

Proof. Apply Proposition 3.115 and Lemma 3.33.(i). □

4 Proof system

Let $\tau = (EVar, SVar, \Sigma)$ be a signature.
We present in the sequel a Hilbert type proof system for AML, that we denote $\mathcal{P}$, obtained by combining the ones from [4] and [1]. $\mathcal{P}$ is also given in Section B.

Definition 4.1. *The **axioms** of $\mathcal{P}$ are the following patterns*

(Tautology)	φ *if φ is a tautology*
($\exists$-Quantifier)	$Subf^x_y \varphi \to \exists x.\varphi$ *if x is free for y in φ*
(Propagation$_\bot$)	$\varphi \cdot \bot \to \bot \qquad \bot \cdot \varphi \to \bot$
(Propagation$_\vee$)	$(\varphi \vee \psi) \cdot \chi \to \varphi \cdot \chi \vee \psi \cdot \chi \qquad \chi \cdot (\varphi \vee \psi) \to \chi \cdot \varphi \vee \chi \cdot \psi$
(Propagation$_\exists$)	$(\exists x.\varphi) \cdot \psi \to \exists x.\varphi \cdot \psi, \qquad \psi \cdot (\exists x.\varphi) \to \exists x.\psi \cdot \varphi$ *if $x \notin FV(\psi)$*
(Pre-Fixpoint)	$Subf^X_{\mu X.\varphi} \varphi \to \mu X.\varphi$ *if φ is positive in X and X is free for $\mu X.\varphi$ in φ*
(existence)	$\exists x.x,$
(Singleton Variable)	$\neg(C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg\varphi]).$

Definition 4.2. *The **deduction rules** (or **inference rules**) of $\mathcal{P}$ are the following:*

(modus ponens)	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}$
($\exists$-quantifier rule)	$\dfrac{\varphi \to \psi}{\exists x.\varphi \to \psi}$ *if $x \notin FV(\psi)$*
(framing)	$\dfrac{\varphi \to \psi}{\varphi \cdot \chi \to \psi \cdot \chi} \qquad \dfrac{\varphi \to \psi}{\chi \cdot \varphi \to \chi \cdot \psi}$
(Set Variable Substitution)	$\dfrac{\varphi}{Subf^X_\psi \varphi}$ *if X is free for ψ in φ*
(Knaster-Tarski)	$\dfrac{Subf^X_\psi \varphi \to \psi}{\mu X.\varphi \to \psi}$ *if X is free for ψ in φ*

Thus, we have two types of inference rules:

(i) $\dfrac{\varphi}{\chi}$. This inference rule is read as follows: from φ infer χ. φ is said to be the **premise** of the rule and χ is the **conclusion** of the rule.

(ii) $\dfrac{\varphi \quad \psi}{\chi}$. This inference rule is read as follows: from φ and ψ infer χ. φ and ψ are said to be the **premise** of the rule and χ is the **conclusion** of the rule.

Let Γ be a set of patterns.

Definition 4.3. *The* Γ***-theorems*** *are the patterns inductively defined as follows:*

(i) Every axiom is a Γ*-theorem.*

(ii) Every pattern of Γ *is a* Γ*-theorem.*

(iii) For every inference rule, the following holds: If its premises are Γ*-theorems, then its conclusion is a* Γ*-theorem.*

(iv) Only the patterns obtained by applying the above rules are Γ*-theorems.*

Definition 4.4. *[Alternative definition for* Γ*-theorems]*
The set of Γ***-theorems*** *is the intersection of all sets* Δ *of patterns that have the following properties:*

(i) Δ *contains all the axioms.*

(ii) Δ *contains all the patterns from* Γ*, that is* $\Gamma \subseteq \Delta$*.*

(iii) For every inference rule, the following holds: If Δ *contains its premise(s), then its conclusion is also in* Δ*.*

The set of Γ-theorems is denoted by $Thm(\Gamma)$. If φ is a Γ-theorem, then we also say that φ is **deduced from the hypotheses** Γ.
As an immediate consequence of Definition 4.4, we get the induction principle for Γ-theorems.

Proposition 4.5. *[Induction principle on* Γ*-theorems]*
Let Δ *be a set of patterns satisfying the following properties:*

(i) Δ *contains all the axioms.*

(ii) $\Gamma \subseteq \Delta$*.*

(iii) For every inference rule, the following holds: If Δ *contains its premise(s), then its conclusion is also in* Δ*.*

Then $Thm(\Gamma) \subseteq \Delta$*.*

Notation 4.6. *Let* Γ, Δ *be sets of patterns and* φ *be a pattern. We use the following notations*

$$\begin{array}{lcl} \Gamma \vdash \varphi & = & \varphi \text{ is a } \Gamma\text{-theorem.} \\ \vdash \varphi & = & \emptyset \vdash \varphi, \\ \Gamma \vdash \Delta & \Leftrightarrow & \Gamma \vdash \varphi \text{ for any } \varphi \in \Delta. \end{array}$$

Definition 4.7. *A pattern* φ *is called a* ***theorem*** *if* $\vdash \varphi$*.*

Proposition 4.8. *Let* Γ, Δ *be sets of patterns.*

(i) Assume that $\Delta \subseteq \Gamma$*. Then for every pattern* φ*,* $Thm(\Delta) \subseteq Thm(\Gamma)$*, that is*

$$\Delta \vdash \varphi \text{ implies } \Gamma \vdash \varphi.$$

(ii) For every pattern φ, $Thm(\emptyset) \subseteq Thm(\Gamma)$, that is

$$\vdash \varphi \text{ implies } \Gamma \vdash \varphi.$$

(iii) Assume that $\Gamma \vdash \Delta$. Then for every pattern φ, $Thm(\Delta) \subseteq Thm(\Gamma)$, that is

$$\Delta \vdash \varphi \text{ implies } \Gamma \vdash \varphi.$$

(iv) For every pattern φ, $Thm(Thm(\Gamma)) = Thm(\Gamma)$, that is

$$Thm(\Gamma) \vdash \varphi \quad \text{iff } \Gamma \vdash \varphi.$$

Proof. (i) As $\Delta \subseteq \Gamma$, one proves immediately by induction on Δ-theorems that $Thm(\Delta) \subseteq Thm(\Gamma)$.

(ii) Apply (i) with $\Delta = \emptyset$.

(iii) As, by hypothesis, $\Delta \subseteq Thm(\Gamma)$, one proves immediately by induction on Δ-theorems that $Thm(\Delta) \subseteq Thm(\Gamma)$.

(iv) $\Leftarrow$ As, by definition, $\Gamma \subseteq Thm(\Gamma)$, we can apply (i) to get that $Thm(\Gamma) \subseteq Thm(Thm(\Gamma))$.
$\Rightarrow$ We have that $\Gamma \vdash Thm(\Gamma)$, so we can apply (iii) with $\Delta = Thm(\Gamma)$ to get that $Thm(Thm(\Gamma)) \subseteq Thm(\Gamma)$.

□

4.1 Γ-proofs

Definition 4.9. *A* ***Γ-proof*** *(or* ***proof from the hypotheses*** *Γ) is a sequence of patterns $\theta_1, \dots, \theta_n$ such that for all $i \in \{1, \dots, n\}$, one of the following holds:*

(i) θ_i is an axiom.

(ii) $\theta_i \in \Gamma$.

(iii) θ_i is the conclusion of an inference rule whose premise(s) are previous pattern(s).

An $\emptyset$-proof is called simply a ***proof****.*

Definition 4.10. *Let φ be a pattern. A* ***Γ-proof of*** *φ or a* ***proof of*** *φ* ***from the hypotheses*** *Γ is a Γ-proof $\theta_1, \dots, \theta_n$ such that $\theta_n = \varphi$.*

Proposition 4.11. *For any pattern φ,*

$$\Gamma \vdash \varphi \quad \text{iff} \quad \text{there exists a } \Gamma\text{-proof of } \varphi.$$

Proof. Let us denote $\Theta = \{\varphi \in Pattern \mid \text{there exists a } \Gamma\text{-proof of } \varphi\}$.
$\Rightarrow$ We prove by induction on Γ-theorems that $Thm(\Gamma) \subseteq \Theta$:

(i) φ is an axiom or a member of Γ. Then φ is a Γ-proof of φ. Hence, $\varphi \in \Theta$.

(ii) Let $\dfrac{\psi}{\varphi}$ be an inference rule such that $\psi \in \Theta$. Then there exists a Γ-proof $\theta_1, \dots, \theta_n = \psi$ of ψ. It follows that $\theta_1, \dots, \theta_n = \psi, \theta_{n+1} = \varphi$ is a Γ-proof of φ. Thus, $\varphi \in \Theta$.

(iii) Let $\dfrac{\psi \quad \chi}{\varphi}$ be an inference rule such that $\psi, \chi \in \Theta$. Then there exist a Γ-proof $\theta_1, \dots, \theta_n = \psi$ of ψ and a Γ-proof $\delta_1, \dots, \delta_p = \chi$ of χ. It follows that $\theta_1, \dots, \theta_n = \psi, \theta_{n+1} = \delta_1, \dots, \theta_{n+p} = \delta_p = \chi, \theta_{n+p+1} = \varphi$ is a Γ-proof of φ. Thus, $\varphi \in \Theta$.

$\Leftarrow$ Assume that φ has a Γ-proof $\theta_1, \ldots, \theta_n = \varphi$. We prove by induction on i that for all $i = 1, \ldots, n$, $\Gamma \vdash \theta_i$. As a consequence, $\Gamma \vdash \theta_n = \varphi$.

(i) $i = 1$. Then θ_1 must be an axiom or a member of Γ. Then obviously $\Gamma \vdash \theta_1$.

(ii) Assume that the induction hypothesis is true for all $j = 1, \ldots, i$. If θ_{i+1} is an axiom or a member of Γ, then obviously $\Gamma \vdash \theta_{i+1}$. Assume that θ_{i+1} is the conclusion of an inference rule whose premise(s) are previous pattern(s). If the inference rule is of type $\dfrac{\theta_j}{\theta_{i+1}}$ with $j \leq i$, then by the induction hypothesis we have that $\Gamma \vdash \theta_j$. By the definition of Γ-theorems, it follows that $\Gamma \vdash \theta_{i+1}$. If the inference rule is of type $\dfrac{\theta_j \quad \theta_k}{\theta_{i+1}}$ with $j, k \leq i$, then by the induction hypothesis we have that $\Gamma \vdash \theta_j$ and $\Gamma \vdash \theta_k$. By the definition of Γ-theorems, it follows that $\Gamma \vdash \theta_{i+1}$.

□

4.2 Derived rules

In the sequel we present different rules that will help us in proving Γ-theorems. These rules are of the form

If A_1, A_2, …, A_n $(n \geq 1)$ hold, then B holds.

and will be written as $\dfrac{A_1 \quad A_2 \quad \ldots \quad A_n}{B}$.

Let Γ be a set of patterns.

Proposition 4.12. *For every patterns φ, ψ,*

$$\frac{\Gamma \vdash \varphi \qquad \models^t \varphi \leftrightarrow \psi}{\Gamma \vdash \psi} \tag{123}$$

Proof.

(1)	Γ	$\vdash \varphi$	hypothesis
(2)		$\vdash \varphi \leftrightarrow \psi$	(TAUTOLOGY): hypothesis
(3)		$\vdash (\varphi \leftrightarrow \psi) \to (\varphi \to \psi)$	(TAUTOLOGY): (12)
(4)		$\vdash \varphi \to \psi$	(MODUS PONENS): (2), (3)
(5)	Γ	$\vdash \varphi \to \psi$	Proposition 4.8.(ii)
(6)	Γ	$\vdash \psi$	(MODUS PONENS): (1), (5).

□

5 Soundness Theorem

5.1 $\models_s$

We consider the proof system

$\mathcal{P}_{\models_s} = \mathcal{P} - \{$(∃-QUANTIFIER RULE), (FRAMING), (SET VARIABLE SUBSTITUTION), (KNASTER-TARSKI)$\}$.

Theorem 5.1. *Let $\Gamma \cup \{\varphi\}$ be a set of patterns. Then*

$\Gamma \vdash \varphi$ implies $\Gamma \models_s \varphi$.

Proof. Let us denote $\Theta = \{\varphi \in \mathit{Pattern} \mid \Gamma \models_s \varphi\}$. We prove that $\mathit{Thm}(\Gamma) \subseteq \Theta$ by induction on Γ-theorems.

Axioms:
(Tautology): By Proposition 3.66 and Remark 3.46.(ii).
($\exists$-Quantifier): By (22) and Remark 3.46.(ii).
(Propagation$_\perp$): By (56), (57) and Remark 3.46.(ii).
(Propagation$_\vee$): By (58), (59) and Remark 3.46.(ii).
(Propagation$_\exists$): By (60), (61) and Remark 3.46.(ii).
(Pre-Fixpoint): By (122) and Remark 3.46.(ii).
(Existence): By (36) and Remark 3.46.(ii).
(Singleton Variable): By (120) and Remark 3.46.(ii).

Rules:
(modus ponens): Assume that $\varphi, \varphi \to \psi \in \Theta$, hence $\Gamma \vDash_s \varphi$ and $\Gamma \vDash_s \varphi \to \psi$. Apply (41) to get that $\Gamma \vDash_s \psi$, hence $\psi \in \Theta$. □

5.2 $\vDash_l$

We consider the proof system

$$\mathcal{P}_{\vDash_l} = \mathcal{P} - \{(\exists\text{-quantifier rule}), (\text{Set Variable Substitution})\}.$$

Theorem 5.2. *Let $\Gamma \cup \{\varphi\}$ be a set of patterns. Then*

$$\Gamma \vdash \varphi \textit{ implies } \Gamma \vDash_l \varphi.$$

Proof. Let us denote $\Theta = \{\varphi \in \mathit{Pattern} \mid \Gamma \vDash_l \varphi\}$. We prove that $\mathit{Thm}(\Gamma) \subseteq \Theta$ by induction on Γ-theorems.

Axioms:
If φ is an axiom, we have by Theorem 5.1 that $\Gamma \vDash_s \varphi$. Apply now Proposition 3.52 to get that $\Gamma \vDash_l \varphi$, hence $\varphi \in \Theta$.

Rules:
(modus ponens): Assume that $\varphi, \psi \in \Theta$, hence $\Gamma \vDash_l \varphi$ and $\Gamma \vDash_l \varphi \to \psi$. Apply (45) to get that $\Gamma \vDash_l \psi$, hence $\psi \in \Theta$.
(framing): Assume that $\varphi \to \psi \in \Theta$, hence $\Gamma \vDash_l \varphi \to \psi$. Apply (74) to get that $\Gamma \vDash_l \varphi \cdot \chi \to \psi \cdot \chi$, hence $\varphi \cdot \chi \to \psi \cdot \chi \in \Theta$, and (76) to get that $\Gamma \vDash_l \chi \cdot \varphi \to \chi \cdot \psi$, hence $\chi \cdot \varphi \to \chi \cdot \psi \in \Theta$.
(Knaster-Tarski): Assume that X is free for ψ in φ and that $\mathit{Subf}_\psi^X \varphi \to \psi \in \Theta$, hence $\Gamma \vDash_l \mathit{Subf}_\psi^X \varphi \to \psi$. Apply Proposition 3.114 to get that $\Gamma \vDash_l \mu X.\varphi \to \psi$, hence $\mu X.\varphi \to \psi \in \Theta$. □

5.3 $\vDash_g$

Theorem 5.3. *Let $\Gamma \cup \{\varphi\}$ be a set of patterns. Then*

$$\Gamma \vdash \varphi \textit{ implies } \Gamma \vDash_g \varphi.$$

Proof. The proof is by induction on Γ-theorems.

Axioms:
If φ is an axiom, we have by Theorem 5.1 that $\Gamma \vDash_s \varphi$. Apply now Proposition 3.52 to get that $\Gamma \vDash_g \varphi$, hence $\varphi \in \Theta$.

Rules:
(modus ponens): Assume that $\varphi, \psi \in \Theta$, hence $\Gamma \vDash_g \varphi$ and $\Gamma \vDash_g \varphi \to \psi$. Apply (49) to get that $\Gamma \vDash_g \psi$, hence $\psi \in \Theta$.
($\exists$-quantifier rule): Assume that $x \notin FV(\psi)$ and $\varphi \to \psi \in \Theta$, hence $\Gamma \vDash_g \varphi \to \psi$. Apply (54) to get that $\Gamma \vDash_g \exists x.\varphi \to \psi$, hence $\exists x.\varphi \to \psi \in \Theta$.

(FRAMING): Assume that $\varphi \to \psi \in \Theta$, hence $\Gamma \vDash_g \varphi \to \psi$. Apply (94) to get that $\Gamma \vDash_g \varphi \cdot \chi \to \psi \cdot \chi$, hence $\varphi \cdot \chi \to \psi \cdot \chi \in \Theta$, and (96) to get that $\Gamma \vDash_g \chi \cdot \varphi \to \chi \cdot \psi$, hence $\chi \cdot \varphi \to \chi \cdot \psi \in \Theta$.
(SET VARIABLE SUBSTITUTION): Assume that X is free for ψ in φ and $\varphi \in \Theta$, hence $\Gamma \vDash_g \varphi$. Apply Proposition 3.111 to get that $\Gamma \vDash_g Subf_\psi^X \varphi$, hence $Subf_\psi^X \varphi \in \Theta$.
(KNASTER-TARSKI): Assume that X is free for ψ in φ and that $Subf_\psi^X \varphi \to \psi \in \Theta$, hence $\Gamma \vDash_g Subf_\psi^X \varphi \to \psi$. Apply Proposition 3.116 to get that $\Gamma \vDash_g \mu X.\varphi \to \psi$, hence $\mu X.\varphi \to \psi \in \Theta$.
□

6 Adding definedness, equality, totality

In the sequel, we consider the signature $\tau_{DEF} = (EVar, SVar, \Sigma)$ satisfying the following:

$DEF \in \Sigma$; DEF is called the **definedness symbol**.

We introduce the derived connective $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$, $\equiv$ by the following abbreviations:

$$\lceil \varphi \rceil = DEF \cdot \varphi \tag{124}$$

$$\lfloor \varphi \rfloor = \neg \lceil \neg \varphi \rceil \tag{125}$$

$$\varphi \equiv \psi = \lfloor \varphi \leftrightarrow \psi \rfloor \tag{126}$$

We assume that $\equiv$ has higher precedence than $\wedge, \vee, \to, \leftrightarrow$.

Definition 6.1. *A τ_{DEF}-structure $\mathcal{A}$ is obtained by defining $DEF^{\mathcal{A}} \subseteq A$ such that*

$$DEF^{\mathcal{A}} \star \{a\} = A \quad \text{for all } a \in A.$$

Lemma 6.2. *Let $\mathcal{A}$ be a τ_{DEF}-structure. Then*

(i) $A \star \{a\} = A$.

(ii) $DEF^{\mathcal{A}} \star B = A$ *for all* $\emptyset \neq B \subseteq A$.

Proof. (i) We have that, for all $a \in A$,

$$A = DEF^{\mathcal{A}} \star \{a\} \overset{(168)}{\subseteq} A \star \{a\} \subseteq A.$$

Thus, $A \star \{a\} = A$.

(ii) Let $\emptyset \neq B \subseteq A$. There exists $a \in B$. Then

$$A = DEF^{\mathcal{A}} \star \{a\} \overset{(168)}{\subseteq} DEF^{\mathcal{A}} \star B \subseteq A.$$

Thus, $DEF^{\mathcal{A}} \star B = A$.

□

Proposition 6.3. *For any pattern φ,*

$$e^+(\lceil \varphi \rceil) = \begin{cases} \emptyset & \text{if } e^+(\varphi) = \emptyset, \\ A & \text{if } e^+(\varphi) \neq \emptyset, \end{cases} \tag{127}$$

$$e^+(\lfloor \varphi \rfloor) = \begin{cases} A & \text{if } e^+(\varphi) = A, \\ \emptyset & \text{if } e^+(\varphi) \neq A. \end{cases} \tag{128}$$

Proof. Let $\mathcal{A}$ be a τ_{DEF}-structure and e be an $\mathcal{A}$-valuation.
We get that

$$\begin{aligned} e^+(\lceil\varphi\rceil) &= e^+(DEF\cdot\varphi) = e^+(DEF)\star e^+(\varphi) = DEF^{\mathcal{A}}\star e^+(\varphi) \\ &= \begin{cases} \emptyset & \text{if } e^+(\varphi)=\emptyset, \quad \text{by (159)} \\ A & \text{if } e^+(\varphi)\neq\emptyset, \quad \text{by Lemma 6.2.(ii)} \end{cases} \end{aligned}$$

$$\begin{aligned} e^+(\lfloor\varphi\rfloor) &= e^+(\neg\lceil\neg\varphi\rceil) = A\setminus e^+(\lceil\neg\varphi\rceil) \overset{(127)}{=} \begin{cases} A\setminus\emptyset & \text{if } e^+(\neg\varphi)=\emptyset \\ A\setminus A & \text{if } e^+(\neg\varphi)\neq\emptyset \end{cases} \\ &= \begin{cases} A & \text{if } e^+(\neg\varphi)=\emptyset \\ \emptyset & \text{if } e^+(\neg\varphi)\neq\emptyset \end{cases} = \begin{cases} A & \text{if } A\setminus e^+(\varphi)=\emptyset \\ \emptyset & \text{if } A\setminus e^+(\varphi)\neq\emptyset \end{cases} = \begin{cases} A & \text{if } e^+(\varphi)=A \\ \emptyset & \text{if } e^+(\varphi)\neq A \end{cases} \end{aligned}$$

□

As an immediate consequence of Proposition 6.3, we get that

Proposition 6.4. *For any pattern φ, $\lceil\varphi\rceil$ and $\lfloor\varphi\rfloor$ are predicate patterns.*

Proposition 6.5. *For any patterns φ, ψ, $\varphi\equiv\psi$ is a predicate pattern.*

Proof. Apply Proposition 6.4 and the definition of $\equiv$. □

Proposition 6.6. *For any pattern φ,*

$$\models \quad \varphi\to\lceil\varphi\rceil, \tag{129}$$

$$\models \quad \lfloor\varphi\rfloor\to\varphi. \tag{130}$$

Proof. Let $\mathcal{A}$ be a τ_{DEF}-structure and e be an $\mathcal{A}$-valuation. We have that

$$\mathcal{A}\models(\varphi\to\lceil\varphi\rceil)[e] \quad \text{iff} \quad e^+(\varphi)\subseteq e^+(\lceil\varphi\rceil), \text{ which is true, by (127)}, \tag{131}$$

$$\mathcal{A}\models(\lfloor\varphi\rfloor\to\varphi)[e] \quad \text{iff} \quad e^+(\lfloor\varphi\rfloor)\subseteq e^+(\varphi), \text{ which is true, by (128)}. \tag{132}$$

□

Proposition 6.7. *For any distinct variables x, y,*

$$\models \exists x.(x\equiv y). \tag{133}$$

Proof. Let $(\mathcal{A},e)$ and $b:=e(y)$. Then

$$e^+(\exists x.(x\equiv y)) = \bigcup_{a\in A}(e_{x\mapsto a})^+(x\equiv y) \supseteq (e_{x\mapsto b})^+(x\equiv y).$$

As

$$(e_{x\mapsto b})^+(x\leftrightarrow y) = A\setminus((e_{x\mapsto b})^+(x)\Delta(e_{x\mapsto b})^+(y)) = A\setminus(\{b\}\Delta\{b\}) = A,$$

we can apply (128) to get that $(e_{x\mapsto b})^+(x\equiv y) = (e_{x\mapsto b})^+(\lfloor x\leftrightarrow y\rfloor) = A$. It follows that $e^+(\exists x(x\equiv y)) = A$, that is $\mathcal{A}\models(\exists x(x\equiv y))[e]$. □

Proposition 6.8. *For any patterns φ, ψ, χ and variables x, y,*

$$\Gamma\models_g x\equiv y\to(\varphi\to\psi) \quad \textit{implies} \quad \Gamma\models_g x\equiv y\to(\chi\cdot\varphi\to\chi\cdot\psi), \tag{134}$$

$$\Gamma\models_g x\equiv y\to(\varphi\to\psi) \quad \textit{implies} \quad \Gamma\models_g x\equiv y\to(\varphi\cdot\chi\to\psi\cdot\chi). \tag{135}$$

Proof. By Proposition 6.5, $x\equiv y$ is a predicate pattern. Apply now Proposition 3.99 with $\sigma := x\equiv y$. □

A Set theory

Let A, B be sets. We use the following notations:

(i) $A \cup B$ for the union of A and B.

(ii) $A \cap B$ for the intersection of A and B.

(iii) $A \setminus B$ for the difference between A and B.

(iv) $A \Delta B$ for the symmetric difference of A and B.

(v) 2^A for the powerset of A.

(vi) $C_A B$ for the complementary of B, when $B \subseteq A$.

(vii) $Fun(A, B)$ for the set of functions from A to B.

(viii) A^n for the set $A \times A \times \ldots \times A$, where $n \geq 2$.

Let $f : A \to B$ be a mapping. For every $C \subseteq A$, we denote by $f|_C$ the restriction of f to C. Thus, $f|_C : C \to B$, $(f|_C)(x) = f(x)$ for every $x \in C$.

A.1 Expressions over a set

An **expression over** A is a finite sequence of elements from A. We denote an expression over A of length $n \in \mathbb{N}^*$ by $a_1 a_2 \ldots a_n$, where $a_i \in A$ for every $i = 1, \ldots, n$. The empty expression (of length 0) is denoted by λ. The concatenation of exoressions over A is defined as follows: if $a = a_1 \ldots a_n$ and $b = b_1 \ldots b_k$, then $ab = a_1 \ldots a_n b_1 \ldots b_k$.
Let $a = a_0 a_1 \ldots a_n$ be an expression over A, where $a_i \in A$ for all $i = 0, \ldots, k-1$.

(i) If $0 \leq i \leq j \leq k-1$, then the expression $a_i \ldots a_j$ is called the (i,j)-**subexpression** of a.

(ii) We say that an expression b **occurs** in a if there exists $0 \leq i \leq j \leq k-1$ such that b is the (i,j)-subexpression of a.

(iii) A **proper initial segment** of a is an expression $a_0 a_1 \ldots a_i$, where $0 \leq i < n-1$.

A.2 Set-theoretic properties used in the lecture notes

Proposition A.1. *Let A, B, C, D be sets. Then*

$$B \subseteq C \text{ and } B \subseteq D \text{ implies } B \subseteq C \cap D \tag{136}$$
$$A \setminus B = \emptyset \text{ iff } A \subseteq B \tag{137}$$
$$A \cup (B \cup C) = (A \cup B) \cup C, \tag{138}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \tag{139}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \tag{140}$$
$$(A \cap D) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \cap (D \cup B) \cap (D \cup C), \tag{141}$$
$$(A \cup D) \cap (B \cup C) = (A \cap B) \cup (B \cap C) \cup (D \cap B) \cup (D \cap C), \tag{142}$$
$$A \Delta B = \emptyset \quad \text{iff} \quad A = B, \tag{143}$$
$$A \setminus C \subseteq (A \setminus B) \cup (B \setminus C) \tag{144}$$

Proposition A.2. *Let $B, D \subseteq A$. Then*

$$B \cup C_A B = A, \tag{145}$$

$$C_A(C_A B) = B \tag{146}$$

$$C_A(B \cup D) = C_A B \cap C_A D \tag{147}$$

$$C_A(B \cap D) = C_A B \cup C_A D \tag{148}$$

$$C_A(C_A B \cup C_A D) = B \cap D \tag{149}$$

$$B \setminus D = B \cap C_A D, \tag{150}$$

$$C_A(B \setminus D) = C_A B \cup D \tag{151}$$

$$B \subseteq D \text{ iff } C_A D \subseteq C_A B \tag{152}$$

$$C_A B = \emptyset \text{ iff } B = A \tag{153}$$

$$C_A B = A \text{ iff } B = \emptyset, \tag{154}$$

$$C_A(B \setminus D) \cap C_A(D \setminus B) = C_A(B \Delta D) \tag{155}$$

A.3 Families of sets

Let $(B_i)_{i \in I}$ be a family of subsets of A.

Remark A.3. *If $I = \emptyset$, then $\bigcap_{i \in I} B_i = A$.*

Proposition A.4.

$$C_A\left(\bigcup_{i \in I} B_i\right) = \bigcap_{i \in I} C_A B_i \tag{156}$$

$$C_A\left(\bigcap_{i \in I} B_i\right) = \bigcup_{i \in I} C_A B_i \tag{157}$$

$$A \cup \bigcap_{i \in I} B_i = \bigcap_{i \in I} (A \cup B_i) \tag{158}$$

A.4 Application

In the sequel, A is a nonempty set and $\_ \star \_ : A \times A \to 2^A$ is a binary **application** function. We extend the application function $\_ \star \_$ as follows:

$$\_ \star \_ : 2^A \times 2^A \to 2^A, \quad B \star C = \bigcup_{b \in B, c \in C} b \star c.$$

Proposition A.5. *Let $B, C, D \subseteq A$ and $(A_i)_{i\in I}$ be a family of subsets of A.*

$$B \star \emptyset = \emptyset \star B = \emptyset, \tag{159}$$
$$(C \cup D) \star B = (C \star B) \cup (D \star B), \tag{160}$$
$$B \star (C \cup D) = (B \star C) \cup (B \star D), \tag{161}$$
$$(C \cap D) \star B \subseteq (C \star B) \cap (D \star B), \tag{162}$$
$$B \star (C \cap D) \subseteq (B \star C) \cap (B \star D), \tag{163}$$
$$\left(\bigcup_{i\in I} A_i\right) \star B = \bigcup_{i\in I}(A_i \star B), \tag{164}$$
$$B \star \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B \star A_i), \tag{165}$$
$$\left(\bigcap_{i\in I} A_i\right) \star B \subseteq \bigcap_{i\in I}(A_i \star B), \tag{166}$$
$$B \star \left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I}(B \star A_i), \tag{167}$$
$$B \subseteq C \textit{ implies } B \star D \subseteq C \star D \textit{ and } D \star B \subseteq D \star C \tag{168}$$
$$B \subseteq C \textit{ and } D \subseteq E \textit{ implies } B \star D \subseteq C \star E \tag{169}$$
$$B \subseteq C \textit{ implies } C_A B \star D \supseteq C_A C \star D \textit{ and } D \star C_A B \supseteq D \star C_A C \tag{170}$$

Proof. (159): Obviously.
(160): $(C \cup D) \star B = \bigcup_{c\in C\cup D, b\in B} c \star b = \left(\bigcup_{c\in C, b\in B} c \star b\right) \cup \left(\bigcup_{c\in D, b\in B} c \star b\right) = (C \star B) \cup (D \star B)$.
(161): $B \star (C \cup D) = \bigcup_{b\in B, c\in C\cup D} b \star c = \left(\bigcup_{b\in B, c\in C} b \star c\right) \cup \left(\bigcup_{b\in B, c\in D} b \star c\right) = (B \star C) \cup (B \star D)$.
(162): Let $a \in (C \cap D) \star B$. Then $a = e \star b$ with $e \in C \cap D$ and $b \in B$. It follows that $a \in (C \star B) \cap (D \star B)$.
(163): Let $a \in B \star (C \cap D)$. Then $a = b \star e$ with $e \in C \cap D$ and $b \in B$. It follows that $a \in (B \star C) \cap (B \star D)$.
(164): $\left(\bigcup_{i\in I} A_i\right) \star B = \bigcup_{a\in\bigcup_{i\in I} A_i, b\in B} a \star b = \bigcup_{i\in I} \bigcup_{a\in A_i, b\in B} a \star b = \bigcup_{i\in I}(A_i \star B)$.
(165): $B \star \left(\bigcup_{i\in I} A_i\right) = \bigcup_{b\in B, a\in\bigcup_{i\in I} A_i} b \star a = \bigcup_{i\in I} \bigcup_{a\in A_i, b\in B} b \star a = \bigcup_{i\in I}(B \star A_i)$.
(166): Let $a \in \left(\bigcap_{i\in I} A_i\right) \star B$. Then $a = e \star b$ with $b \in B$ and $e \in \bigcap_{i\in I} A_i$, hence $a \in A_i$ for all $i \in I$. It follows that $a \in \bigcap_{i\in I}(A_i \star B)$.
(167): Let $a \in B \star \left(\bigcap_{i\in I} A_i\right)$. Then $a = b \star e$ with $b \in B$ and $e \in \bigcap_{i\in I} A_i$, hence $a \in A_i$ for all $i \in I$. It follows that $a \in \bigcap_{i\in I}(B \star A_i)$.
(168): $B \star D = \bigcup_{b\in B, d\in D} b \star d \subseteq \bigcup_{b\in C, d\in D} b \star d = C \star D$ and, similarly, $D \star B \subseteq D \star C$.
(169): Apply (168) twice to get that $B \star D \subseteq C \star D \subseteq C \star E$.
(170) As $B \subseteq C$, we have that $C_A B \supseteq C_A C$. Apply (168). □

A.5 Knaster-Tarski Theorem

In the sequel A is a nonempty set and $F : 2^A \to 2^A$ is a mapping.

Definition A.6. *A subset D of A is a* ***fixpoint*** *of F if $F(D) = D$.*

Definition A.7. *A subset D of A is*

(i) ***the least fixpoint*** *of F if D is a fixpoint of F and for every fixpoint D' of F, we have that $D \subseteq D'$.*

(ii) ***the greatest fixpoint*** *of F if D is a fixpoint of F and for every fixpoint D' of F, we have that $D \supseteq D'$.*

Definition A.8. *F is said to be* ***monotone*** *if for every $B, C \subseteq A$,*

$$B \subseteq C \quad \textit{implies} \quad F(B) \subseteq F(C).$$

Theorem A.9 (Knaster-Tarski [11])**.**
Let $F : 2^A \to 2^A$ be a monotone function. Define

$$\mu F = \bigcap \{B \subseteq A \mid F(B) \subseteq B\} \quad \textit{and} \quad \nu F = \bigcup \{C \subseteq A \mid C \subseteq F(C)\}. \tag{171}$$

Then μF is the least fixpoint of F and νF is the greatest fixpoint of F. Furthermore,

$$\mu F = \bigcap \{D \subseteq A \mid F(D) = D\} \quad \textit{and} \quad \nu F = \bigcup \{D \subseteq A \mid F(D) = D\}. \tag{172}$$

Proof. Let us denote $\mathcal{B} = \{B \subseteq A \mid F(B) \subseteq B\}$ and $\mathcal{C} = \{C \subseteq A \mid C \subseteq F(C)\}$.

Claim 1: $F(\mu F) = \mu F$.

Proof of claim: "$\subseteq$" For every $B \in \mathcal{B}$, we have that $\mu F \subseteq B$, hence $F(\mu F) \subseteq F(B) \subseteq B$. It follows that $F(\mu F) \subseteq \bigcap \mathcal{B} = \mu F$.
"$\supseteq$" As $F(\mu F) \subseteq \mu F$, we have that $F(F(\mu F)) \subseteq F(\mu F)$, so $F(\mu F) \in \mathcal{B}$. It follows that $\mu F \subseteq F(\mu F)$. ■

Claim 2: μF is the least fixpoint of F.

Proof of claim: If D is another fixpoint of F, we have, in particular, that $F(D) \subseteq D$, so $D \in \mathcal{B}$. It follows that $\mu F \subseteq D$. ■

Claim 3: $F(\nu F) = \nu F$.

Proof of claim: "$\supseteq$" For every $C \in \mathcal{C}$, we have that $C \subseteq \nu F$. It follows that $F(C) \subseteq F(\nu F)$, so $C \subseteq F(C) \subseteq F(\nu F)$. It follows that $\nu F = \bigcup \mathcal{C} \subseteq F(\nu F)$.
"$\subseteq$" As $\nu F \subseteq F(\nu F)$, we have that $F(\nu F) \subseteq F(F(\nu F))$, so $F(\nu F) \in \mathcal{C}$. It follows that $F(\nu F) \subseteq \nu F$. ■

Claim 4: νF is the greatest fixpoint of F.

Proof of claim: If D is another fixpoint of F, we have, in particular, that $D \subseteq F(D)$, so $D \in \mathcal{C}$. It follows that $D \subseteq \nu F$. ■

Let us denote $\mathcal{D} = \{D \subseteq A \mid F(D) = D\}$.

Claim 5: $\mu F = \bigcap \mathcal{D}$.

Proof of claim: "$\supseteq$" We have, by Claim 1, that $\mu F \in \mathcal{D}$, hence $\mu F \supseteq \bigcap \mathcal{D}$.
"$\subseteq$" For every $D \in \mathcal{D}$, we have, by Claim 2, that $\mu F \subseteq D$. Hence, $\mu F \subseteq \bigcap \mathcal{D}$. ■

Claim 6: $\nu F = \bigcup \mathcal{D}$.

Proof of claim: "$\subseteq$" We have, by Claim 3, that $\nu F \in \mathcal{D}$, hence $\nu F \subseteq \bigcup \mathcal{D}$.
"$\supseteq$" For every $D \in \mathcal{D}$, we have, by Claim 4, that $D \subseteq \nu F$. Hence, $\bigcup \mathcal{D} \subseteq \nu F$. □

Thus, if $F : 2^A \to 2^A$ is monotone, then $F(\mu F) = \mu F$, $F(\nu F) = \nu F$ and for every $B \subseteq A$ such that $F(B) = B$,

$$\mu F \subseteq B \subseteq \nu F.$$

B AML proof system $\mathcal{P}$

(Tautology)	φ	if φ is a tautology
($\exists$-Quantifier)	$Subf^x_y\varphi \to \exists x.\varphi$	if x is free for y in φ
(modus ponens)	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}$	
($\exists$-quantifier rule)	$\dfrac{\varphi \to \psi}{\exists x.\varphi \to \psi}$	if $x \notin FV(\psi)$
(Propagation$_\bot$)	$\varphi \cdot \bot \to \bot \quad \bot \cdot \varphi \to \bot$	
(Propagation$_\vee$)	$(\varphi \vee \psi) \cdot \chi \to \varphi \cdot \chi \vee \psi \cdot \chi \quad \chi \cdot (\varphi \vee \psi) \to \chi \cdot \varphi \vee \chi \cdot \psi$	
(Propagation$_\exists$)	$(\exists x.\varphi) \cdot \psi \to \exists x.\varphi \cdot \psi \qquad \psi \cdot (\exists x.\varphi) \to \exists x.\psi \cdot \varphi$	if $x \notin FV(\psi)$
(framing)	$\dfrac{\varphi \to \psi}{\varphi \cdot \chi \to \psi \cdot \chi} \qquad \dfrac{\varphi \to \psi}{\chi \cdot \varphi \to \chi \cdot \psi}$	
(Set Variable Substitution)	$\dfrac{\varphi}{Subf^X_\psi \varphi}$	if X is free for ψ in φ
(Pre-Fixpoint)	$Subf^X_{\mu X.\varphi}\varphi \to \mu X.\varphi$	if φ is positive in X and X is free for $\mu X.\varphi$ in φ
(Knaster-Tarski)	$\dfrac{Subf^X_\psi \varphi \to \psi}{\mu X.\varphi \to \psi}$	if X is free for ψ in φ
(Existence)	$\exists x.x$	
(Singleton Variable)	$\neg(C_1[x \wedge \varphi] \wedge C_2[x \wedge \neg\varphi])$	

Figure 1: Proof system $\mathcal{P}$ for AML

References

[1] P. Bereczky, X. Chen, D. Horpácsi, T. B. Mizsei, L. Peña, and J. Tusil. Mechanizing matching logic in Coq. In V. Rusu, editor, *Proceedings of the Sixth Working Formal Methods Symposium (FROM 2022)*, volume 369 of *Electronic Proceedings in Theoretical Computer Science, EPTCS*, pages 17–36. 2022.

[2] X. Chen, D. Lucanu, and G. Roşu. Matching logic explained. *Journal of Logical and Algebraic Methods in Programming*, 120(100638), 2021.

[3] X. Chen and G. Roşu. Applicative matching logic. Technical Report http://hdl.handle.net/2142/104616, University of Illinois at Urbana-Champaign, July 2019.

[4] X. Chen and G. Roşu. Matching μ-logic. In *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27*, pages 1–13. IEEE, 2019. https://doi.org/10.1109/LICS.2019.8785675.

[5] X. Chen and G. Roşu. Applicative matching logic: Semantics of K. https://fsl.cs.illinois.edu/publications/chen-rosu-2019-trb.pdf, University of Illinois at Urbana-Champaign, March 2020.

[6] H. Cheval and B. Macovei. Lean formalization of applicative matching logic. https://gitlab.com/ilds/aml-lean/MatchingLogic/, 2022.

[7] L. Leuştean. Notes on applicative matching logic. Lecture Notes, 2024.

[8] J. D. Monk. *Mathematical Logic*. Graduate Texts in Mathematics. Springer, 1976.

[9] G. Roşu. Matching logic. *Logical Methods in Computer Science*, 13(4):1–61, 2017.

[10] G. Roşu, C. Ellison, and W. Schulte. Matching Logic: An Alternative to Hoare/Floyd Logic. In *AMAST*, pages 142–162, 2010.

[11] A. Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.