Abstract matching logic

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Abstract

In these notes we present an abstract version of matching logic.

Contents

1 Introduction

Applicative matching logic (AML) was introduced recently by Xiaohong Chen and Grigore Roşu [1, 2] as a variant of matching logic (ML), developed by Grigore Roşu and collaborators [7, 6]. In [3], the first author gives a theoretical introduction to AML. In these notes we develop an abstract version of matching logic, based on the observation that different basic definitions and results from [3] can be obtained in a much more general setting. As in the case of [3], Monk's textbook [5] has a huge influence on these notes.

We consider a language $\mathcal{L}$ for abstract matching logic that contains a countable set of element variables, sets of set variables and constants, and finite sets of propositional constants, unary and binary connectives, equality symbols, first-order quantifiers and second-order binders. Examples of such languages are given in [4]. We define $\mathcal{L}$-patterns, prove unique readability results, give a recursion principle on patterns. In Section 3 we prove useful properties of $\mathcal{L}$-contexts and in Section 4 we define a general notion of congruence that is used in the next section to prove important replacement theorems. In Sections 6-14 we define and prove properties of free, bound, fresh element/set variables, substitution of free occurrences of element/set variables, bounded substitution, variables free for patterns, positive and negative occurences of set variables. One of the main results is the bounded substitution theorem (Section 9), whose proof is inspired by the proof of [5, Theorem 10.59] for first-order logic. Finally, in Section 15, we study general proof systems and define an abstract matching logic.

The general setting from these notes is applied by the author and Dafina Trufaş in [4] for first-order matching logic with application (and definedness).

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2 Language

Definition 2.1. *A* ***language*** $\mathcal{L}$ ***for abstract matching logic*** *consists of:*

(i) a countable set $EVar$ *of* ***element variables****;*

(ii) a set $SVar$ *of* ***set variables****;*

(iii) a set Σ *of* ***constants****;*

(iv) a finite set $\mathcal{P}_C$ *of* ***propositional constants****;*

(v) a finite set $\mathcal{P}_1$ *of* ***unary connectives****;*

(vi) a finite set $\mathcal{P}_2$ *of* ***binary connectives****;*

(vii) a finite set $Equal$ *of* ***equality symbols****;*

(viii) a finite set $\mathcal{Q}$ *of* ***first-order quantifiers****;*

(ix) a finite set $\overline{\mathcal{Q}}$ *of* ***second-order binders****.*

Furthermore, the following holds:

$$\text{If } \overline{\mathcal{Q}} \neq \emptyset, \text{ then } SVar \text{ is a countable set.}$$

Each two of the sets $EVar$, $SVar$, Σ, $\mathcal{P}_C$, $\mathcal{P}_1$, $\mathcal{P}_2$, $Equal$, $\mathcal{Q}$, $\overline{\mathcal{Q}}$ are pairwise disjoint. We denote element variables by $x, y, z, x_1, x_2, \ldots$ and set variables by $X, Y, Z, X_1, X_2, \ldots$.

In the sequel, $\mathcal{L}$ is a language for abstract matching logic.

Definition 2.2. *The set $Sym_{\mathcal{L}}$ of **$\mathcal{L}$-symbols** is defined as*

$$Sym_{\mathcal{L}} = EVar \cup SVar \cup \Sigma \cup \mathcal{P}_C \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup Equal \cup \mathcal{Q} \cup \overline{\mathcal{Q}}.$$

Definition 2.3. *The set $Expr_{\mathcal{L}}$ of **$\mathcal{L}$-expressions** is the set of all expressions over $Sym_{\mathcal{L}}$.*

Definition 2.4. *The set $AtomicPattern_{\mathcal{L}}$ of **atomic $\mathcal{L}$-patterns** is defined as follows:*

$$AtomicPattern_{\mathcal{L}} = EVar \cup SVar \cup \Sigma \cup \mathcal{P}_C.$$

Let Γ be a set of $\mathcal{L}$-expressions. We say that

(i) Γ is closed to $\mathcal{P}_1$ if for every $- \in \mathcal{P}_1$ and every $\mathcal{L}$-expression φ,

$$\varphi \in \Gamma \quad \text{implies} \quad -\varphi \in \Gamma.$$

(ii) Γ is closed to $\mathcal{P}_2$ if for every $\circ \in \mathcal{P}_2$ and every $\mathcal{L}$-expressions φ, ψ,

$$\varphi, \psi \in \Gamma \quad \text{implies} \quad \circ\varphi\psi \in \Gamma.$$

(iii) Γ is closed to $Equal$ if for every $\sim \in Equal$ and every $\mathcal{L}$-expressions φ, ψ,

$$\varphi, \psi \in \Gamma \quad \text{implies} \quad \sim\varphi\psi \in \Gamma.$$

(iv) Γ is closed to $\mathcal{Q}$ if for every $Q \in \mathcal{Q}$, $x \in EVar$ and $\mathcal{L}$-expression φ,

$$\varphi \in \Gamma \quad \text{implies} \quad Qx\varphi \in \Gamma.$$

(v) Γ is closed to $\overline{\mathcal{Q}}$ if for every $\overline{Q} \in \overline{\mathcal{Q}}$, $X \in SVar$ and $\mathcal{L}$-expression φ,

$$\varphi \in \Gamma \quad \text{implies} \quad \overline{Q}X\varphi \in \Gamma.$$

Definition 2.5. *The set $Pattern_{\mathcal{L}}$ of **$\mathcal{L}$-patterns** is the intersection of all sets Γ of $\mathcal{L}$-expressions that have the following properties:*

(i) Γ contains all atomic $\mathcal{L}$-patterns.

(ii) Γ is closed to $\mathcal{P}_1$, $\mathcal{P}_2$, $Equal$, $\mathcal{Q}$ and $\overline{\mathcal{Q}}$.

We use the Polish notation in the definition of $\mathcal{L}$-patterns as this notation allows us to obtain the unique readability of $\mathcal{L}$-patterns (see Proposition 2.8), a fundamental property.
$\mathcal{L}$-patterns are denoted by $\varphi, \psi, \chi, \ldots$.
For any $\mathcal{L}$-pattern φ, we use the following notations

$$EVar(\varphi) = \{x \in EVar \mid x \text{ occurs in } \varphi\},$$
$$SVar(\varphi) = \{X \in SVar \mid X \text{ occurs in } \varphi\}.$$

Proposition 2.6 (Induction principle on patterns)**.**
Let Γ be a set of $\mathcal{L}$-patterns satisfying the following properties:

(i) Γ contains all atomic $\mathcal{L}$-patterns.

(ii) Γ is closed to $\mathcal{P}_1$, $\mathcal{P}_2$, $Equal$, $\mathcal{Q}$ and $\overline{\mathcal{Q}}$.

Then $\Gamma = Pattern_{\mathcal{L}}$.

Proof. By hypothesis, $\Gamma \subseteq Pattern_{\mathcal{L}}$. By Definition 2.5, we get that $Pattern_{\mathcal{L}} \subseteq \Gamma$. □

Induction principle on patterns is used to prove that all patterns have a property $\mathcal{P}$: we define Γ as the set of all patterns satisfying $\mathcal{P}$ and apply induction on patterns to obtain that $\Gamma = Pattern_{\mathcal{L}}$.

Definition 2.7 (Alternative definition for $\mathcal{L}$-patterns)**.** *The* ***$\mathcal{L}$-patterns*** *are the $\mathcal{L}$-expressions inductively defined as follows:*

(i) Every atomic $\mathcal{L}$-pattern is an $\mathcal{L}$-pattern.

(ii) If φ is an $\mathcal{L}$-pattern and $- \in \mathcal{P}_1$, then $-\varphi$ is an $\mathcal{L}$-pattern.

(iii) If φ and ψ are $\mathcal{L}$-patterns and $\circ \in \mathcal{P}_2$, then $\circ\varphi\psi$ is an $\mathcal{L}$-pattern.

(iv) If φ and ψ are $\mathcal{L}$-patterns and $\sim \in Equal$, then $\sim \varphi\psi$ is an $\mathcal{L}$-pattern.

(v) If φ is an $\mathcal{L}$-pattern, $Q \in \mathcal{Q}$ and $x \in EVar$, then $Qx\varphi$ is an $\mathcal{L}$-pattern.

(vi) If φ is an $\mathcal{L}$-pattern, $\overline{Q} \in \mathcal{Q}$ and $X \in SVar$, then $\overline{Q}X\varphi$ is an $\mathcal{L}$-pattern.

(vii) Only the expressions obtained by applying the above rules are $\mathcal{L}$-patterns.

When the signature $\mathcal{L}$ is clear from the context, we shall write simply expression(s), pattern(s) and we shall denote the set of expressions by *Expr*, the set of patterns by *Pattern*, the set of atomic patterns by *AtomicPattern*, etc..

2.1 Unique readability

Proposition 2.8 (Unique readability of patterns)**.**

(i) Any pattern has a positive length.

(ii) If φ is a pattern, then one of the following holds:

(a) $\varphi = x$, where $x \in EVar$.
(b) $\varphi = X$, where $X \in SVar$.
(c) $\varphi = \sigma$, where $\sigma \in \Sigma$.
(d) $\varphi = P$, where $P \in \mathcal{P}_C$.
(e) $\varphi = -\psi$, where $- \in \mathcal{P}_1$ and ψ is a pattern.
(f) $\varphi = \circ\psi\chi$, where $\circ \in \mathcal{P}_2$ and ψ, χ are patterns.
(g) $\varphi = \sim \psi\chi$, where $\sim \in Equal$ and ψ, χ are patterns.
(h) $\varphi = Qx\psi$, where $Q \in \mathcal{Q}$, $x \in EVar$ and ψ is a pattern.
(i) $\varphi = \overline{Q}X\psi$, where $\overline{Q} \in \overline{\mathcal{Q}}$, $X \in SVar$ and ψ is a pattern.

(iii) Any proper initial segment of a pattern is not a pattern.

(iv) If φ is a pattern, then exactly one of the cases from (ii) holds. Moreover, φ can be written in a unique way in one of these forms.

Proof. (i) Let Γ be the set of patterns of positive length. We prove that $\Gamma = Pattern$ using the Induction principle on patterns (Proposition 2.6).

(a) If φ is an atomic pattern, then its length is 1, so $\varphi \in \Gamma$.

(b) If $\varphi, \psi \in \Gamma$, hence they have positive length, then obviously the patterns $-\psi$ ($- \in \mathcal{P}_1$), $\circ\varphi\psi$ ($\circ \in \mathcal{P}_2$), $\sim \varphi\psi$ ($\sim \in Equal$), $Qx\varphi$ ($Q \in \mathcal{Q}$, $x \in EVar$), $\overline{Q}X\varphi \in \Gamma$ ($\overline{Q} \in \overline{\mathcal{Q}}$, $X \in SVar$) have positive length, hence they are in Γ.

(ii) Let $\Gamma_1 = \{-\psi \mid - \in \mathcal{P}_1 \text{ and } \psi \in Pattern\}$, $\Gamma_2 = \{\circ\psi\chi \mid \circ \in \mathcal{P}_2 \text{ and } \psi, \chi \in Pattern\}$, $\Gamma_3 = \{\sim \psi\chi \mid \sim \in Equal \text{ and } \psi, \chi \in Pattern\}$, $\Gamma_4 = \{Qx\psi \mid Q \in \mathcal{Q}, x \in EVar \text{ and } \psi \in Pattern\}$ and $\Gamma_5 = \{\overline{Q}X\psi \mid \overline{Q} \in \overline{\mathcal{Q}}, X \in SVar \text{ and } \psi \in Pattern\}$. Define

$$\Gamma = EVar \cup SVar \cup \Sigma \cup \mathcal{P}_C \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5.$$

Then obviously, $\Gamma \subseteq Pattern$. We prove that $\Gamma = Pattern$ using the Induction principle on patterns (Proposition 2.6).

(a) As $EVar \cup SVar \cup \Sigma \cup \mathcal{P}_C \subseteq \Gamma$, we have that Γ contains all atomic patterns.

(b) Let $\psi, \chi \in \Gamma$, $- \in \mathcal{P}_1$, $\circ \in \mathcal{P}_2$, $\sim \in Equal$, $Q \in \mathcal{Q}$, $x \in EVar$, $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$. Then $-\psi \in \Gamma_1 \subseteq \Gamma$, $\circ\psi\chi \in \Gamma_2 \subseteq \Gamma$, $\sim \psi\chi \in \Gamma_3$, $Qx\psi \in \Gamma_4 \subseteq \Gamma$, and $\overline{Q}X\psi \in \Gamma_5 \subseteq \Gamma$. Thus, Γ is closed to $\mathcal{P}_1$, $\mathcal{P}_2$, $Equal$, $\mathcal{Q}$ and $\overline{\mathcal{Q}}$.

(iii) As, by (i), patterns have positive length, it follows that we have to prove that for all $n \geq 1$,

(P) if $\varphi = \varphi_0 \dots \varphi_{n-1}$ is a pattern of length n, then for any $0 \leq i < n-1$, $\varphi = \varphi_0 \dots \varphi_i$ is not a pattern.

The proof is by induction on n.

$n = 1$: Then one cannot have $0 \leq i < 0$, hence (P) holds.

Assume that $n > 1$ and that (P) holds for any pattern of length $< n$. Let $\varphi = \varphi_0 \dots \varphi_{n-1}$ be a pattern of length n. By (ii), we have the following cases:

(a) $\varphi = -\psi$, where $- \in \mathcal{P}_1$ and ψ is a pattern. Then $\varphi_0 = -$ and $\psi = \varphi_1 \dots \varphi_{n-1}$. Let $0 \leq i < n-1$ and assume, by contradiction, that $\varphi_0 \dots \varphi_i$ is a pattern. Applying again (ii), it follows that $\varphi_0 \dots \varphi_i = -\psi^1$, where $\psi^1 = \varphi_1 \dots \varphi_i$ is a pattern. Then ψ^1 is a proper initial segment of ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get that ψ^1 is not a pattern. We have obtained a contradiction.

(b) $\varphi = \ominus\psi\chi$, where $\ominus \in \mathcal{P}_2 \cup Equal$ and ψ, χ are patterns. Thus, $\varphi_0 = \ominus$, $\psi = \varphi_1 \dots \varphi_{k-1}$ and $\chi = \varphi_k \dots \varphi_{n-1}$, where $2 \leq k \leq n-1$.
Let $0 \leq i < n-1$ and assume, by contradiction, that $\varphi_0 \dots \varphi_i$ is a pattern. Applying again (ii), it follows that $\varphi_0 \dots \varphi_i = \ominus\psi^1\chi^1$, where ψ^1, χ^1 are patterns. Thus, $\psi^1 = \varphi_1 \dots \varphi_{p-1}$, $\chi^1 = \varphi_p \dots \varphi_i$, where $2 \leq p \leq i$. We have the following cases:

(1) $p < k$. Then ψ^1 is a proper initial segment of ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get that ψ^1 is not a pattern. We have obtained a contradiction.

(2) $p = k$. Then $\psi^1 = \psi$ and χ^1 is a proper initial segment of χ. As the length of χ is $< n$, we can apply the induction hypothesis to get that χ^1 is not a pattern. We have obtained a contradiction.

(3) $p > k$. Then ψ is a proper initial segment of ψ^1. As the length of ψ^1 is $< n$, we can apply the induction hypothesis to get that ψ is not a pattern. We have obtained a contradiction.

(c) $\varphi = \theta\psi$, where ψ is a pattern and $\theta \in \{Qx \mid Q \in FolQ, x \in EVar\} \cup \{\overline{Q}X \mid \overline{Q} \in \overline{\mathcal{Q}}, X \in SVar\}$. Then $\varphi_0\varphi_1 = \theta$ and $\psi = \varphi_2 \dots \varphi_{n-1}$. Let $0 \leq i < n-1$ and assume, by contradiction, that $\varphi_0 \dots \varphi_i$ is a pattern. Applying again (ii), it follows that $\varphi_0 \dots \varphi_i = \theta\psi^1$, where $\psi^1 = \varphi_2 \dots \varphi_i$ is a pattern. Then ψ^1 is a proper initial segment of ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get that ψ^1 is not a pattern. We have obtained a contradiction.

(iv) is an immediate consequence of (ii) and (iii).

□

Proposition 2.9. *Let $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ be a pattern and suppose that $\varphi_i \in \mathcal{Q} \cup \overline{\mathcal{Q}}$ for some $i = 0, \dots, n-1$. Then there exists a unique j such that $i < j \leq n-1$ and $\varphi_i \dots \varphi_j$ is a pattern.*

Proof. Let us prove first the uniqueness. Assume, by contradiction, that $i < j < k \leq n-1$ are such that $\varphi_i \ldots \varphi_j$ and $\varphi_i \ldots \varphi_k$ are both patterns. As $\varphi_i \ldots \varphi_j$ is a proper initial segment of $\varphi_i \ldots \varphi_k$, it follows, by Proposition 2.8.(iii) that $\varphi_i \ldots \varphi_j$ is not a pattern. We have obtained a contradiction.
Let us prove in the sequel the existence.
As, by Proposition 2.8.(i), patterns have positive length, it follows that we have to prove that for all $n \geq 1$,

(P) if $\varphi = \varphi_0 \ldots \varphi_{n-1}$ is a pattern of length n and $\varphi_i \in \mathcal{Q} \cup \overline{\mathcal{Q}}$ for some $i = 0, \ldots, n-1$, then there exists j such that $i < j \leq n-1$ and $\varphi_i \ldots \varphi_j$ is a pattern.

The proof is by induction on n.
$n = 1$. Then $\varphi = \varphi_0$ is an atomic pattern, so there exists no i satisfying the premise in (P), hence (P) holds.
Assume that $n > 1$ and that (P) holds for any pattern of length $< n$. Let $\varphi = \varphi_0 \ldots \varphi_{n-1}$ be a pattern of length n such that $\varphi_i \in \mathcal{Q} \cup \overline{\mathcal{Q}}$ for some $i = 0, \ldots, n-1$. By Proposition 2.8.(ii), we have the following cases:

(i) $\varphi = -\psi$, where $- \in \mathcal{P}_1$ and ψ is a pattern. Then $\varphi_0 = -$ and $\psi = \varphi_1 \ldots \varphi_{n-1}$. It follows that $i \geq 1$, hence φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get the existence of j such that $i < j \leq n-1$ and $\varphi_i \ldots \varphi_j$ is a pattern.

(ii) $\varphi = \ominus\psi\chi$, where $\ominus \in \mathcal{P}_2 \cup Equal$ and ψ, χ are patterns. Thus, $\varphi_0 = \ominus$, $\psi = \varphi_1 \ldots \varphi_{k-1}$ and $\chi = \varphi_k \ldots \varphi_{n-1}$, where $2 \leq k \leq n-1$. We have the following cases:

   (a) $i \leq k-1$. Then φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get the existence of j such that $i < j \leq k-1$ and $\varphi_i \ldots \varphi_j$ is a pattern.

   (b) $i \geq k$. Then φ_i occurs in χ. As the length of χ is $< n$, we can apply the induction hypothesis to get the existence of j such that $i < j \leq n-1$ and $\varphi_i \ldots \varphi_j$ is a pattern.

(iii) $\varphi = \theta\psi$, where ψ is a pattern and $\theta \in \{Qx \mid Q \in \mathcal{Q}, x \in EVar\} \cup \{\overline{Q}X \mid \overline{Q} \in \overline{\mathcal{Q}}, X \in SVar\}$. Then $\varphi_0\varphi_1 = \theta$ and $\psi = \varphi_2 \ldots \varphi_{n-1}$. As $i \neq 1$, we have the following cases:

   (a) $i = 0$. Then $j = n-1$ and $\varphi_i \ldots \varphi_j = \varphi$ is a pattern.

   (b) $2 \leq i \leq n-1$. Then φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis to get the existence of j such that $i < j \leq n-1$ and $\varphi_i \ldots \varphi_j$ is a pattern.

□

Proposition 2.10. *Let $\varphi = \varphi_0\varphi_1 \ldots \varphi_{n-1}$ be a pattern and suppose that $\varphi_i \in \mathcal{P}_2 \cup Equal$ for some $i = 0, \ldots, n-1$. Then there exist unique j, l such that $i < j < l \leq n-1$ and $\varphi_{i+1} \ldots \varphi_j$, $\varphi_{j+1} \ldots \varphi_l$ are patterns.*

Proof. Let us prove first the uniqueness. Assume, by contradiction, that $i < j < l \leq n-1$ and $i < j_1 < l_1 \leq n-1$ are such that $\psi = \varphi_{i+1} \ldots \varphi_j$, $\chi = \varphi_{j+1} \ldots \varphi_l$, $\psi^1 = \varphi_{i+1} \ldots \varphi_{j_1}$, $\chi = \varphi_{j+1} \ldots \varphi_{l_1}$ are patterns. If $j \neq j_1$, then either $j < j_1$ or $j_1 < j$, hence one of ψ, ψ^1 is a proper initial segment of the other one. By Proposition 2.8.(iii), we get that one of ψ, ψ^1 is not a pattern. We have obtained a contradiction. Thus, we must have $j = j_1$. We prove similarly that we must have $l = l_1$.
Let us prove in the sequel the existence. As, by Proposition 2.8.(i), patterns have positive length, it follows that we have to prove that for all $n \geq 1$,

(P) if $\varphi = \varphi_0 \ldots \varphi_{n-1}$ is a pattern of length n and $\varphi_i \in \mathcal{P}_2 \cup Equal$ for some $i = 0, \ldots, n-1$, then there exist j, l such that $i < j < l \leq n-1$ and $\varphi_{i+1} \ldots \varphi_j$, $\varphi_{j+1} \ldots \varphi_l$ are patterns.

The proof is by induction on n.
$n = 1$. Then $\varphi = \varphi_0$ is an atomic pattern, so there exists no i satisfying the premise in (P), hence (P) holds.

Assume that $n > 1$ and that (P) holds for any pattern of length $< n$. Let $\varphi = \varphi_0 \dots \varphi_{n-1}$ be a pattern of length n such that $\varphi_i \in \mathcal{P}_2 \cup Equal$ for some $i = 0, \dots, n-1$. By Proposition 2.8.(ii), we have the following cases:

(i) $\varphi = -\psi$, where $- \in \mathcal{P}_1$ and ψ is a pattern. We get that $i \geq 1$, hence φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis for ψ to get the conclusion.

(ii) $\varphi = \ominus\psi\chi$, where $\ominus \in \mathcal{P}_2 \cup Equal$ and ψ, χ are patterns. Thus, $\varphi_0 = \ominus$, $\psi = \varphi_1 \dots \varphi_{k-1}$ and $\chi = \varphi_k \dots \varphi_{n-1}$, where $2 \leq k \leq n-1$. We have the following cases:

   (a) $i = 0$. Then we can take $j = k - 1$ and $l = n - 1$.

   (b) $i \in \{1, \dots, k-1\}$. Then φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis for ψ to get the conclusion.

   (c) $i \geq k$. Then φ_i occurs in χ. As the length of χ is $< n$, we can apply the induction hypothesis for χ to get the conclusion

(iii) $\varphi = \theta\psi$, where ψ is a pattern and $\theta \in \{Qx \mid Q \in \mathcal{Q}, x \in EVar\} \cup \{\overline{Q}X \mid \overline{Q} \in \overline{\mathcal{Q}}, X \in SVar\}$. Then $\varphi_0\varphi_1 = \theta$ and $\psi = \varphi_2 \dots \varphi_{n-1}$. As φ_i does not occur in θ, we have that φ_i occurs in ψ. As the length of ψ is $< n$, we can apply the induction hypothesis for ψ to get the conclusion.

□

2.2 Recursion principle on patterns

Proposition 2.11 (Recursion principle on patterns)**.** *Let D be a set and the mappings*

$$\begin{aligned}
G_0 &: AtomicPattern \to D, \\
G_- &: D \times Pattern \to D \quad \text{for any } - \in \mathcal{P}_1 \\
G_\circ &: D^2 \times Pattern^2 \to D \quad \text{for any } \circ \in \mathcal{P}_2, \\
G_\sim &: D^2 \times Pattern^2 \to D \quad \text{for any } \sim \in Equal, \\
G_Q &: D \times EVar \times Pattern \to D \quad \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\
G_{\overline{Q}} &: D \times SVar \times Pattern \to D \quad \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar.
\end{aligned}$$

Then there exists a unique mapping

$$F : Pattern \to D$$

that satisfies the following properties:

(i) $F(\varphi) = G_0(\varphi)$ *for any atomic pattern* φ.

(ii) $F(-\varphi) = G_-(F(\varphi), \varphi)$ *for any* $- \in \mathcal{P}_1$ *and any pattern* φ.

(iii) $F(\circ\varphi\psi) = G_\circ(F(\varphi), F(\psi), \varphi, \psi)$ *for any* $\circ \in \mathcal{P}_2$ *and any patterns* φ, ψ.

(iv) $F(\sim \varphi\psi) = G_\sim(F(\varphi), F(\psi), \varphi, \psi)$ *for any* $\sim \in Equal$ *and any patterns* φ, ψ.

(v) $F(Qx\varphi) = G_Q(F(\varphi), x, \varphi)$ *for any* $Q \in \mathcal{Q}$, $x \in EVar$ *and any pattern* φ.

(vi) $F(\overline{Q}X\varphi) = G_{\overline{Q}}(F(\varphi), X, \varphi)$ *for any* $\overline{Q} \in \overline{\mathcal{Q}}$, $X \in SVar$ *and any pattern* φ.

Proof. Apply Proposition 2.8. □

2.3 Subpatterns

Definition 2.12. *Let φ be a pattern. A subpattern of φ is a pattern ψ that occurs in φ.*

Notation 2.13. *We denote by $SubPattern(\varphi)$ the set of subpatterns of φ.*

Proposition 2.14 (Definition by recursion)**.**
The mapping

$$SubPattern : Pattern \to 2^{Pattern}, \quad \varphi \mapsto SubPattern(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcl}
SubPattern(\varphi) & = & \{\varphi\} \quad \textit{if } \varphi \textit{ is an atomic pattern}, \\
SubPattern(-\varphi) & = & SubPattern(\varphi) \cup \{-\varphi\} \quad \textit{for any } - \in \mathcal{P}_1, \\
SubPattern(\circ\varphi\psi) & = & SubPattern(\varphi) \cup SubPattern(\psi) \cup \{\circ\varphi\psi\} \quad \textit{for any } \circ \in \mathcal{P}_2, \\
SubPattern(\sim \varphi\psi) & = & SubPattern(\varphi) \cup SubPattern(\psi) \cup \{\sim \varphi\psi\} \quad \textit{for any } \sim \in Equal, \\
SubPattern(Qx\varphi) & = & SubPattern(\varphi) \cup \{Qx\varphi\} \quad \textit{for any } Q \in \mathcal{Q} \textit{ and } x \in EVar, \\
SubPattern(\overline{Q}X\varphi) & = & SubPattern(\varphi) \cup \{\overline{Q}X\varphi\} \quad \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = 2^{Pattern}$ and

$$\begin{array}{lcll}
G_0(\varphi) & = & \{\varphi\} & \text{if } \varphi \text{ is an atomic pattern}, \\
G_-(\Gamma, \varphi) & = & \Gamma \cup \{-\varphi\} & \text{for any } - \in \mathcal{P}_1, \\
G_\circ(\Gamma, \Delta, \varphi, \psi) & = & \Gamma \cup \Delta \cup \{\circ\varphi\psi\} & \text{for any } \circ \in \mathcal{P}_2, \\
G_\sim(\Gamma, \Delta, \varphi, \psi) & = & \Gamma \cup \Delta \cup \{\sim \varphi\psi\} & \text{for any } \sim \in Equal, \\
G_Q(\Gamma, x, \varphi) & = & \Gamma \cup \{Qx\varphi\} & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\
G_{\overline{Q}}(\Gamma, X, \varphi) & = & \Gamma \cup \{\overline{Q}X\varphi\} & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar.
\end{array}$$

Then

(i) $SubPattern(\varphi) = \{\varphi\} = G_0(\varphi)$ if φ is an atomic pattern.

(ii) For $- \in \mathcal{P}_1$, we have that

$$SubPattern(-\varphi) = SubPattern(\varphi) \cup \{-\varphi\} = G_-(SubPattern(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$\begin{aligned} SubPattern(\circ\varphi\psi) &= SubPattern(\varphi) \cup SubPattern(\psi) \cup \{\circ\varphi\psi\} \\ &= G_\circ(SubPattern(\varphi), SubPattern(\psi), \varphi, \psi). \end{aligned}$$

(iv) For $\sim \in Equal$, we have that

$$\begin{aligned} SubPattern(\sim \varphi\psi) &= SubPattern(\varphi) \cup SubPattern(\psi) \cup \{\sim \varphi\psi\} \\ &= G_\sim(SubPattern(\varphi), SubPattern(\psi), \varphi, \psi). \end{aligned}$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$SubPattern(Qx\varphi) = SubPattern(\varphi) \cup \{Qx\varphi\} = G_Q(SubPattern(\varphi), x, \varphi).$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$SubPattern(\overline{Q}X\varphi) = SubPattern(\varphi) \cup \{\overline{Q}X\varphi\} = G_{\overline{Q}}(SubPattern(\varphi), X, \varphi).$$

Thus, $SubPattern : Pattern \to 2^{Pattern}$ is the unique mapping given by Proposition 2.11. □

Lemma 2.15. *Let φ, ψ be patterns such that ψ is a subpattern of φ. Then $SubPattern(\psi) \subseteq SubPattern(\varphi)$.*

Proof. Obviously. If $\chi \in SubPattern(\psi)$, then χ is a pattern that occurs in ψ. As ψ occurs in φ, it follows that χ occurs also in φ. Thus, $\chi \in SubPattern(\varphi)$. □

Lemma 2.16. *Let φ, ψ, χ be patterns such that ψ, χ are subpattern of φ. Then one of the following holds:*

(i) (ψ is a subpattern of χ) or (χ is a subpattern of ψ).

(ii) $Occur_\psi(\varphi) \cap Occur_\chi(\varphi) = \emptyset$.

Proof. The proof is by induction on φ.

(i) φ is an atomic pattern. Then we must have $\psi = \chi = \varphi$. The conclusion is obvious.

(ii) $\varphi = -\delta$, where $- \in \mathcal{P}_1$ and δ is a pattern. If $\psi = \varphi$ or $\chi = \varphi$, then (i) holds. Assume that $\psi \neq \varphi$ and $\chi \neq \varphi$. Then ψ, χ are subpatterns of δ. Applying the inductive hypothesis for δ, we get that either (i) holds or $Occur_\psi(\delta) \cap Occur_\chi(\delta) = \emptyset$. It follows that

$$Occur_\psi(\varphi) \cap Occur_\chi(\varphi) = (Occur_\psi(\delta) + 1) \cap (Occur_\chi(\delta) + 1) = \emptyset.$$

(iii) $\varphi = \ominus\delta\theta$, where $\ominus \in \mathcal{P}_2 \cup Equal$ and δ, θ are patterns. If $\psi = \varphi$ or $\chi = \varphi$, then (i) holds. Assume that $\psi \neq \varphi$ and $\chi \neq \varphi$. Then $\psi, \chi \in SubPattern(\delta) \cup SubPattern(\theta)$. We have the following cases:

(a) ψ, χ are subpattern of δ. Applying the inductive hypothesis for δ, we get that either (i) holds or $Occur_\psi(\delta) \cap Occur_\chi(\delta) = \emptyset$. It follows that

$$Occur_\psi(\varphi) \cap Occur_\chi(\varphi) = (Occur_\psi(\delta) + 1) \cap (Occur_\chi(\delta) + 1) = \emptyset.$$

(b) ψ, χ are subpattern of θ. Applying the inductive hypothesis for θ, we get that either(i) holds or $Occur_\psi(\theta) \cap Occur_\chi(\theta) = \emptyset$. It follows that

$$Occur_\psi(\varphi) \cap Occur_\chi(\varphi) = (Occur_\psi(\theta) + \ell(\delta) + 1) \cap (Occur_\chi(\theta) + \ell(\delta) + 1) = \emptyset.$$

(c) ψ is a subpattern of δ and χ is a subpattern of θ. Then $Occur_\psi(\varphi) \subseteq [1, \ell(\delta) - 1]$ and $Occur_\chi(\varphi) \subseteq [\ell(\delta), \ell(\varphi) - 1]$. Thus, (ii) holds.

(iv) $\varphi = \theta\delta$, where δ is a pattern and $\theta \in \{Qx \mid Q \in \mathcal{Q}, x \in EVar\} \cup \{\overline{Q}X \mid \overline{Q} \in \overline{\mathcal{Q}}, X \in SVar\}$. If $\psi = \varphi$ or $\chi = \varphi$, then (i) holds. Assume that $\psi \neq \varphi$ and $\chi \neq \varphi$. Then ψ, χ are subpatterns of δ. Applying the inductive hypothesis for δ, we get that either (i) holds or $Occur_\psi(\delta) \cap Occur_\chi(\delta) = \emptyset$. It follows that

$$Occur_\psi(\varphi) \cap Occur_\chi(\varphi) = (Occur_\psi(\delta) + 2) \cap (Occur_\chi(\delta) + 2) = \emptyset.$$

□

3 $\mathcal{L}$-contexts

Let $\mathcal{L}$ be a language for abstract matching logic and $\Box$ be a new symbol and let us denote

$$Sym_\Box = Sym_\mathcal{L} \cup \{\Box\}.$$

Definition 3.1.
*The set $\mathcal{C}_\mathcal{L}$ of **$\mathcal{L}$-contexts** is the intersection of all sets Γ of expressions over $Sym_\Box$ that have the following properties:*

(i) $\Box \in \Gamma$.

(ii) *For every* $- \in \mathcal{P}_1$,

$$C_\Box \in \Gamma \quad \textit{implies} \quad -C_\Box \in \Gamma.$$

(iii) *For every* $\circ \in \mathcal{P}_2$ *and every* $\mathcal{L}$*-pattern* φ,

$$C_\Box \in \Gamma \quad \textit{implies} \quad \circ C_\Box\varphi, \circ\varphi C_\Box \in \Gamma.$$

(iv) *For every* $\sim \in Equal$ *and every* $\mathcal{L}$*-pattern* φ,

$$C_\Box \in \Gamma \quad \textit{implies} \quad \sim C_\Box\varphi, \sim \varphi C_\Box \in \Gamma.$$

(v) *For every* $Q \in \mathcal{Q}$, $x \in EVar$,

$$C_\Box \in \Gamma \quad \textit{implies} \quad QxC_\Box \in \Gamma.$$

(vi) *For every* $\overline{Q} \in \overline{\mathcal{Q}}$, $X \in SVar$,

$$C_\Box \in \Gamma \quad \textit{implies} \quad \overline{Q}XC_\Box \in \Gamma.$$

Proposition 3.2. *[Induction principle on* $\mathcal{L}$*-contexts]*
Let Γ *be a set of* $\mathcal{L}$*-contexts satisfying (i)-(vi) from Definition 3.1.*
Then $\Gamma = \mathcal{C}_\mathcal{L}$.

Proof. By hypothesis, $\Gamma \subseteq \mathcal{C}_\mathcal{L}$. By Definition 3.1, we get that $\mathcal{C}_\mathcal{L} \subseteq \Gamma$. □

Proposition 3.3 (Unique readability of $\mathcal{L}$-contexts)**.**

(i) *Any* $\mathcal{L}$*-context has a positive length.*

(ii) *If* $C_\Box$ *is an* $\mathcal{L}$*-context, then one of the following hold:*

- *(a)* $C_\Box = \Box$.
- *(b)* $C_\Box = -D_\Box$, *where* $D_\Box$ *is an* $\mathcal{L}$*-context and* $- \in \mathcal{P}_1$.
- *(c)* $C_\Box = \circ D_\Box\varphi$, *where* $D_\Box$ *is an* $\mathcal{L}$*-context,* φ *is an* $\mathcal{L}$*-pattern and* $\circ \in \mathcal{P}_2$.
- *(d)* $C_\Box = \circ\varphi D_\Box$, *where* $D_\Box$ *is an* $\mathcal{L}$*-context,* φ *is an* $\mathcal{L}$*-pattern and* $\circ \in \mathcal{P}_2$.
- *(e)* $C_\Box = \sim D_\Box\varphi$, *where* $D_\Box$ *is an* $\mathcal{L}$*-context,* φ *is an* $\mathcal{L}$*-pattern and* $\sim \in Equal$.
- *(f)* $C_\Box = \sim \varphi D_\Box$, *where* $D_\Box$ *is an* $\mathcal{L}$*-context,* φ *is an* $\mathcal{L}$*-pattern and* $\sim \in Equal$.
- *(g)* $C_\Box = QxD_\Box$, *where* $D_\Box$ *is an* $\mathcal{L}$*-context,* $Q \in \mathcal{Q}$ *and* $x \in EVar$.
- *(h)* $C_\Box = \overline{Q}XD_\Box$, *where* $D_\Box$ *is an* $\mathcal{L}$*-context,* $\overline{Q} \in \overline{\mathcal{Q}}$, $X \in SVar$.

(iii) *Any proper initial segment of an* $\mathcal{L}$*-context is not an* $\mathcal{L}$*-context.*

(iv) *If* $C_\Box$ *is an* $\mathcal{L}$*-context, then exactly one of the cases from (ii) holds. Moreover,* $C_\Box$ *can be written in a unique way in one of these forms.*

Proof. Similarly with the proof of Proposition 2.8. □

Proposition 3.4 (Recursion principle on $\mathcal{L}$-contexts).
Let A be a set, $\Box^A \in A$ and the mappings

$$
\begin{aligned}
G_- &: A \to A \quad \textit{for any } - \in \mathcal{P}_1\\
G^1_\circ &: A \times Pattern \to A \quad \textit{for any } \circ \in \mathcal{P}_2,\\
G^2_\circ &: A \times Pattern \to A \quad \textit{for any } \circ \in \mathcal{P}_2,\\
G^1_\sim &: A \times Pattern \to A \quad \textit{for any } \sim \in Equal,\\
G^2_\sim &: A \times Pattern \to A \quad \textit{for any } \sim \in Equal,\\
G_Q &: A \times EVar \to A \quad \textit{for any } Q \in \mathcal{Q} \textit{ and } x \in EVar,\\
G_{\overline{Q}} &: A \times SVar \to A \quad \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{aligned}
$$

Then there exists a unique mapping

$$F : \mathcal{C}_\mathcal{L} \to A$$

that satisfies the following properties:

(i) $F(\Box) = \Box^A$.

(ii) $F(-C_\Box) = G_-(F(C_\Box))$ *for any $\mathcal{L}$-context $C_\Box$ and any $- \in \mathcal{P}_1$.*

(iii) $F(\circ\varphi C_\Box) = G^1_\circ(F(C_\Box), \varphi)$ *for any $\mathcal{L}$-context $C_\Box$, any $\mathcal{L}$-pattern φ and any $\circ \in \mathcal{P}_2$.*

(iv) $F(\circ C_\Box\varphi) = G^2_\circ(F(C_\Box), \varphi)$ *for any $\mathcal{L}$-context $C_\Box$, any $\mathcal{L}$-pattern φ and any $\circ \in \mathcal{P}_2$.*

(v) $F(\sim \varphi C_\Box) = G^1_\sim(F(C_\Box), \varphi)$ *for any $\mathcal{L}$-context $C_\Box$, any $\mathcal{L}$-pattern φ and any $\sim \in Equal$.*

(vi) $F(\sim C_\Box\varphi) = G^2_\sim(F(C_\Box), \varphi)$ *for any $\mathcal{L}$-context $C_\Box$, any $\mathcal{L}$-pattern φ and any $\sim \in Equal$.*

(vii) $F(QxC_\Box) = G_Q(F(C_\Box), x)$ *for any $\mathcal{L}$-context $C_\Box$, any $Q \in \mathcal{Q}$ and any $x \in EVar$.*

(viii) $F(\overline{Q}XC_\Box) = G_Q(F(C_\Box), X)$ *for any $\mathcal{L}$-context $C_\Box$, any $\overline{Q} \in \overline{\mathcal{Q}}$ and any $X \in SVar$*

Proof. Apply Proposition 3.3. ☐

3.1 ☐ occurs exactly once in every $\mathcal{L}$-context

Proposition 3.5. ☐ *occurs exactly once in every $\mathcal{L}$-context $C_\Box$.*

Proof. The proof is by induction on the context $C_\Box$:

(i) $C_\Box = \Box$. Obviously.

(ii) $C_\Box = -D_\Box$, where $D_\Box$ is an $\mathcal{L}$-context and $- \in \mathcal{P}_1$. By the induction hypothesis, ☐ occurs exactly once in $D_\Box$. Obviously, ☐ occurs exactly once in $C_\Box$.

(iii) $C_\Box = \ominus D_\Box\varphi$, where $D_\Box$ is an $\mathcal{L}$-context, φ is an $\mathcal{L}$-pattern and $\circ \in \mathcal{P}_2 \cup Equal$. By the induction hypothesis, ☐ occurs exactly once in $D_\Box$. As φ is an $\mathcal{L}$-pattern, ☐ does not occur in φ. Thus, ☐ occurs exactly once in $C_\Box$.

(iv) $C_\Box = \ominus\varphi D_\Box$, where $D_\Box$ is an $\mathcal{L}$-context, φ is an $\mathcal{L}$-pattern and $\circ \in \mathcal{P}_2 \cup Equal$. By the induction hypothesis, ☐ occurs exactly once in $D_\Box$. As φ is an $\mathcal{L}$-pattern, ☐ does not occur in φ. Thus, ☐ occurs exactly once in $C_\Box$.

(v) $C_\Box = QxD_\Box$, where $D_\Box$ is an $\mathcal{L}$-context, $Q \in \mathcal{Q}$ and $x \in EVar$. By the induction hypothesis, ☐ occurs exactly once in $D_\Box$. Obviously, ☐ occurs exactly once in $C_\Box$.

(vi) $C_\Box = \overline{Q}XD_\Box$, where $D_\Box$ is an $\mathcal{L}$-context, $\overline{Q} \in \overline{\mathcal{Q}}$, $X \in SVar$. By the induction hypothesis, ☐ occurs exactly once in $D_\Box$. Obviously, ☐ occurs exactly once in $C_\Box$.

☐

3.2 Replacement in an $\mathcal{L}$-context

Definition 3.6. *Let $C_\Box$ be an $\mathcal{L}$-context and δ be an $\mathcal{L}$-pattern. We denote by $C_\Box[\delta]$ the $\mathcal{L}$-expression obtained by replacing the unique occurrence of $\Box$ with δ.*

Remark 3.7. $C_\Box[\delta] = Replall^{\Box}_{\delta}(C_\Box)$.

Proposition 3.8 (Definition by recursion)**.**
$C_\Box[\delta]$ can be defined by recursion on $\mathcal{L}$-contexts as follows:

$$\begin{array}{lcll} \Box[\delta] & = & \delta, & \\ (-C_\Box)[\delta] & = & -C_\Box[\delta] & \text{for any } - \in \mathcal{P}_1 \\ (\circ C_\Box\varphi)[\delta] & = & \circ C_\Box[\delta]\varphi & \text{for any } \circ \in \mathcal{P}_2, \\ (\circ\varphi C_\Box)[\delta] & = & \circ\varphi C_\Box[\delta] & \text{for any } \circ \in \mathcal{P}_2, \\ (\sim C_\Box\varphi)[\delta] & = & \sim C_\Box[\delta]\varphi & \text{for any } \sim\in Equal, \\ (\sim \varphi C_\Box)[\delta] & = & \sim \varphi C_\Box[\delta] & \text{for any } \sim\in Equal, \\ (QxC_\Box)[\delta] & = & QxC_\Box[\delta] & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\ (\overline{Q}XC_\Box)[\delta] & = & \overline{Q}XC_\Box[\delta] & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar. \end{array}$$

Proof. Let

$$F : \mathcal{C}_{\mathcal{L}} \to Fun(Pattern_{\mathcal{L}}, Expr_{\mathcal{L}}), \quad F(C_\Box)(\delta) = C_\Box[\delta].$$

Apply Proposition 3.4 with $A = Fun(Pattern_{\mathcal{L}}, Expr_{\mathcal{L}})$, $\Box^A(\delta) = \delta$ and, for any $f \in A$, $\delta \in Pattern_{\mathcal{L}}$

$$\begin{array}{lcll} G_-(f)(\delta) & = & -f(\delta) & \text{for any } - \in \mathcal{P}_1, \\ G^1_\circ(f,\varphi)(\delta) & = & \circ f(\delta)\varphi & \text{for any } \circ \in \mathcal{P}_2 \text{ and } \varphi \in Pattern_{\mathcal{L}}, \\ G^2_\circ(f,\varphi)(\delta) & = & \circ\varphi f(\delta) & \text{for any } \circ \in \mathcal{P}_2 \text{ and } \varphi \in Pattern_{\mathcal{L}}, \\ G^1_\sim(f,\varphi)(\delta) & = & \sim f(\delta)\varphi & \text{for any } \sim\in Equal \text{ and } \varphi \in Pattern_{\mathcal{L}}, \\ G^2_\sim(f,\varphi)(\delta) & = & \sim \varphi f(\delta) & \text{for any } \sim\in Equal \text{ and } \varphi \in Pattern_{\mathcal{L}}, \\ G_Q(f,x)(\delta) & = & Qxf(\delta) & \text{for any } Q \in \mathcal{Q},\ x \in EVar, \\ G_{\overline{Q}}(f,X)(\delta) & = & \overline{Q}Xf(\delta) & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar. \end{array}$$

Then

(i) $F(\Box)(\delta) = \delta = \Box^A(\delta)$ for every $\mathcal{L}$-pattern δ. It follows that $F(\Box) = \Box^A$.

(ii) For $- \in \mathcal{P}_1$, we have that for any $\mathcal{L}$-context $C_\Box$ and any $\mathcal{L}$-pattern δ,

$$F(-C_\Box)(\delta) = (-C_\Box)[\delta] = -C_\Box[\delta] = -F(C_\Box)(\delta) = G_-(F(C_\Box))(\delta).$$

It follows that $F(-C_\Box) = G_-(F(C_\Box))$.

(iii) For $\circ \in \mathcal{P}_2$, we have that for any $\mathcal{L}$-context $C_\Box$ and any $\mathcal{L}$-patterns φ, δ,

$$F(\circ C_\Box\varphi)(\delta) = (\circ C_\Box\varphi)[\delta] = \circ C_\Box[\delta]\varphi = \circ F(C_\Box)(\delta)\varphi = G^1_\circ(F(C_\Box),\varphi)(\delta),$$
$$F(\circ\varphi C_\Box)(\delta) = (\circ\varphi C_\Box)[\delta] = \circ\varphi C_\Box[\delta] = \circ\varphi F(C_\Box)(\delta) = G^2_\circ(F(C_\Box),\varphi)(\delta).$$

It follows that for any $\mathcal{L}$-context $C_\Box$ and any $\mathcal{L}$-pattern φ, $F(\circ C_\Box\varphi) = G^1_\circ(F(C_\Box),\varphi)$ and $F(\circ\varphi C_\Box) = G^2_\circ(F(C_\Box),\varphi)$.

(iv) For $\sim\in Equal$, we have that for any $\mathcal{L}$-context $C_\Box$ and any $\mathcal{L}$-patterns φ, δ,

$$F(\sim C_\Box\varphi)(\delta) = (\sim C_\Box\varphi)[\delta] =\sim C_\Box[\delta]\varphi =\sim F(C_\Box)(\delta)\varphi = G^1_\sim(F(C_\Box),\varphi)(\delta),$$
$$F(\sim \varphi C_\Box)(\delta) = (\sim \varphi C_\Box)[\delta] =\sim \varphi C_\Box[\delta] =\sim \varphi F(C_\Box)(\delta) = G^2_\sim(F(C_\Box),\varphi)(\delta).$$

It follows that for any $\mathcal{L}$-context $C_\Box$ and any $\mathcal{L}$-pattern φ, $F(\sim C_\Box\varphi) = G^1_\sim(F(C_\Box),\varphi)$ and $F(\sim \varphi C_\Box) = G^2_\sim(F(C_\Box),\varphi)$.

(v) For $Q \in \mathcal{Q}$, we have that for any $x \in EVar$, any $\mathcal{L}$-context $C_\Box$ and any $\mathcal{L}$-pattern δ,

$$F(QxC_\Box)(\delta) = (QxC_\Box)[\delta] = QxC_\Box[\delta] = QxF(C_\Box)(\delta) = G_Q(F(C_\Box), x)(\delta).$$

It follows that for any $x \in EVar$ and any $\mathcal{L}$-context $C_\Box$, $F(QxC_\Box) = G_Q(F(C_\Box), x)$.

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$, we have that for any $X \in SVar$, any $\mathcal{L}$-context $C_\Box$ and any $\mathcal{L}$-pattern δ,

$$F(\overline{Q}XC_\Box)(\delta) = (\overline{Q}XC_\Box)[\delta] = \overline{Q}XC_\Box[\delta] = \overline{Q}XF(C_\Box)(\delta) = G_{\overline{Q}}(F(C_\Box), X)(\delta).$$

It follows that for any $X \in SVar$ and any $\mathcal{L}$-context $C_\Box$, $F(\overline{Q}XC_\Box) = G_{\overline{Q}}(F(C_\Box), X)$.

Thus, F is the unique mapping given by Proposition 3.4. □

Proposition 3.9. *For any $\mathcal{L}$-context $C_\Box$ and any $\mathcal{L}$-pattern δ, $C_\Box[\delta]$ is an $\mathcal{L}$-pattern.*

Proof. The proof is by induction on $\mathcal{L}$-context $C_\Box$, using Proposition 3.8. □

Remark 3.10. *Let $C_\Box$, $D_\Box$ be $\mathcal{L}$-contexts such that $Occur_\Box(C_\Box) = Occur_\Box(D_\Box)$. If $C_\Box[\delta] = D_\Box[\delta]$ for some $\mathcal{L}$-pattern δ, then $C_\Box = D_\Box$.*

Proof. Apply Remark 3.7 and Lemma A.11. □

Proposition 3.11. *Let φ, δ be $\mathcal{L}$-patterns such that δ is a subpattern of φ.*

(i) For any occurrence of δ in φ, there exists an $\mathcal{L}$-context $C_\Box$ such that $C_\Box[\delta] = \varphi$.

(ii) If k is an occurrence of δ in φ and $C_\Box$ is as in (i), then $\Box$ occurs uniquely at place k of $C_\Box$ and $C_\Box = Repl^{\delta}_{\Box}(\varphi; \{k\})$. Thus, $C_\Box$ is unique satisfying (i).

Proof. (i) If $\delta = \varphi$, then we can take $C_\Box = \Box$. Assume in the sequel that $\delta \neq \varphi$. The proof is by induction on φ, using the definition by recursion of subpatterns (Proposition 2.14).

(a) φ is an atomic $\mathcal{L}$-pattern. Obviously, as the unique subpattern is φ.

(b) $\varphi = -\psi$, where ψ is an $\mathcal{L}$-pattern and $- \in \mathcal{P}_1$. As $\delta \neq \varphi$, we have that the occurrence of δ in φ is an occurrence of δ in ψ. Apply the induction hypothesis for ψ to get the $\mathcal{L}$-context $D_\Box$ such that $D_\Box[\delta] = \psi$. Take $C_\Box := -D_\Box$. Then

$$C_\Box[\delta] = -D_\Box[\delta] = -\psi = \varphi.$$

(c) $\varphi = \ominus\psi\chi$, where ψ, χ are $\mathcal{L}$-patterns and $\ominus \in \mathcal{P}_2 \cup Equal$. As $\delta \neq \varphi$, we have two cases:

(1) The occurrence of δ in φ is an occurrence of δ in ψ. Apply the induction hypothesis for ψ to get the $\mathcal{L}$-context $D_\Box$ such that $D_\Box[\delta] = \psi$. Take $C_\Box := \ominus D_\Box\chi$. Then

$$C_\Box[\delta] = \ominus D_\Box[\delta]\chi = \ominus\psi\chi = \varphi.$$

(2) The occurrence of δ in φ is an occurrence of δ in χ. Apply the induction hypothesis for χ to get the $\mathcal{L}$-context $D_\Box$ such that $D_\Box[\delta] = \chi$. Take $C_\Box := \ominus\psi D_\Box$. Then

$$C_\Box[\delta] = \ominus\psi D_\Box[\delta] = \ominus\psi\chi = \varphi.$$

(d) $\varphi = Qx\psi$, where ψ is an $\mathcal{L}$-pattern, $Q \in \mathcal{Q}$ and $x \in EVar$. As $\delta \neq \varphi$, we have that the occurrence of δ in φ is an occurrence of δ in ψ. Apply the induction hypothesis for ψ to get the $\mathcal{L}$-context $D_\Box$ such that $D_\Box[\delta] = \psi$. Take $C_\Box := QxD_\Box$. Then

$$C_\Box[\delta] = QxD_\Box[\delta] = Qx\psi = \varphi.$$

(e) $\varphi = \overline{Q}X\psi$, where ψ is an $\mathcal{L}$-pattern, $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$. As $\delta \neq \varphi$, we have that the occurrence of δ in φ is an occurrence of δ in ψ. Apply the induction hypothesis for ψ to get the $\mathcal{L}$-context $D_\Box$ such that $D_\Box[\delta] = \psi$. Take $C_\Box := \overline{Q}XD_\Box$. Then

$$C_\Box[\delta] = \overline{Q}XD_\Box[\delta] = \overline{Q}X\psi = \varphi.$$

(ii) By (i) and Remark 3.7, we have that

$$\varphi = C_\Box[\delta] = Replall_\delta^\Box(C_\Box) = Repl_\delta^\Box(C_\Box; \{k\}).$$

Apply Lemma A.13 with $\mathbf{a} := C_\Box$, $\mathbf{b} := \Box$, $\mathbf{c} := \delta$, $\mathbf{a}^* := \varphi$ and k to get that

$$C_\Box = Repl_\Box^\delta(\varphi; \{k\}).$$

□

4 $\mathcal{L}$-congruences

Let $\mathcal{L}$ be a language and **Cong** be a binary relation on the set $Pattern_\mathcal{L}$ of $\mathcal{L}$-patterns.

Definition 4.1. *We say that* ***Cong*** *is an* ***$\mathcal{L}$-congruence*** *if it satisfies the following conditions:*

(i) ***Cong*** *is* ***compatible*** *with $\mathcal{P}_1$, that is for any $- \in \mathcal{P}_1$ and any $\mathcal{L}$-patterns φ_1, φ_2,*

$$\varphi_1 \, \textbf{Cong} \, \varphi_2 \quad \textit{implies} \quad -\varphi_1 \, \textbf{Cong} -\varphi_2.$$

(ii) ***Cong*** *is* ***compatible*** *with $\mathcal{P}_2$, that is for any $\circ \in \mathcal{P}_2$ and any $\mathcal{L}$-patterns φ_1,φ_2, ψ,*

$$\varphi_1 \, \textbf{Cong} \, \varphi_2 \quad \textit{implies} \quad \circ\varphi_1\psi \, \textbf{Cong} \circ\varphi_2\psi \textit{ and } \circ\psi\varphi_1 \, \textbf{Cong} \circ\psi\varphi_2.$$

(iii) ***Cong*** *is* ***compatible*** *with Equal, that is for any $\sim \in Equal$ and any $\mathcal{L}$-patterns φ_1,φ_2, ψ,*

$$\varphi_1 \, \textbf{Cong} \, \varphi_2 \quad \textit{implies} \quad \sim \varphi_1\psi \, \textbf{Cong} \sim \varphi_2\psi \textit{ and } \sim \psi\varphi_1 \, \textbf{Cong} \sim \psi\varphi_2.$$

(iv) ***Cong*** *is* ***compatible*** *with $\mathcal{Q}$, that is for every $Q \in \mathcal{Q}$, $x \in EVar$ and any $\mathcal{L}$-patterns φ_1, φ_2,*

$$\varphi_1 \, \textbf{Cong} \, \varphi_2 \quad \textit{implies} \quad Qx\varphi_1 \, \textbf{Cong} \, Qx\varphi_2.$$

(v) ***Cong*** *is* ***compatible*** *with $\overline{\mathcal{Q}}$, that is for every $\overline{Q} \in \overline{\mathcal{Q}}$, $X \in SVar$ and any $\mathcal{L}$-patterns φ_1, φ_2,*

$$\varphi_1 \, \textbf{Cong} \, \varphi_2 \quad \textit{implies} \quad \overline{Q}X\varphi_1 \, \textbf{Cong} \overline{Q}X\varphi_2.$$

Lemma 4.2.
Let ***Cong*** *be a transitive $\mathcal{L}$-congruence. Then for any $\ominus \in \mathcal{P}_2 \cup Equal$ and any $\mathcal{L}$-patterns φ_1,φ_2, ψ_1,*

$$\varphi_1 \, \textbf{Cong} \, \varphi_2 \textit{ and } \psi_1 \, \textbf{Cong} \, \psi_2 \quad \textit{imply} \quad \ominus\varphi_1\psi_1 \, \textbf{Cong} \ominus\varphi_2\psi_2.$$

Proof. Assume that φ_1 **Cong** φ_2 and ψ_1 **Cong** ψ_2. It follows that

$$\ominus\varphi_1\psi_1 \, \mathbf{Cong} \ominus\varphi_2\psi_1 \text{ and } \ominus\varphi_2\psi_1 \, \mathbf{Cong} \ominus\varphi_2\psi_2.$$

Since **Cong** is transitive, it follows that $\ominus\varphi_1\psi_1 \, \mathbf{Cong} \ominus\varphi_2\psi_2$. □

5 Replacement theorems

Let **Cong** be an $\mathcal{L}$-congruence.

Theorem 5.1 (Replacement Theorem for $\mathcal{L}$-contexts)**.**
For any $\mathcal{L}$-context $C_\square$ and any $\mathcal{L}$-patterns φ, ψ,

$$\varphi\, \textbf{\textit{Cong}}\, \psi \text{ implies } C_\square[\varphi]\, \textbf{\textit{Cong}}\, C_\square[\psi].$$

Proof. The proof is by induction on the $\mathcal{L}$-context $C_\square$. Let φ, ψ be $\mathcal{L}$-patterns such that $\varphi\,\mathbf{Cong}\,\psi$.

(i) $C_\square = \square$. Then $C_\square[\varphi] = \varphi$ and $C_\square[\psi] = \psi$.
By hypothesis, $\varphi\,\mathbf{Cong}\,\psi$, hence $C_\square[\varphi]\,\mathbf{Cong}\,C_\square[\psi]$.

(ii) $C_\square = -D_\square$, where $D_\square$ is an $\mathcal{L}$-context and $- \in \mathcal{P}_1$. By the induction hypothesis, we have that $D_\square[\varphi]\,\mathbf{Cong}\,D_\square[\psi]$. As **Cong** is compatible with $\mathcal{P}_1$, we get that $-D_\square[\varphi]\,\mathbf{Cong}\,-D_\square[\psi]$, that is $C_\square[\varphi]\,\mathbf{Cong}\,C_\square[\psi]$.

(iii) $C_\square = \ominus D_\square \chi$, where $D_\square$ is an $\mathcal{L}$-context, χ is an $\mathcal{L}$-pattern and $\ominus \in \mathcal{P}_2 \cup Equal$. By the induction hypothesis, we have that $D_\square[\varphi]\,\mathbf{Cong}\,D_\square[\psi]$. As **Cong** is compatible with $\mathcal{P}_2$ and $Equal$, we get that $\ominus D_\square[\varphi]\chi\,\mathbf{Cong}\,\ominus D_\square[\psi]\chi$, that is $C_\square[\varphi]\,\mathbf{Cong}\,C_\square[\psi]$.

(iv) $C_\square = \ominus \chi D_\square$, where $D_\square$ is an $\mathcal{L}$-context, χ is an $\mathcal{L}$-pattern and $\ominus \in \mathcal{P}_2 \cup Equal$. By the induction hypothesis, we have that $D_\square[\varphi]\,\mathbf{Cong}\,D_\square[\psi]$. As **Cong** is compatible with $\mathcal{P}_2$ and $Equal$, we get that $\ominus \chi D_\square[\varphi]\,\mathbf{Cong}\,\ominus \chi D_\square[\psi]$, that is $C_\square[\varphi]\,\mathbf{Cong}\,C_\square[\psi]$.

(v) $C_\square = QxD_\square$, where $D_\square$ is an $\mathcal{L}$-context, $Q \in \mathcal{Q}$ and $x \in EVar$. By the induction hypothesis, we have that $D_\square[\varphi]\,\mathbf{Cong}\,D_\square[\psi]$. As **Cong** is compatible with $\mathcal{Q}$, we get that $QxD_\square[\varphi]\,\mathbf{Cong}\,QxD_\square[\psi]$, that is $C_\square[\varphi]\,\mathbf{Cong}\,C_\square[\psi]$.

(vi) $C_\square = \overline{Q}XD_\square$, where $D_\square$ is an $\mathcal{L}$-context, $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$. By the induction hypothesis, we have that $D_\square[\varphi]\,\mathbf{Cong}\,D_\square[\psi]$. As **Cong** is compatible with $\overline{\mathcal{Q}}$, we get that $\overline{Q}XD_\square[\varphi]\,\mathbf{Cong}\,\overline{Q}XD_\square[\psi]$, that is $C_\square[\varphi]\,\mathbf{Cong}\,C_\square[\psi]$.

□

Theorem 5.2 (Replacement Theorem - one occurrence)**.**
Let $\varphi, \psi, \chi, \theta$ be $\mathcal{L}$-patterns such that φ is a subpattern of χ and θ is obtained from χ by replacing an occurrence of φ with ψ. Then θ is an $\mathcal{L}$-pattern and, moreover,

$$\varphi\, \textbf{\textit{Cong}}\, \psi \text{ implies } \chi\, \textbf{\textit{Cong}}\, \theta.$$

Proof. Asume that k is the occurrence of φ in χ that is replaced with ψ. Then

$$\theta = Repl^{\varphi}_{\psi}(\chi; \{k\}).$$

By Proposition 3.11, there exists an $\mathcal{L}$-context $C_\square$ such that $\square$ occurs uniquely at place k of $C_\square$ and

$$\chi = C_\square[\varphi] = Replall^{\square}_{\varphi}(C_\square) = Repl^{\square}_{\varphi}(C_\square; \{k\}).$$

Thus,

$$\theta = Repl^{\varphi}_{\psi}(Repl^{\square}_{\varphi}(C_\square; \{k\}); \{k\}).$$

Apply Lemma A.12 with $\mathbf{a} := C_\square$, $\mathbf{b} := \square$, $\mathbf{c} := \varphi$, $\mathbf{d} := \psi$ and k to get that

$$\theta = Repl^{\square}_{\psi}(C_\square; \{k\}) = Replall^{\square}_{\psi}(C_\square) = C_\square[\psi].$$

It follows, by Proposition 3.9, that θ is an $\mathcal{L}$-pattern and, by Theorem 5.1, that $\chi\,\mathbf{Cong}\,\theta$. □

Theorem 5.3 (Replacement Theorem)**.**
Let $\varphi, \psi, \chi, \theta$ be $\mathcal{L}$-patterns such that φ is a subpattern of χ and θ is obtained from χ by replacing one or more occurrences of φ with ψ. Then θ is an $\mathcal{L}$-pattern and

$$\varphi\, \textbf{\textit{Cong}}\, \psi \text{ implies } \chi\, \textbf{\textit{Cong}}\, \theta.$$

Proof. It follows immediately from Theorem 5.2 by induction on the number of occurrences of φ in χ that are replaced with ψ. □

6 Free, bound, fresh element variables

Assume that $EVar \neq \emptyset$ and $\mathcal{Q} \neq \emptyset$.

Definition 6.1. *Let $\varphi = \varphi_0\varphi_1 \ldots \varphi_{n-1}$ be a pattern and $x \in EVar$.*

*(i) We say that $Q \in \mathcal{Q}$ is a **quantifier on** x **at place** i **of** φ **with scope** ψ if $\varphi_i = Q$, $\varphi_{i+1} = x$ and $\psi = \varphi_i \ldots \varphi_j$ is the unique pattern given by Proposition 2.9.*

*(ii) We say that x **occurs bound at place** k **of** φ if $\varphi_k = x$ and there exist $Q \in \mathcal{Q}$ and $i \leq j \leq n-1$ such that $k \in (i, j]$ and Q is a quantifier on x at place i of φ with scope $\psi = \varphi_i \ldots \varphi_j$.*

*(iii) If $\varphi_k = x$ but x does not occur bound at place k of φ, we say that x **occurs free at place** k **of** φ.*

*(iv) x is a **bound variable** of φ (or that x **occurs bound** in φ) if there exists k such that x occurs bound at place k of of φ.*

*(v) x is a **free variable** of φ (or that x **occurs free** in φ) if there exists k such that x occurs free at place k of φ.*

*(vi) A **bound occurrence** of x in φ is a $k \in [0, n-1]$ such that x occurs bound at place k in φ.*

*(vii) A **free occurrence** of x in φ is a $k \in [0, n-1]$ such that x occurs free at place k in φ.*

Notation 6.2. *Let us denote, for every pattern φ,*

$$
\begin{aligned}
FV(\varphi) &= \{x \in EVar(\varphi) \mid x \text{ is free in } \varphi\},\\
BV(\varphi) &= \{x \in EVar(\varphi) \mid x \text{ is bound in } \varphi\},\\
NotFV(\varphi) &= EVar(\varphi) \setminus FV(\varphi),\\
NotBV(\varphi) &= EVar(\varphi) \setminus BV(\varphi),\\
FreshF(\varphi) &= EVar \setminus FV(\varphi),\\
FreshB(\varphi) &= EVar \setminus BV(\varphi),
\end{aligned}
$$

Lemma 6.3. *(i) $EVar(\varphi) = FV(\varphi) \cup BV(\varphi)$.*

(ii) $FV(\varphi)$ and $BV(\varphi)$ are not disjoint, in general, as $x \in EVar$ can be both free and bound in a pattern φ.

(iii) $NotFV(\varphi) \subseteq BV(\varphi)$.

(iv) $FreshF(\varphi) = (EVar \setminus EVar(\varphi)) \cup NotFV(\varphi)$.

(v) $FreshB(\varphi) = (EVar \setminus EVar(\varphi)) \cup NotBV(\varphi)$.

Proof. We have that

$$
\begin{aligned}
FreshF(\varphi) = EVar \setminus FV(\varphi) &= \Big(EVar(\varphi) \cup \big(EVar \setminus EVar(\varphi)\big)\Big) \setminus FV(\varphi)\\
&\overset{(16)}{=} \big(EVar \setminus EVar(\varphi)\big) \cup \big(EVar(\varphi) \setminus FV(\varphi)\big)\\
&= (EVar \setminus EVar(\varphi)) \cup NotFV(\varphi),\\
FreshB(\varphi) = EVar \setminus BV(\varphi) &= \Big(EVar(\varphi) \cup \big(EVar \setminus EVar(\varphi)\big)\Big) \setminus BV(\varphi)\\
&\overset{(16)}{=} \big(EVar \setminus EVar(\varphi)\big) \cup \big(EVar(\varphi) \setminus BV(\varphi)\big)\\
&= (EVar \setminus EVar(\varphi)) \cup NotBV(\varphi).
\end{aligned}
$$

□

Notation 6.4. *Let us denote, for every pattern φ and element variable x,*

$$\begin{array}{lcl} FreeOccur_x(\varphi) & = & \text{the set of all free occurrences of } x \text{ in } \varphi, \\ BoundOccur_x(\varphi) & = & \text{the set of all bound occurrences of } x \text{ in } \varphi. \end{array}$$

The following lemma contains some obvious useful properties.

Lemma 6.5. *(i) $x \in FV(\varphi)$ iff $FreeOccur_x(\varphi) \neq \emptyset$.*

(ii) $x \in FreshF(\varphi)$ iff $FreeOccur_x(\varphi) = \emptyset$.

(iii) $x \in BV(\varphi)$ iff $BoundOccur_x(\varphi) \neq \emptyset$.

(iv) $x \in FreshB(\varphi)$ iff $BoundOccur_x(\varphi) = \emptyset$.

(v) $FreeOccur_x(\varphi) \cap BoundOccur_x(\varphi) = \emptyset$.

(vi) $Occur_x(\varphi) = FreeOccur_x(\varphi) \cup BoundOccur_x(\varphi)$.

(vii) $FreeOccur_x(\varphi) = Occur_x(\varphi) \setminus BoundOccur_x(\varphi)$ and $BoundOccur_x(\varphi) = Occur_x(\varphi) \setminus FreeOccur_x(\varphi)$.

Proposition 6.6 (Definition by recursion of FV)**.**
The mapping

$$FV : Pattern \to 2^{EVar}, \quad \varphi \mapsto FV(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll} FV(\varphi) & = & EVar(\varphi) & \text{if } \varphi \text{ is an atomic pattern}, \\ FV(-\varphi) & = & FV(\varphi) & \text{for any } - \in \mathcal{P}_1, \\ FV(\circ\varphi\psi) & = & FV(\varphi) \cup FV(\psi) & \text{for any } \circ \in \mathcal{P}_2, \\ FV(\sim\varphi\psi) & = & FV(\varphi) \cup FV(\psi) & \text{for any } \sim \in Equal, \\ FV(Qx\varphi) & = & FV(\varphi) \setminus \{x\} & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\ FV(\overline{Q}X\varphi) & = & FV(\varphi) & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar. \end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = 2^{EVar}$ and

$$\begin{array}{lcll} G_0(\varphi) & = & EVar(\varphi), & \\ G_-(V, \varphi) & = & V & \text{for any } - \in \mathcal{P}_1, \\ G_\circ(V_1, V_2, \varphi, \psi) & = & V_1 \cup V_2 & \text{for any } \circ \in \mathcal{P}_2, \\ G_\sim(V_1, V_2, \varphi, \psi) & = & V_1 \cup V_2 & \text{for any } \sim \in Equal, \\ G_Q(V, x, \varphi) & = & V \setminus \{x\} & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\ G_{\overline{Q}}(V, X, \varphi) & = & V & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar. \end{array}$$

Then

(i) $FV(\varphi) = EVar(\varphi) = G_0(\varphi)$ if φ is an atomic pattern.

(ii) For $- \in \mathcal{P}_1$, we have that

$$FV(-\varphi) = FV(\varphi) = G_-(FV(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$FV(\circ\varphi\psi) = FV(\varphi) \cup FV(\psi) = G_\circ(FV(\varphi), FV(\psi), \varphi, \psi).$$

(iv) For $\sim \in Equal$, we have that

$$FV(\sim \varphi\psi) = FV(\varphi) \cup FV(\psi) = G_{\sim}(FV(\varphi), FV(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$FV(Qx\varphi) = FV(\varphi) \setminus \{x\} = G_Q(FV(\varphi), x, \varphi).$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$FV(\overline{Q}X\varphi) = FV(\varphi) = G_{\overline{Q}}(FV(\varphi), X, \varphi).$$

Thus, $FV : Pattern \to 2^{EVar}$ is the unique mapping given by Proposition 2.11. □

Remark 6.7 (Definition by recursion of BV)**.**
The mapping

$$BV : Pattern \to 2^{EVar}, \quad \varphi \mapsto BV(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
BV(\varphi) & = & \emptyset & \textit{if } \varphi \textit{ is an atomic pattern,} \\
BV(-\varphi) & = & BV(\varphi) & \textit{for any } - \in \mathcal{P}_1, \\
BV(\circ\varphi\psi) & = & BV(\varphi) \cup BV(\psi) & \textit{for any } \circ \in \mathcal{P}_2, \\
BV(\sim \varphi\psi) & = & BV(\varphi) \cup BV(\psi) & \textit{for any } \sim \in Equal, \\
BV(Qx\varphi) & = & BV(\varphi) \cup \{x\} & \textit{for any } Q \in \mathcal{Q} \textit{ and } x \in EVar, \\
BV(\overline{Q}X\varphi) & = & BV(\varphi) & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = 2^{EVar}$ and

$$\begin{array}{lcll}
G_0(\varphi) & = & \emptyset, & \\
G_-(V, \varphi) & = & V & \text{for any } - \in \mathcal{P}_1, \\
G_\circ(V_1, V_2, \varphi, \psi) & = & V_1 \cup V_2 & \text{for any } \circ \in \mathcal{P}_2, \\
G_\sim(V_1, V_2, \varphi, \psi) & = & V_1 \cup V_2 & \text{for any } \sim \in Equal, \\
G_Q(V, x, \varphi) & = & V \cup \{x\} & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\
G_{\overline{Q}}(V, X, \varphi) & = & V & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar.
\end{array}$$

Then

(i) $BV(\varphi) = \emptyset = G_0(\varphi)$ if φ is an atomic pattern.

(ii) For $- \in \mathcal{P}_1$, we have that

$$BV(-\varphi) = BV(\varphi) = G_-(BV(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$BV(\circ\varphi\psi) = BV(\varphi) \cup BV(\psi) = G_\circ(BV(\varphi), BV(\psi), \varphi, \psi).$$

(iv) For $\sim \in Equal$, we have that

$$BV(\sim \varphi\psi) = BV(\varphi) \cup BV(\psi) = G_\sim(BV(\varphi), BV(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$BV(Qx\varphi) = BV(\varphi) \cup \{x\} = G_Q(BV(\varphi), x, \varphi).$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$BV(\overline{Q}X\varphi) = BV(\varphi) = G_{\overline{Q}}(BV(\varphi), X, \varphi).$$

Thus, $BV : Pattern \to 2^{EVar}$ is the unique mapping given by Proposition 2.11. □

Remark 6.8 (Definition by recursion of $NotFV$)**.**
The mapping

$$NotFV : Pattern \to 2^{EVar}, \quad \varphi \mapsto NotFV(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
NotFV(\varphi) & = & \emptyset & \textit{if } \varphi \textit{ is an atomic pattern,} \\
NotFV(-\varphi) & = & NotFV(\varphi) & \textit{for any } - \in \mathcal{P}_1, \\
NotFV(\circ\varphi\psi) & = & NotFV(\varphi) \cap NotFV(\psi) & \textit{for any } \circ \in \mathcal{P}_2, \\
NotFV(\sim \varphi\psi) & = & NotFV(\varphi) \cap NotFV(\psi) & \textit{for any } \sim \in Equal, \\
NotFV(Qx\varphi) & = & NotFV(\varphi) \cup \{x\} & \textit{for any } Q \in \mathcal{Q} \textit{ and } x \in EVar, \\
NotFV(\overline{Q}X\varphi) & = & NotFV(\varphi) & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = 2^{EVar}$ and

$$\begin{array}{lcll}
G_0(\varphi) & = & \emptyset, & \\
G_-(V, \varphi) & = & V & \text{for any } - \in \mathcal{P}_1, \\
G_\circ(V_1, V_2, \varphi, \psi) & = & V_1 \cap V_2 & \text{for any } \circ \in \mathcal{P}_2, \\
G_\sim(V_1, V_2, \varphi, \psi) & = & V_1 \cap V_2 & \text{for any } \sim \in Equal, \\
G_Q(V, x, \varphi) & = & V \cup \{x\} & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\
G_{\overline{Q}}(V, X, \varphi) & = & V & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar.
\end{array}$$

Then

(i) $NotFV(\varphi) = \emptyset = G_0(\varphi)$ if φ is an atomic pattern.

(ii) For $- \in \mathcal{P}_1$, we have that

$$NotFV(-\varphi) = NotFV(\varphi) = G_-(NotFV(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$NotFV(\circ\varphi\psi) = NotFV(\varphi) \cap NotFV(\psi) = G_\circ(NotFV(\varphi), NotFV(\psi), \varphi, \psi).$$

(iv) For $\sim \in Equal$, we have that

$$NotFV(\sim \varphi\psi) = NotFV(\varphi) \cap NotFV(\psi) = G_\sim(NotFV(\varphi), NotFV(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$\begin{aligned}
NotFV(Qx\varphi) = EVar(Qx\varphi) \setminus FV(Qx\varphi) &= \Big(EVar(\varphi) \cup \{x\}\Big) \setminus \Big(FV(\varphi) \setminus \{x\}\Big) \\
&\overset{(14)}{=} \Big(\big(EVar(\varphi) \cup \{x\}\big) \setminus FV(\varphi)\Big) \cup \{x\} \\
&\overset{(15)}{=} \big(EVar(\varphi) \setminus FV(\varphi)\big) \cup \{x\} = NotFV(\varphi) \cup \{x\} = G_Q(NotFV(\varphi), x, \varphi)
\end{aligned}$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$NotFV(\overline{Q}X\varphi) = NotFV(\varphi) = G_{\overline{Q}}(NotFV(\varphi), X, \varphi).$$

Thus, $NotFV : Pattern \to 2^{EVar}$ is the unique mapping given by Proposition 2.11. $\square$

Lemma 6.9. *Let φ be a pattern and $x \in EVar$ such that x occurs bound in φ. Then there exists a subpattern $Qx\psi$ (with $Q \in \mathcal{Q}$) of φ such that x does not occur bound in ψ.*

Proof. The proof is by induction on φ, using the definition by recursion of subpatterns (Proposition 2.14) and the definition by recursion of bounded variables (Proposition 6.7).

(i) φ is an atomic pattern. Then x does not occur bound in φ.

(ii) $\varphi = -\delta$, where δ is a pattern and $- \in \mathcal{P}_1$. Then x occurs bound in δ. Apply the induction hypothesis for δ to get a subpattern $Qx\psi$ (with $Q \in \mathcal{Q}$) of δ such that x does not occur bound in ψ. We have that δ is a subpattern of φ, hence, by Lemma 2.15, $SubPattern(\delta) \subseteq SubPattern(\varphi)$. Thus, $Qx\psi$ is a subpattern of φ such that x does not occur bound in ψ.

(iii) $\varphi = \ominus\delta\chi$, where δ, χ are patterns and $\ominus \in \mathcal{P}_2 \cup Equal$. We have two cases:

  (a) x occurs bound in δ. Apply the induction hypothesis for δ to get a subpattern $Qx\psi$ (with $Q \in \mathcal{Q}$) of δ such that x does not occur bound in ψ. We have that δ is a subpattern of φ, hence, by Lemma 2.15, $SubPattern(\delta) \subseteq SubPattern(\varphi)$. Thus, $Qx\psi$ is a subpattern of φ such that x does not occur bound in ψ.

  (b) x occurs bound in χ. Apply the induction hypothesis for χ to get a subpattern $Qx\psi$ (with $Q \in \mathcal{Q}$) of χ such that x does not occur bound in ψ. We have that χ is a subpattern of φ, hence, by Lemma 2.15, $SubPattern(\chi) \subseteq SubPattern(\varphi)$. Thus, $Qx\psi$ is a subpattern of φ such that x does not occur bound in ψ.

(iv) $\varphi = Qx\delta$, where δ is a pattern, $Q \in \mathcal{Q}$ and $x \in EVar$. We have two cases:

  (a) x does not occur bound in δ. Then we can take $Qx\psi = \varphi$, hence $\psi = \delta$.

  (b) x occurs bound in δ. Apply the induction hypothesis for δ to get a subpattern $Qx\psi$ (with $Q \in \mathcal{Q}$) of δ such that x does not occur bound in ψ. We have that δ is a subpattern of φ, hence, by Lemma 2.15, $SubPattern(\delta) \subseteq SubPattern(\varphi)$. Thus, $Qx\psi$ is a subpattern of φ such that x does not occur bound in ψ.

(v) $\varphi = \overline{Q}X\delta$, where δ is a pattern, $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$. Then x occurs bound in δ. Apply the induction hypothesis for δ to get a subpattern $Qx\psi$ (with $Q \in \mathcal{Q}$) of δ such that x does not occur bound in ψ. We have that δ is a subpattern of φ, hence, by Lemma 2.15, $SubPattern(\delta) \subseteq SubPattern(\varphi)$. Thus, $Qx\psi$ is a subpattern of φ such that x does not occur bound in ψ.

$\square$

7 Substitution of free occurrences of element variables

Assume that $EVar \neq \emptyset$ and $\mathcal{Q} \neq \emptyset$.
Let $x \in EVar$ and φ, δ be patterns.

Definition 7.1. *We define $Subf^x_\delta\varphi$ to be the expression obtained from φ by replacing every free occurrence of x in φ with δ.*

Remark 7.2.

$$Subf^x_\delta\varphi = Repl^x_\delta(\varphi; FreeOccur_x(\varphi)).$$

Proposition 7.3 (Definition by recursion)**.**
The mapping

$$Subf^x_\delta : Pattern \to Expr, \quad Subf^x_\delta(\varphi) = Subf^x_\delta\varphi$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
Subf^x_\delta(z) & = & \begin{cases} \delta & \textit{if } x = z \\ z & \textit{if } x \neq z \end{cases} & \textit{if } z \in EVar, \\
Subf^x_\delta(\varphi) & = & \varphi & \textit{if } \varphi \in SVar \cup \Sigma \cup \mathcal{P}_C, \\
Subf^x_\delta(-\varphi) & = & -Subf^x_\delta(\varphi) & \textit{for any } - \in \mathcal{P}_1, \\
Subf^x_\delta(\circ\varphi\psi) & = & \circ Subf^x_\delta(\varphi) Subf^x_\delta(\psi) & \textit{for any } \circ \in \mathcal{P}_2, \\
Subf^x_\delta(\sim \varphi\psi) & = & \sim Subf^x_\delta(\varphi) Subf^x_\delta(\psi) & \textit{for any } \sim \in Equal, \\
Subf^x_\delta(Qz\varphi) & = & \begin{cases} Qz\varphi & \textit{if } x = z \\ QzSubf^x_\delta(\varphi) & \textit{if } x \neq z \end{cases} & \textit{for any } Q \in \mathcal{Q} \textit{ and } z \in EVar, \\
Subf^x_\delta(\overline{Q}X\varphi) & = & \overline{Q}XSubf^x_\delta(\varphi) & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = Expr$ and

$$G_0(\varphi) = \begin{cases} \delta & \text{if } \varphi = x \\ \varphi & \text{if } \varphi \in (EVar \setminus \{x\}) \cup SVar \cup \Sigma \cup \mathcal{P}_C \end{cases}$$

$$\begin{array}{lcll}
G_-(\theta, \varphi) & = & -\theta & \text{for any } - \in \mathcal{P}_1, \\
G_\circ(\theta, \tau, \varphi, \psi) & = & \circ\theta\tau & \text{for any } \circ \in \mathcal{P}_2, \\
G_\sim(\theta, \tau, \varphi, \psi) & = & \sim \theta\tau & \text{for any } \sim \in Equal, \\
G_Q(\theta, z, \varphi) & = & \begin{cases} Qz\varphi & \text{if } x = z \\ Qz\theta & \text{if } x \neq z \end{cases} & \text{for any } Q \in \mathcal{Q} \text{ and } z \in EVar, \\
G_{\overline{Q}}(\theta, X, \varphi) & = & \overline{Q}X\theta & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar.
\end{array}$$

Then

(i) If φ is an atomic pattern, we have the following cases:

(a) $\varphi = x$. Then $Subf^x_\delta(\varphi) = Subf^x_\delta(x) = \delta = G_0(\varphi)$.

(b) $\varphi \in (EVar \setminus \{x\}) \cup SVar \cup \Sigma \cup \mathcal{P}_C$. Then $Subf^x_\delta(\varphi) = \varphi = G_0(\varphi)$.

(ii) For $- \in \mathcal{P}_1$, we have that

$$Subf^x_\delta(-\varphi) = -Subf^x_\delta(\varphi) = G_-(Subf^x_\delta(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$Subf^x_\delta(\circ\varphi\psi) = \circ Subf^x_\delta(\varphi) Subf^x_\delta(\psi) = G_\circ(Subf^x_\delta(\varphi), Subf^x_\delta(\psi), \varphi, \psi).$$

(iv) For $\sim \in Equal$, we have that

$$Subf^x_\delta(\sim \varphi\psi) = \sim Subf^x_\delta(\varphi) Subf^x_\delta(\psi) = G_\sim(Subf^x_\delta(\varphi), Subf^x_\delta(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $z \in EVar$, we have that

$$Subf^x_\delta(Qz\varphi) = \begin{cases} Qz\varphi & \text{if } x = z \\ QzSubf^x_\delta(\varphi) & \text{if } x \neq z \end{cases} = G_Q(Subf^x_\delta(\varphi), z, \varphi).$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$Subf^x_\delta(\overline{Q}X\varphi) = \overline{Q}XSubf^x_\delta(\varphi) = G_{\overline{Q}}(Subf^x_\delta(\varphi), X, \varphi).$$

Thus, $Subf^x_\delta$ is the unique mapping given by Proposition 2.11. □

Proposition 7.4. *$Subf^x_\delta\varphi$ is a pattern.*

Proof. The proof is immediate by induction on φ, using Proposition 7.3. □

Lemma 7.5. *$Subf^x_x\varphi = \varphi$.*

Proof. Apply Remark 7.2 and Lemma A.8.(i) with $\mathbf{a} := \varphi$, $\mathbf{b} := x$, $\mathbf{c} := x$, $I := FreeOccur_x(\varphi)$. □

Lemma 7.6. *If x does not occur (free) in φ, then $Subf^x_\delta\varphi = \varphi$.*

Proof. Apply Remark 7.2, the fact that $FreeOccur_x(\varphi) = \emptyset$ and Lemma A.8.(ii) with $\mathbf{a} := \varphi$, $\mathbf{b} := x$, $\mathbf{c} := \delta$. □

7.1 Substitution of free occurrences of x with y

Remark 7.7.

$$Subf^x_y\varphi = Repl^x_y(\varphi; FreeOccur_x(\varphi)).$$

Proof. By Remark 7.2. □

Lemma 7.8. *Let φ be a pattern and x, y be element variables. Then $\ell(\varphi) = \ell(Subf^x_y\varphi)$.*

Proof. Apply Remark 7.7 and Lemma A.18 with x, y, $\mathbf{a} := \varphi$ and $I := FreeOccur_x(\varphi)$. □

Lemma 7.9. *Let φ be a pattern and x, y be element variables. Then*

(i) $Occur_y(Subf^x_y\varphi) = Occur_y(\varphi) \cup FreeOccur_x(\varphi)$.

(ii) If y does not occur bound in φ, then y does not occur bound in $Subf^x_y\varphi$.

(iii) If y does not occur in φ, then

$$Occur_y(Subf^x_y\varphi) = FreeOccur_y(Subf^x_y\varphi) = FreeOccur_x(\varphi).$$

(iv) Assume that $x \neq y$. Then $Occur_x(Subf^x_y\varphi) = BoundOccur_x(\varphi)$.

(v) Assume that $x \neq y$. Then x does not occur free in $Subf^x_y\varphi$.

(vi) Assume that $x \neq y$. If x does not occur bound in φ, then x does not occur in $Subf^x_y\varphi$.

Proof. Denote, for simplicity, $\psi := Subf^x_y\varphi$. Let $\varphi = \varphi_0\varphi_1\ldots\varphi_{n-1}$, $n \geq 1$.
Then $\psi = \psi_0\psi_1\ldots\psi_{n-1}$, where, for all $k \in [0, n-1]$,

$$\psi_k = \begin{cases} y & \text{if } k \in FreeOccur_x(\varphi) \\ \varphi_k & \text{otherwise} \end{cases}$$

(i) For all $k \in [0, n-1]$, we have that

$$\begin{aligned} k \in Occur_y(Subf^x_y\varphi) \quad &\text{iff} \quad \psi_k = y \text{ iff } (k \in FreeOccur_x(\varphi) \text{ or } \varphi_k = y) \\ &\text{iff} \quad (k \in FreeOccur_x(\varphi) \text{ or } k \in Occur_y(\varphi)). \end{aligned}$$

(ii) Assume by contradiction that y occurs bound in ψ. It follows that for some $Q \in \mathcal{Q}$ and $i \in [0, n-2)$, we have that $\psi_i = Q$ and $\psi_{i+1} = y$. Obviously, $\varphi_i = \psi_i = Q$. If $\varphi_{i+1} = x$, then $i+1 \notin FreeOccur_x(\varphi)$, so $\psi_{i+1} = \varphi_{i+1} = x$, contradiction, as $\psi_{i+1} = y$ and x, y are distinct. Thus, we must have that $\varphi_{i+1} = y$. It follows that y occurs bound in φ at $i+1$, a contradiction with the hypothesis.

(iii) We have that $Occur_y(\varphi) = \emptyset$ and, by (ii), $BoundOccur_y(Subf^x_y\varphi) = \emptyset$. It follows that

$$FreeOccur_y(Subf^x_y\varphi) = Occur_y(Subf^x_y\varphi) \overset{(i)}{=} Occur_y(\varphi) \cup FreeOccur_x(\varphi) = FreeOccur_x(\varphi).$$

(iv) $\subseteq$ Let $k \in Occur_x(Subf^x_y\varphi)$. Then $\psi_k = x$. As $x \neq y$, we must have that $k \notin FreeOccur_x(\varphi)$ and $\psi_k = \varphi_k = x$. Thus, $k \in Occur_x(\varphi) \setminus FreeOccur_x(\varphi) = BoundOccur_x(\varphi)$.

$\supseteq$ Assume that $k \in BoundOccur_x(\varphi)$. Then $\varphi_k = x$ and $k \notin FreeOccur_x(\varphi)$, so we must have $\psi_k = \varphi_k = x$. Thus, $k \in Occur_x(Subf^x_y\varphi)$.

(v) By (iv), we have that $FreeOccur_x(Subf^x_y\varphi) = \emptyset$.

(vi) Apply (iv) and the hypothesis to get that $Occur_x(Subf^x_y\varphi) = BoundOccur_x(\varphi) = \emptyset$.

□

Lemma 7.10. *Let φ be a pattern and x, y, z be variables such that $y \neq z$ and y does not occur in φ.*
Then y does not occur in $Subf^x_z\varphi$.

Proof. Apply Lemma A.14 with $\mathbf{a} := y$, $\mathbf{b} := \varphi$, $\mathbf{c} := x$, $\mathbf{c}_1 := z$, $\mathbf{b}_1 := Subf^x_z\varphi$. □

Lemma 7.11. *Let φ be a pattern and x, y, z be element variables such that z does not occur in φ.*
Then $Subf^z_y Subf^x_z\varphi = Subf^x_y\varphi$.

Proof. If x does not occur free in φ, then, by Lemma 7.6, $Subf^x_y\varphi = Subf^x_z\varphi = \varphi$. Thus, $Subf^z_y Subf^x_z\varphi = Subf^z_y\varphi$. As z does not occur in φ, we can apply again Lemma 7.6 to conclude that $Subf^z_y\varphi = \varphi$.
Assume now that x occurs free in φ and denote, for simplicity,

$$\delta := Subf^x_z\varphi, \quad \chi := Subf^z_y Subf^x_z\varphi = Subf^z_y\delta.$$

By Lemma 7.9.(iii),

$$FreeOccur_z(\delta) = FreeOccur_x(\varphi).$$

We can apply Remark 7.2 to get that

$$\begin{aligned} Subf^x_y\varphi &= Repl^x_y(\varphi; FreeOccur_x(\varphi)), \\ \delta &= Repl^x_z(\varphi; FreeOccur_x(\varphi)), \\ \chi &= Repl^z_y(\delta; FreeOccur_z(\delta)) = Repl^z_y(\delta; FreeOccur_x(\varphi)). \end{aligned}$$

Apply now Lemma A.27 with $\mathbf{a} := \varphi$, $\mathbf{b} := Subf^x_y\varphi$, $\mathbf{c} := \delta$ and $I := FreeOccur_x(\varphi)$ to get that $Subf^x_y\varphi = \chi$. □

Lemma 7.12. *Let φ be a pattern and x, z be element variables such that z does not occur in φ. Then $Subf^z_x Subf^x_z\varphi = \varphi$.*

Proof. Apply Lemma 7.11 with $y := x$ and the fact that $Subf^x_x\varphi = \varphi$, by Lemma 7.5. □

8 Bounded substitution of element variables

Assume that $EVar \neq \emptyset$ and $\mathcal{Q} \neq \emptyset$.
Let φ be a pattern and x, y be element variables.

Definition 8.1. *We define $Subb^x_y\varphi$ to be the expression obtained from φ by replacing every bound occurrence of x in φ with y.*

Remark 8.2. *Then*

$$Subb^x_y\varphi = Repl^x_y(\varphi; BoundOccur_x(\varphi)).$$

Proposition 8.3 (Definition by recursion)**.**
If $x = y$, then obviously $Subb^x_y\varphi = \varphi$. Assume that $x \neq y$. Then the mapping

$$Subb^x_y : Pattern \to Expr, \quad Subb^x_y(\varphi) = Subb^x_y\varphi$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
Subb^x_y(\varphi) & = & \varphi & \textit{if } \varphi \textit{ is an atomic pattern,}\\
Subb^x_y(-\varphi) & = & -Subb^x_y(\varphi) & \textit{for any } - \in \mathcal{P}_1,\\
Subb^x_y(\circ\varphi\psi) & = & \circ Subb^x_y(\varphi)Subb^x_y(\psi) & \textit{for any } \circ \in \mathcal{P}_2,\\
Subb^x_y(\sim\varphi\psi) & = & \sim Subb^x_y(\varphi)Subb^x_y(\psi) & \textit{for any } \sim \in Equal,\\
Subb^x_y(Qz\varphi) & = & \begin{cases} QySubf^x_y\big(Subb^x_y(\varphi)\big) & \textit{if } x = z\\ QzSubb^x_y(\varphi) & \textit{if } x \neq z\end{cases} & \textit{for any } Q \in \mathcal{Q} \textit{ and } z \in EVar,\\
Subb^x_y(\overline{Q}X\varphi) & = & \overline{Q}XSubb^x_y(\varphi) & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = Expr$ and

$$\begin{array}{lcll}
G_0(\varphi) & = & \varphi & \text{if } \varphi \text{ is an atomic pattern,}\\
G_-(\theta,\varphi) & = & -\theta & \text{for any } - \in \mathcal{P}_1,\\
G_\circ(\theta,\tau,\varphi,\psi) & = & \circ\theta\tau & \text{for any } \circ \in \mathcal{P}_2,\\
G_\sim(\theta,\tau,\varphi,\psi) & = & \sim\theta\tau & \text{for any } \sim \in Equal,\\
G_Q(\theta,z,\varphi) & = & \begin{cases} QySubf^x_y(\theta) & \text{if } x = z\\ Qz\theta & \text{if } x \neq z\end{cases} & \text{for any } Q \in \mathcal{Q} \text{ and } z \in EVar,\\
G_{\overline{Q}}(\theta,X,\varphi) & = & \overline{Q}X\theta & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar.
\end{array}$$

Then

(i) If φ is an atomic pattern, $Subb^x_y(\varphi) = \varphi = G_0(\varphi)$.

(ii) For $- \in \mathcal{P}_1$, we have that

$$Subb^x_y(-\varphi) = -Subb^x_y(\varphi) = G_-(Subb^x_y(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$Subb^x_y(\circ\varphi\psi) = \circ Subb^x_y(\varphi)Subb^x_y(\psi) = G_\circ(Subb^x_y(\varphi), Subb^x_y(\psi), \varphi, \psi).$$

(iv) For $\sim \in Equal$, we have that

$$Subb^x_y(\sim\varphi\psi) = \sim Subb^x_y(\varphi)Subb^x_y(\psi) = G_\sim(Subb^x_y(\varphi), Subb^x_y(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $z \in EVar$, we have that

$$Subb^x_y(Qz\varphi) = \begin{cases} QySubf^x_y\big(Subb^x_y(\varphi)\big) & \text{if } x = z\\ QzSubb^x_y(\varphi) & \text{if } x \neq z\end{cases} = G_Q(Subb^x_y(\varphi), z, \varphi).$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$Subb_y^x(\overline{Q}X\varphi) = \overline{Q}X\,Subb_y^x(\varphi) = G_{\overline{Q}}(Subb_y^x(\varphi), X, \varphi).$$

Thus, $Subb_y^x$ is the unique mapping given by Proposition 2.11. □

Proposition 8.4. *$Subb_y^x\varphi$ is a pattern.*

Proof. The proof is immediate by induction on φ, using Proposition 8.3. □

Lemma 8.5. *$Subb_x^x\varphi = \varphi$.*

Proof. Apply Remark 8.2 and Lemma A.8.(i) with $\mathbf{a} := \varphi$, $\mathbf{b} := x$, $\mathbf{c} := x$, $I := BoundOccur_x(\varphi)$. □

Lemma 8.6. *If x does not occur (bound) in φ, then $Subb_y^x\varphi = \varphi$.*

Proof. Apply Remark 8.2, the fact that $BoundOccur_x(\varphi) = \emptyset$ and Lemma A.8.(ii) with $\mathbf{a} := \varphi$, $\mathbf{b} := x$, $\mathbf{c} := y$. □

Lemma 8.7. *Let φ be a pattern and x, y be distinct element variables. Then*

(i) x does not occur bound in $Subb_y^x\varphi$.

(ii) $Occur_x(Subb_y^x\varphi) = FreeOccur_x(Subb_y^x\varphi) = FreeOccur_x(\varphi)$.

(iii) $x \in FreshF(\varphi)$ iff x does not occur in $Subb_y^x\varphi$.

(iv) $Occur_y(Subb_y^x\varphi) = Occur_y(\varphi) \cup BoundOccur_x(\varphi)$.

(v) $BoundOccur_x(\varphi) \subseteq BoundOccur_y(Subb_y^x\varphi)$.

(vi) If y does not occur in φ, then

$$Occur_y(Subb_y^x\varphi) = BoundOccur_y(Subb_y^x\varphi) = BoundOccur_x(\varphi).$$

In particular, $FreeOccur_y(Subb_y^x\varphi) = \emptyset$.

(vii) If y does not occur in φ, then y does not occur free in $Subb_y^x\varphi$.

(viii) For every $Q \in \mathcal{Q}$, x does not occur in $Subb_y^xQx\varphi$.

Proof. Denote, for simplicity, $\psi := Subb_y^x\varphi$. Let $\varphi = \varphi_0\varphi_1\ldots\varphi_{n-1}$, $n \geq 1$.
Then $\psi = \psi_0\psi_1\ldots\psi_{n-1}$, where, for all $k \in [0, n-1]$,

$$\psi_k = \begin{cases} y & \text{if } k \in BoundOccur_x(\varphi) \\ \varphi_k & \text{otherwise} \end{cases}$$

(i) Assume by contradiction that x occurs bound in ψ. It follows that for some $Q \in \mathcal{Q}$ and $i \in [0, n-2)$, we have that $\psi_i = Q$ and $\psi_{i+1} = x$. Obviously, $\varphi_i = \psi_i = Q$. We have two cases:

(a) $\varphi_{i+1} = x$. Then $i+1 \in BoundOccur_x(\varphi)$, hence $\psi_{i+1} = y \neq x$, a contradiction.

(b) $\varphi_{i+1} \neq x$. Then $i+1 \notin Occur_x(\varphi)$, in particular $i+1 \notin BoundOccur_x(\varphi)$. Thus, we have $\psi_{i+1} = \varphi_{i+1}$, a contradiction.

(ii) The first equality follows from (i). Let us prove now that $Occur_x(\psi) = FreeOccur_x(\varphi)$.

$\supseteq$: Let $k \in FreeOccur_x(\varphi)$. Then $k \notin BoundOccur_x(\varphi)$, hence we must have that $\psi_k = \varphi_k = x$. Thus, $k \in Occur_x(\psi)$.

$\subseteq$: Let $k \in Occur_x(\psi)$, so $\psi_k = x$. If $k \in BoundOccur_x(\varphi)$, then we must have $\psi_k = y \neq x$, a contradiction. Thus, $k \notin BoundOccur_x(\varphi)$. Then $\psi_k = \varphi_k$, so $\varphi_k = x$. It follows that $k \in Occur_x(\varphi) \setminus BoundOccur_x(\varphi) = FreeOccur_x(\varphi)$.

(iii) Apply (ii) and the fact that $FreeOccur_x(\varphi) = \emptyset$.

(iv) $\supseteq$: If $k \in Occur_y(\varphi)$, then $\varphi_k = y \neq x$, so $\psi_k = \varphi_k = y$. Thus, $k \in Occur_y(\psi)$. If $k \in BoundOccur_x(\varphi)$, then $\psi_k = y$. Thus, $k \in Occur_y(\psi)$.

$\subseteq$: Let $k \in Occur_y(\psi)$, so $\psi_k = y$. Assume that $k \notin BoundOccur_x(\varphi)$. Then we must have $\psi_k = \varphi_k$, that is $\varphi_k = y$, hence $k \in Occur_y(\varphi)$.

(v) Let $k \in BoundOccur_x(\varphi)$. Then $\varphi_k = x$ and there exist $Q \in \mathcal{Q}$ and $0 \leq i, j \leq n-1$ such that $i < k \leq j$ and Q is a quantifier on x at place i of φ with scope $\chi = \varphi_i \dots \varphi_j$, so $\varphi_i = Q$, $\varphi_{i+1} = x$. Then $\psi_k = y$, $\psi_i = Q$, $\psi_{i+1} = y$. Let $J := Occur_x(\chi)$ and $\theta := Repl_y^x(\chi; J)$. As $J \subseteq BoundOccur_x(\varphi)$, we get that $\theta = \psi_i \dots \psi_j$ and Q is a quantifier on y at place k of ψ with scope θ. Thus, $k \in BoundOccur_y(\psi)$.

(vi) As $Occur_y(\varphi) = \emptyset$, we have that

$$Occur_y(\psi) \overset{(iv)}{=} BoundOccur_x(\varphi) \overset{(v)}{\subseteq} BoundOccur_y(\psi) \subseteq Occur_y(\psi).$$

It follows that we must have equalities instead of inclusions.

(vii) By (vi).

(viii) By (ii), we have that $Occur_x(Subb_y^x Qx\varphi) = FreeOccur_x(Qx\varphi) = \emptyset$.

□

Lemma 8.8. *Let φ be a pattern and x, y, z be element variables such that $x \neq y, z$. Then x does not occur in $Subf_y^x Subb_z^x \varphi$.*

Proof. Let us denote $\psi = Subf_y^x Subb_z^x \varphi$. We have that

$$\begin{aligned} \psi &= Repl_y^x(Subb_z^x\varphi; FreeOccur_x(Subb_z^x\varphi)) \quad \text{by Remark 7.2} \\ &= Repl_y^x(Subb_z^x\varphi; FreeOccur_x(\varphi)) \quad \text{by Lemma 8.7(ii)} \\ &= Repl_y^x(Repl_z^x(\varphi; BoundOccur_x(\varphi)); FreeOccur_x(\varphi)) \quad \text{by Remark 8.2} \end{aligned}$$

Apply now Lemma A.28 with x, y, z, $\mathbf{a} := \varphi$, $I := FreeOccur_x(\varphi)$ and $J := BoundOccur_x(\varphi)$ to get that x does not occur in ψ. □

Lemma 8.9. *Let φ be a pattern and x, y, z be distinct variables. Then*

$$Subf_y^x Subb_z^x \varphi = Subb_z^x Subf_y^x \varphi.$$

Proof. Let us denote

$$\psi = Subf_y^x Subb_z^x \varphi, \quad \chi = Subb_z^x Subf_y^x \varphi.$$

Then

$$\begin{aligned} \psi &= Repl_y^x(Subb_z^x\varphi; FreeOccur_x(Subb_z^x\varphi)) \quad \text{by Remark 7.2} \\ &= Repl_y^x(Subb_z^x\varphi; FreeOccur_x(\varphi)) \quad \text{by Lemma 8.7(ii)} \\ &= Repl_y^x(Repl_z^x(\varphi; BoundOccur_x(\varphi)); FreeOccur_x(\varphi)) \quad \text{by Remark 8.2} \\ \chi &= Repl_z^x(Subf_z^x\varphi; BoundOccur_x(Subf_z^x\varphi)) \quad \text{by Remark 8.2} \\ &= Repl_z^x(Subf_z^x\varphi; BoundOccur_x(\varphi)) \quad \text{by Lemma 7.9(iv)} \\ &= Repl_z^x(Repl_y^x(\varphi; FreeOccur_x(\varphi)); BoundOccur_x(\varphi)) \quad \text{by Remark 7.2} \end{aligned}$$

Apply Lemma A.28 with x, y, z, $\mathbf{a} := \varphi$, $I := FreeOccur_x(\varphi)$ and $J := BoundOccur_x(\varphi)$ to get that $\psi = \chi$. □

Lemma 8.10. *Let φ be a pattern and x, z be distinct variables such that z does not occur in φ. Then*

$$Subb^z_x Subb^x_z \varphi = \varphi.$$

Proof. Let $\psi := Subb^x_z \varphi$. Then, by Remark 8.2,

$$\psi = Repl^x_z(\varphi; BoundOccur_x(\varphi)).$$

Apply Lemma A.22 with x, $y := z$, $\mathbf{a} := \varphi$, $\mathbf{a}^* := \psi$ and $I := BoundOccur_x(\varphi)$ to get that

$$\varphi = Repl^z_x(\psi; BoundOccur_x(\varphi)).$$

As z does not occur in φ, we have, by Lemma 8.7(vi), that $BoundOccur_x(\varphi) = BoundOccur_z(\psi)$. Thus,

$$\begin{aligned}\varphi &= Repl^z_x(\psi; BoundOccur_x(\varphi)) = Repl^z_x(\psi; BoundOccur_z(\psi))\\ &= Subb^z_x \psi \quad \text{by Remark 8.2}\\ &= Subb^z_x Subb^x_z \varphi.\end{aligned}$$

□

Lemma 8.11. *Let φ be a pattern and x, y, z be distinct variables such that z does not occur in φ. Then*

$$Subb^z_x Subf^x_y Subb^x_z \varphi = Subf^x_y \varphi.$$

Proof. We get, by Lemma 8.9, that

$$Subb^z_x Subf^x_y Subb^x_z \varphi = Subb^z_x Subb^x_z Subf^x_y \varphi$$

As z does not occur in φ and $y \neq z$, we get, by Lemma 7.10, that z does not occur in $Subf^x_y \varphi$. Thus, we can apply Lemma 8.10 with x, z and $\varphi := Subf^x_y \varphi$ to get that

$$Subb^z_x Subb^x_z Subf^x_y \varphi = Subf^x_y \varphi.$$

□

9 Bounded substitution theorem

Theorem 9.1. *Let* ***Cong*** *be a reflexive and transitive $\mathcal{L}$-congruence. Assume that the following holds:*

(ASSUMPTION) For any pattern φ, distinct variables x, z such that x does not occur bound in φ and z does not occur in φ,

$$Qx\varphi \,\boldsymbol{Cong}\, QzSubf^x_z\varphi \quad \text{for all } Q \in \mathcal{Q}.$$

Then for any $\mathcal{L}$-pattern φ and any variables x, y such that y does not occur in φ,

$$\varphi \,\boldsymbol{Cong}\, Subb^x_y \varphi. \tag{1}$$

Proof. The case $x = y$ is obvious, as, by Lemma 8.5, $Subb^x_x \varphi = \varphi$ and **Cong** is reflexive. Assume that x and y are distinct variables.
We prove (1) by induction on the number m of bound occurrences of x in φ.
The case $m = 0$ is obvious, as, by Lemma 8.6, $Subb^x_y \varphi = \varphi$ and **Cong** is reflexive.
Assume that $m \geq 1$ and that (1) holds for all patterns with fewer than m bound occurrences of x.
As $m \geq 1$, x occurs bound in φ. Apply Proposition 6.9 to get a subpattern $Qx\psi$ (with $Q \in \mathcal{Q}$) of φ such that x does not occur bound in ψ.

Let z be a new variable, that is, z is distinct from x, y and z does not occur in φ (hence not in ψ). We can apply (ASSUMPTION) for x, z and ψ to get that

$$Qx\psi \mathbf{Cong}\, Qz Subf_z^x \psi. \tag{2}$$

Let χ be the pattern obtained from φ by replacing an occurrence of $Qx\psi$ in φ by $QzSubf_z^x\psi$. Then, by (2) and Theorem 5.2 we get that

$$\varphi \mathbf{Cong}\, \chi. \tag{3}$$

Furthermore, χ has fewer than m bound occurrences of x, so we can apply the induction hypothesis to conclude that

$$\chi \mathbf{Cong}\, Subb_y^x \chi. \tag{4}$$

Let δ be the pattern obtained from $Subb_y^x\chi$ by replacing an occurrence of $QzSubf_z^x\psi$ in $Subb_y^x\chi$ by $QySubf_y^x\psi$.

Then we have that (see the detailed proof in Subsubsection 9.1)

$$Subb_y^x \varphi = \delta. \tag{5}$$

Apply Theorem 5.2 to conclude that

$$Subb_y^x \chi \mathbf{Cong}\, Subb_y^x \varphi. \tag{6}$$

Using now (3), (4), (6) and the transitivity of **Cong**, we get that

$$\varphi \mathbf{Cong}\, Subb_y^x \varphi.$$

□

9.1 Proof of (5)

Claim 1: x, y do not occur in $QzSubf_z^x\psi$.
Proof of claim: As y does not occur in φ (hence not in ψ) and $y \neq z$, we get that y does not occur in $Subf_z^x\psi$, by Lemma 7.10. As $y \neq z$, we can apply Lemma A.9 with $\mathbf{a} := y$, $\mathbf{b} := Qz$ and $\mathbf{c} := Subf_z^x\psi$ to conclude that y does not occur in $QzSubf_z^x\psi$.
As x does not occur bound in ψ and $x \neq z$, we can apply Lemma 7.9.(vi) to get that x does not occur in $Subf_z^x\psi$. As $x \neq z$, we can apply Lemma A.9 with $\mathbf{a} := x$, $\mathbf{b} := Qz$ and $\mathbf{c} := Subf_z^x\psi$ to conclude that x does not occur in $QzSubf_z^x\psi$. ■

Claim 2: y does not occur in χ.
Proof of claim: Apply Lemma A.14 with $\mathbf{a} := y$, $\mathbf{b} := \varphi$, $\mathbf{c} := Qx\psi$, $\mathbf{c}_1 := QzSubf_z^x\psi$, $\mathbf{b}_1 := \chi$. ■

Claim 3:

(i) $QzSubf_z^x\psi = Replall_z^x(Qx\psi)$ and $QySubf_y^x\psi = Replall_y^x(Qx\psi)$.

(ii) $QySubf_y^x\psi = Replall_y^z(QzSubf_z^x\psi)$.

(iii) $\ell(Qx\psi) = \ell(QySubf_y^x\psi) = \ell(QzSubf_z^x\psi)$.

(iv) $Occur_x(Qx\psi) = Occur_y(QySubf_y^x\psi) = Occur_z(QzSubf_z^x\psi)$.

Proof of claim:

(i) Let $u \in \{x, z\}$. Denote $\tilde{\psi} = Subf_u^x\psi$. As x does not occur bound in ψ, we have that $FreeOccur_x(\psi) = Occur_x(\psi)$. Hence, $Subf_u^x\psi = Replall_u^x(\psi)$.

As obviously $Qu = Replall_u^x(Qx)$, we can apply Lemma A.10 to get the conclusion.

(ii) By (i), we have that $QzSubf_z^x\psi = Replall_z^x(Qx\psi)$, hence

$$Replall_y^z(QzSubf_z^x\psi) = Replall_y^z(Replall_z^x(Qx\psi)).$$

Apply Lemma A.21 with $\mathbf{a} := Qx\psi$, x, $y := z$, $z :=$ and $I := Occur_x(Qx\psi)$ to get that

$$Replall_y^z(Replall_z^x(Qx\psi)) = Replall_y^x(Qx\psi).$$

Apply again (i) to get the conclusion.

(iii) Apply (i) and Lemma A.18.

(iv) Using (i), the fact that y, z do not occur in $Qx\psi$ and that x does not occur bound in ψ, we can apply Lemma 7.9(iii) to get that

$$Occur_y(Subf_y^x\psi) = Occur_z(Subf_z^x\psi) = FreeOccur_x(\psi) = Occur_x(\psi).$$

As $Occur_x(Qx) = Occur_y(Qy) = Occur_z(Qz) = \{1\}$ and $\ell(Qx) = \ell(Qy) = \ell(Qz) = 2$, we can apply Lemma A.24 to get that

$$Occur_x(Qx\psi) = Occur_y(QySubf_y^x\psi) = Occur_z(QzSubf_z^x\psi) = \{1\} \cup (Occur_x(\psi) + 2).$$

■

Assume, in the sequel, that χ is obtained by replacing the occurrence of $Qx\psi$ at place k in φ by $QzSubf_z^x\psi$. Thus,

$$\chi = Repl_{QzSubf_z^x\psi}^{Qx\psi}(\varphi; \{k\}). \tag{7}$$

Claim 4: $\varphi = Repl_{Qx\psi}^{QzSubf_z^x\psi}(\chi; \{k\})$.
Proof of claim: Using Claim 3(iii) and (7), we can apply Lemma A.16(v) with $\mathbf{a} := \varphi$, $\mathbf{b} := Qx\psi$, $\mathbf{c} := QzSubf_z^x\varphi$, $\mathbf{a}^* := \chi$ and $I := \{k\}$. ■

Claim 5: $\chi = Repl_z^x(\varphi; Occur_x(Qx\psi) + k)$.
Proof of claim: Using (7) and Claim 3(i), we can apply Lemma A.26 with $\mathbf{a} := \varphi$, $\mathbf{b} := Qx\psi$, $\mathbf{c} := QzSubf_z^x\varphi$, $\mathbf{d} := \chi$. ■

Claim 6: $QzSubf_z^x\psi$ occurs uniquely in χ at place k.
Proof of claim: As z is a new variable, we can apply Lemma A.15 with $\mathbf{a} := \varphi$, $\mathbf{b} := Qx\psi$, $\mathbf{c} := QzSubf_z^x\psi$, $\mathbf{d} := \chi$. ■

Claim 7: $QzSubf_z^x\psi$ occurs uniquely in $Subb_y^x\chi$ at place k.
Proof of claim: Apply Claims 1, 6 and Lemma A.25 with x, y, $\mathbf{a} := \chi$, $\mathbf{b} := QzSubf_z^x\psi$, $\mathbf{d} := Subb_y^x\chi$. ■

Thus,

$$\delta = Repl_{QySubf_y^x\psi}^{QzSubf_z^x\psi}(Subb_y^x\chi; \{k\}). \tag{8}$$

Claim 8: $\delta = Repl_y^z(Subb_y^x\chi; Occur_z(QzSubf_z^x\psi) + k) = Repl_y^z(Subb_y^x\chi; Occur_x(Qx\psi) + k)$.
Proof of claim: Using (8) and Claims 1, 3(i), we can apply Lemma A.26 with $x := z$, $z := y$, $\mathbf{a} := Subb_y^x\chi$, $\mathbf{b} := QzSubf_z^x\psi$, $\mathbf{c} := QySubf_y^x\psi$ and $\mathbf{d} := \delta$ to get that

$$\delta = Repl_y^z(Subb_y^x\chi; Occur_z(QzSubf_z^x\psi) + k).$$

By Claim 3, $Occur_y(QySubf_y^x\psi) = Occur_x(Qx\psi)$. ■

Claim 9: The following hold:

$$BoundOccur_x(\varphi) = BoundOccur_x(\chi) \cup (Occur_x(Qx\psi) + k), \quad (9)$$

$$BoundOccur_x(\chi) \cap (Occur_x(Qx\psi) + k) = \emptyset. \quad (10)$$

Proof of claim: Denote $n = \ell(\varphi)$ and $p = \ell(Qx\psi)$. Let $\varphi = \varphi_0 \ldots \varphi_{n-1}$. As $Qx\psi$ occurs in φ at place k, we have that

$$Qx\psi = \varphi_k \ldots \varphi_{k+p-1}.$$

It follows that $Occur_x(Qx\psi) + k = Occur_x(\varphi) \cap [k, k+p-1]$. Furthermore, $Q = \varphi_k$ is a quantifier at place k of φ with scope ψ, so, if x occurs in φ at place $j \in [k, k+p-1]$, then x occurs bound in φ at place j. Thus, $Occur_x(\varphi) \cap [k, k+p-1] = BoundOccur_x(\varphi) \cap [k, k+p-1]$. We have got that

$$Occur_x(Qx\psi) + k = BoundOccur_x(\varphi) \cap [k, k+p-1] \quad (11)$$

By Claim 3(iii), we have that $p = \ell(QzSubf^x_z\psi)$. Hence $\chi = \chi_0 \ldots \chi_{n-1}$, where

$$\chi_i = \varphi_i \text{ for all } i \in [0, k-1] \cup [k+p, n-1] \text{ and } QzSubf^x_z\psi = \chi_k \ldots \chi_{k+p-1}.$$

By Claim 1, we have that $\chi_i \neq x$ for all $i \in [k, k+p-1]$, hence

$$BoundOccur_x(\chi) \subseteq Occur_x(\chi) \subseteq [0, k-1] \cup [k+p, n-1]. \quad (12)$$

As an application of (11) and (12) we get (10).
Subclaim: $BoundOccur_x(\chi) = BoundOccur_x(\varphi) \cap ([0, k-1] \cup [k+p, n-1])$.
Proof of subclaim:
$\subseteq$ Assume that $j \in BoundOccur_x(\chi)$. Then $j \in [0, k-1] \cup [k+p, n-1]$, $\chi_j = \varphi_j = x$, and there exists $Q \in \mathcal{Q}$ and $i < j \leq l \leq n-1$ such that Q is a quantifier on x at place i of χ with scope $\theta = \chi_i \ldots \chi_l$. Thus, if $\theta = \theta_0 \ldots \theta_{l-i+1}$, then $\theta_0 = \chi_i = \varphi_i = Q$, $\theta_1 = \chi_{i+1} = \varphi_{i+1} = x$ and $\theta_{j-i} = \chi_j = \varphi_j = x$.
Applying Lemma 2.16 for the subpatterns θ and $QzSubf^x_z\psi = \chi_k \ldots \chi_{k+p-1}$ of χ and taking into account that θ cannot be a subpattern of $QzSubf^x_z\psi$ (as x occurs in θ and x does not occur in $QzSubf^x_z\psi$), we have the following cases:

(i) $QzSubf^x_z\psi$ is a subpattern of θ. We have that $i \leq k < k+p-1 \leq l \leq n-1$ and $QzSubf^x_z\psi$ occurs in θ at place $k-i$. Let $\theta^* := Repl^{QzSubf^x_z\psi}_{Qx\psi}(\theta; \{k-i\})$. Then θ^* is a pattern, by Theorem 5.2. Moreover, we get that θ^* is subpattern of φ, by applying Lemma A.17(i) with k, $\mathbf{a} := \varphi$, $\mathbf{b} := Qx\psi$, $\mathbf{c} := QzSubf^x_z\psi$, $\mathbf{d} := \chi$, $\mathbf{d}^* := \theta$ and $\mathbf{a}^* := \theta^*$.

Furthermore, $\theta^*_0 = \theta_0 = \varphi_i = Q$, $\theta^*_1 = \theta_1 = \varphi_{i+1} = x$, $\theta^*_{j-i} = \theta_{j-i} = \varphi_j = x$. We get, by Proposition 2.9, that Q is a quantifier on x at place i of φ with scope θ^*. Thus, $j \in BoundOccur_x(\varphi)$.

(ii) $Occur_\theta(\chi) \cap Occur_{QzSubf^x_z\psi}(\chi) = \emptyset$. Then $\theta = \varphi_i \ldots \varphi_l$. Hence, by Proposition 2.9, Q is a quantifier on x at place i of φ with scope θ. Thus, $j \in BoundOccur_x(\varphi)$.

$\supseteq$ Assume that $j \in BoundOccur_x(\varphi) \cap ([0, k-1] \cup [k+p, n-1])$. Then $j \in [0, k-1] \cup [k+p, n-1]$, $\varphi_j = x$, and there exists $Q \in \mathcal{Q}$ and $i < j \leq l \leq n-1$ such that Q is a quantifier on x at place i of φ with scope $\theta = \varphi_i \ldots \varphi_l$. Thus, if $\theta = \theta_0 \ldots \theta_{l-i+1}$, then $\theta_0 = \varphi_i = Q$, $\theta_1 = \varphi_{i+1} = x$ and $\theta_{j-i} = \varphi_j = x$.
Applying Lemma 2.16 for the subpatterns θ and $Qx\psi = \varphi_k \ldots \varphi_{k+p-1}$ of φ and taking into account that θ cannot be a subpattern of $Qx\psi$ (as $\theta_{j-i} = \varphi_j = x$ and $j \notin [k, k+p-1]$), we have the following cases:

(i) $Qx\psi$ is a subpattern of θ. We have that $i \leq k < k+p-1 \leq l \leq n-1$ and $Qx\psi$ occurs in θ at place $k-i$. Let $\theta^* := Repl^{Qx\psi}_{QzSubf^x_z\psi}(\theta; \{k-i\})$. Then θ^* is a pattern (by Theorem 5.2). Moreover, we get that θ^* is subpattern of φ, by applying Lemma A.17(i) with k, $\mathbf{a} := \varphi$, $\mathbf{b} := Qx\psi$, $\mathbf{c} := QzSubf^x_z\psi$, $\mathbf{d} := \chi$, $\mathbf{a}^* := \theta$ and $\mathbf{d}^* := \theta^*$. Furthermore, $\theta^*_0 = \theta_0 = \chi_i = Q$, $\theta^*_1 = \theta_1 = \chi_{i+1} = x$, $\theta^*_{j-i} = \theta_{j-i} = \chi_j = x$. We get that Q is a quantifier on x at place i of χ with scope θ^*. Thus, $j \in BoundOccur_x(\chi)$.

(ii) $Occur_\theta(\varphi) \cap Occur_{Qx\psi}(\varphi) = \emptyset$. Then $\theta = \chi_i \dots \chi_l$. Hence, by Proposition 2.9, Q is a quantifier on x at place i of χ with scope θ. Thus, $j \in BoundOccur_x(\chi)$.

■

It follows that

$$\begin{aligned} BoundOccur_x(\varphi) &= BoundOccur_x(\varphi) \cap [0, n-1] \\ &= BoundOccur_x(\varphi) \cap ([0, k-1] \cup [k, k+p-1] \cup [k+p, n-1]) \\ &= \big(BoundOccur_x(\varphi) \cap [k, k+p-1]\big) \cup \\ &\quad \cup \big(BoundOccur_x(\varphi) \cap ([0, k-1] \cup [k+p, n-1])\big) \\ &= BoundOccur_x(\chi) \cup (Occur_x(Qx\psi) + k). \end{aligned}$$

Thus, (9) holds. ■

Remark now that

$$\begin{aligned} Subb^x_y\chi &= Repl^x_y(\chi; BoundOccur_x(\chi)) \qquad \text{by Remark 8.2} \\ &= Repl^z_y(Repl^x_z(\chi; BoundOccur_x(\chi)); BoundOccur_x(\chi)) \\ &\quad \text{by Lemma A.21 with } \mathbf{a} := \chi,\ x,\ y := z,\ z := y,\ I := BoundOccur_x(\chi) \\ &= Repl^z_y(Repl^x_z(Repl^x_z(\varphi; Occur_x(Qx\psi) + k); BoundOccur_x(\chi)); BoundOccur_x(\chi)) \\ &\quad \text{by Claim 5} \\ &= Repl^z_y(Repl^x_z(\varphi; (Occur_x(Qx\psi) + k) \cup BoundOccur_x(\chi)); BoundOccur_x(\chi)) \\ &\quad \text{by Lemma A.20 with } \mathbf{a} := \varphi,\ x,\ y := z,\ I := Occur_x(Qx\psi) + k,\ J := BoundOccur_x(\chi) \\ &= Repl^z_y(Repl^x_z(\varphi; BoundOccur_x(\varphi)); BoundOccur_x(\chi)) \qquad \text{by Claim 9} \end{aligned}$$

Thus, by Remark 8.2

$$Subb^x_y\chi = Repl^z_y(Subb^x_z\varphi; BoundOccur_x(\chi)) \tag{13}$$

It follows that

$$\begin{aligned} \delta &= Repl^z_y(Subb^x_y\chi; Occur_x(Qx\psi) + k) \qquad \text{by Claim 8} \\ &= Repl^z_y(Repl^z_y(Subb^x_z\varphi; BoundOccur_x(\chi)); Occur_x(Qx\psi) + k) \qquad \text{by (13)} \\ &= Repl^z_y(Subb^x_z\varphi; BoundOccur_x(\chi) \cup (Occur_x(Qx\psi) + k)) \\ &\quad \text{by Lemma A.20 with } \mathbf{a} := Subb^x_z\varphi,\ x,\ y := z,\ I := BoundOccur_x(\chi),\ J := Occur_x(Qx\psi) + k \\ &= Repl^z_y(Subb^x_z\varphi; BoundOccur_x(\varphi)) \qquad \text{by Claim 9} \\ &= Repl^z_y(Repl^x_z(\varphi; BoundOccur_x(\varphi)); BoundOccur_x(\varphi)) \qquad \text{by Remark 8.2} \\ &= Repl^x_y(\varphi; BoundOccur_x(\varphi)) \\ &\quad \text{by Lemma A.21 with } \mathbf{a} := \varphi,\ x,\ y := z,\ z := y,\ I := BoundOccur_x(\varphi) \\ &= Subb^x_y\varphi \qquad \text{by Remark 8.2} \end{aligned}$$

□

10 Free and bound set variables

Assume that $SVar \neq \emptyset$ and $\overline{\mathcal{Q}} \neq \emptyset$.

Definition 10.1. *Let $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ be a pattern and X be a set variable.*

*(i) We say that $\overline{Q} \in \overline{\mathcal{Q}}$ is **a binder on X at the ith place with scope** ψ if $\varphi_i = \overline{Q}$, $\varphi_{i+1} = X$ and $\psi = \varphi_i \dots \varphi_j$ is the unique pattern given by Proposition 2.9.*

(ii) *We say that* X ***occurs bound at the*** k***th place of*** φ *if* $\varphi_k = X$ *and there exist* $\overline{Q} \in \overline{\mathcal{Q}}$ *and* $0 \leq i, j \leq n-1$ *such that* $i < k \leq j$ *and* $\overline{Q}$ *is a binder on* X *at the* i*th place with scope* $\psi = \varphi_i \ldots \varphi_j$.

(iii) *If* $\varphi_k = X$ *but* X *does not occur bound at the* k*th place of* φ*, we say that* X ***occurs free at the*** k***th place of*** φ.

(iv) X *is a* ***bound variable*** *of* φ *if there exists* k *such that* X *occurs bound at the* k*th place of* φ.

(v) X *is a* ***free variable*** *of* φ *if there exists* k *such that* X *occurs free at the* k*th place of* φ.

Notation 10.2. *Let us denote, for every pattern* φ,

$$\begin{aligned} FV_{SVar}(\varphi) &= \{X \in SVar(\varphi) \mid X \text{ is free in } \varphi\}, \\ BV_{SVar}(\varphi) &= \{X \in SVar(\varphi) \mid X \text{ is bound in } \varphi\}, \\ NotFV_{SVar}(\varphi) &= SVar(\varphi) \setminus FV_{SVar}(\varphi), \\ Fresh_{SVar}(\varphi) &= SVar \setminus FV_{SVar}(\varphi). \end{aligned}$$

Remark 10.3. *(i)* $SVar(\varphi) = FV_{SVar}(\varphi) \cup BV_{SVar}(\varphi)$.

(ii) $FV_{SVar}(\varphi)$ *and* $BV_{SVar}(\varphi)$ *are not disjoint, in general, as* $X \in SVar$ *can be both free and bound in a pattern* φ.

(iii) $NotFV_{SVar}(\varphi) \subseteq BV_{SVar}(\varphi)$.

(iv) $Fresh_{SVar}(\varphi) = (SVar \setminus SVar(\varphi)) \cup NotFV_{SVar}(\varphi)$.

Proof. We have that

$$\begin{aligned} Fresh_{SVar}(\varphi) = SVar \setminus FV_{SVar}(\varphi) &= \Big(SVar(\varphi) \cup \big(SVar \setminus SVar(\varphi)\big)\Big) \setminus FV_{SVar}(\varphi) \\ &\overset{(16)}{=} \big(SVar \setminus SVar(\varphi)\big) \cup \big(SVar(\varphi) \setminus FV_{SVar}(\varphi)\big) \\ &= (SVar \setminus SVar(\varphi)) \cup NotFV_{SVar}(\varphi). \end{aligned}$$

□

Notation 10.4. *Let us denote, for every pattern* φ *and set variable* X,

$$\begin{aligned} FreeOccur_X(\varphi) &= \text{the set of all free occurrences of } X \text{ in } \varphi, \\ BoundOccur_X(\varphi) &= \text{the set of all bound occurrences of } X \text{ in } \varphi. \end{aligned}$$

Lemma 10.5. *(i)* $X \in FV_{SVar}(\varphi)$ *iff* $FreeOccur_X(\varphi) \neq \emptyset$.

(ii) $X \in BV_{SVar}(\varphi)$ *iff* $BoundOccur_X(\varphi) \neq \emptyset$.

(iii) $FreeOccur_X(\varphi) \cap BoundOccur_X(\varphi) = \emptyset$.

(iv) $Occur_X(\varphi) = FreeOccur_X(\varphi) \cup BoundOccur_X(\varphi)$.

(v) $FreeOccur_X(\varphi) = Occur_X(\varphi) \setminus BoundOccur_X(\varphi)$ *and* $BoundOccur_X(\varphi) = Occur_X(\varphi) \setminus FreeOccur_X(\varphi)$.

Remark 10.6 (Definition by recursion of FV_{SVar})**.**
The mapping

$$FV_{SVar} : Pattern \to 2^{SVar}, \quad \varphi \mapsto FV_{SVar}(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
FV_{SVar}(\varphi) & = & SVar(\varphi) & \textit{if } \varphi \textit{ is an atomic pattern,} \\
FV_{SVar}(-\varphi) & = & FV_{SVar}(\varphi) & \textit{for any } - \in \mathcal{P}_1, \\
FV_{SVar}(\circ\varphi\psi) & = & FV_{SVar}(\varphi) \cup FV_{SVar}(\psi) & \textit{for any } \circ \in \mathcal{P}_2, \\
FV_{SVar}(\sim\varphi\psi) & = & FV_{SVar}(\varphi) \cup FV_{SVar}(\psi) & \textit{for any } \sim \in Equal, \\
FV_{SVar}(Qx\varphi) & = & FV_{SVar}(\varphi) & \textit{for any } Q \in \mathcal{Q} \textit{ and } x \in EVar, \\
FV_{SVar}(\overline{Q}X\varphi) & = & FV_{SVar}(\varphi) \setminus \{X\} & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = 2^{SVar}$ and

$$\begin{array}{lcll}
G_0(\varphi) & = & SVar(\varphi), & \\
G_-(V,\varphi) & = & V & \text{for any } - \in \mathcal{P}_1, \\
G_\circ(V_1,V_2,\varphi,\psi) & = & V_1 \cup V_2 & \text{for any } \circ \in \mathcal{P}_2, \\
G_\sim(V_1,V_2,\varphi,\psi) & = & V_1 \cup V_2 & \text{for any } \sim \in Equal, \\
G_Q(V,x,\varphi) & = & V & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\
G_{\overline{Q}}(V,X,\varphi) & = & V \setminus \{X\} & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar.
\end{array}$$

Then

(i) $FV_{SVar}(\varphi) = SVar(\varphi) = G_0(\varphi)$ if φ is an atomic pattern.

(ii) For $- \in \mathcal{P}_1$, we have that

$$FV_{SVar}(-\varphi) = FV_{SVar}(\varphi) = G_-(FV_{SVar}(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$FV_{SVar}(\circ\varphi\psi) = FV_{SVar}(\varphi) \cup FV_{SVar}(\psi) = G_\circ(FV_{SVar}(\varphi), FV_{SVar}(\psi), \varphi, \psi).$$

(iv) For $\sim \in Equal$, we have that

$$FV_{SVar}(\sim\varphi\psi) = FV_{SVar}(\varphi) \cup FV_{SVar}(\psi) = G_\sim(FV_{SVar}(\varphi), FV_{SVar}(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$FV_{SVar}(Qx\varphi) = FV_{SVar}(\varphi) = G_Q(FV_{SVar}(\varphi), x, \varphi).$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$FV_{SVar}(\overline{Q}X\varphi) = FV_{SVar}(\varphi) \setminus \{X\} = G_{\overline{Q}}(FV_{SVar}(\varphi), X, \varphi).$$

Thus, $FV_{SVar} : Pattern \to 2^{SVar}$ is the unique mapping given by Proposition 2.11. □

Remark 10.7 (Definition by recursion of BV_{SVar})**.**
The mapping

$$BV_{SVar} : Pattern \to 2^{SVar}, \quad \varphi \mapsto BV_{SVar}(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
BV_{SVar}(\varphi) & = & \emptyset & \textit{if } \varphi \textit{ is an atomic pattern,} \\
BV_{SVar}(-\varphi) & = & BV_{SVar}(\varphi) & \textit{for any } - \in \mathcal{P}_1, \\
BV_{SVar}(\circ\varphi\psi) & = & BV_{SVar}(\varphi) \cup BV_{SVar}(\psi) & \textit{for any } \circ \in \mathcal{P}_2, \\
BV_{SVar}(\sim\varphi\psi) & = & BV_{SVar}(\varphi) \cup BV_{SVar}(\psi) & \textit{for any } \sim \in Equal, \\
BV_{SVar}(Qx\varphi) & = & BV_{SVar}(\varphi) & \textit{for any } Q \in \mathcal{Q} \textit{ and } x \in EVar, \\
BV_{SVar}(\overline{Q}X\varphi) & = & BV_{SVar}(\varphi) \cup \{X\} & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = 2^{SVar}$ and

$$\begin{array}{lcll} G_0(\varphi) & = & \emptyset, & \\ G_-(V,\varphi) & = & V & \text{for any } - \in \mathcal{P}_1, \\ G_\circ(V_1,V_2,\varphi,\psi) & = & V_1 \cup V_2 & \text{for any } \circ \in \mathcal{P}_2, \\ G_\sim(V_1,V_2,\varphi,\psi) & = & V_1 \cup V_2 & \text{for any } \sim \in Equal, \\ G_Q(V,x,\varphi) & = & V & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\ G_{\overline{Q}}(V,X,\varphi) & = & V \cup \{X\} & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar. \end{array}$$

Then

(i) $BV_{SVar}(\varphi) = \emptyset = G_0(\varphi)$ if φ is an atomic pattern.

(ii) For $- \in \mathcal{P}_1$, we have that

$$BV_{SVar}(-\varphi) = BV_{SVar}(\varphi) = G_-(BV_{SVar}(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$BV_{SVar}(\circ\varphi\psi) = BV_{SVar}(\varphi) \cup BV_{SVar}(\psi) = G_\circ(BV_{SVar}(\varphi), BV_{SVar}(\psi), \varphi, \psi).$$

(iv) For $\sim \in Equal$, we have that

$$BV_{SVar}(\sim \varphi\psi) = BV_{SVar}(\varphi) \cup BV_{SVar}(\psi) = G_\sim(BV_{SVar}(\varphi), BV_{SVar}(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$BV_{SVar}(Qx\varphi) = BV_{SVar}(\varphi) = G_Q(BV_{SVar}(\varphi), x, \varphi).$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$BV_{SVar}(\overline{Q}X\varphi) = BV_{SVar}(\varphi) \cup \{X\} = G_{\overline{Q}}(BV_{SVar}(\varphi), X, \varphi).$$

Thus, $BV_{SVar} : Pattern \to 2^{SVar}$ is the unique mapping given by Proposition 2.11. ☐

Remark 10.8 (Definition by recursion of $NotFV_{SVar}$)**.**
The mapping

$$NotFV_{SVar} : Pattern \to 2^{SVar}, \quad \varphi \mapsto NotFV_{SVar}(\varphi)$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll} NotFV_{SVar}(\varphi) & = & \emptyset & \textit{if } \varphi \textit{ is an atomic pattern,} \\ NotFV_{SVar}(-\varphi) & = & NotFV_{SVar}(\varphi) & \textit{for any } - \in \mathcal{P}_1, \\ NotFV_{SVar}(\circ\varphi\psi) & = & NotFV_{SVar}(\varphi) \cap NotFV_{SVar}(\psi) & \textit{for any } \circ \in \mathcal{P}_2, \\ FV_{SVar}(\sim \varphi\psi) & = & NotFV_{SVar}(\varphi) \cap NotFV_{SVar}(\psi) & \textit{for any } \sim \in Equal, \\ NotFV_{SVar}(Qx\varphi) & = & NotFV_{SVar}(\varphi) & \textit{for any } Q \in \mathcal{Q} \textit{ and } x \in EVar, \\ NotFV_{SVar}(\overline{Q}X\varphi) & = & NotFV_{SVar}(\varphi) \cup \{X\} & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar. \end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = 2^{SVar}$ and

$$\begin{array}{lcll} G_0(\varphi) & = & \emptyset, & \\ G_-(V,\varphi) & = & V & \text{for any } - \in \mathcal{P}_1, \\ G_\circ(V_1,V_2,\varphi,\psi) & = & V_1 \cap V_2 & \text{for any } \circ \in \mathcal{P}_2, \\ G_\sim(V_1,V_2,\varphi,\psi) & = & V_1 \cap V_2 & \text{for any } \sim \in Equal, \\ G_Q(V,x,\varphi) & = & V & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\ G_{\overline{Q}}(V,X,\varphi) & = & V \cup \{X\} & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar. \end{array}$$

Then

(i) $NotFV_{SVar}(\varphi) = \emptyset = G_0(\varphi)$ if φ is an atomic pattern.

(ii) For $- \in \mathcal{P}_1$, we have that

$$NotFV_{SVar}(-\varphi) = NotFV_{SVar}(\varphi) = G_-(NotFV_{SVar}(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$\begin{aligned} NotFV_{SVar}(\circ\varphi\psi) &= NotFV_{SVar}(\varphi) \cap NotFV_{SVar}(\psi) \\ &= G_\circ(NotFV_{SVar}(\varphi), NotFV_{SVar}(\psi), \varphi, \psi). \end{aligned}$$

(iv) For $\sim\in Equal$, we have that

$$\begin{aligned} NotFV_{SVar}(\sim \varphi\psi) &= NotFV_{SVar}(\varphi) \cap NotFV_{SVar}(\psi) \\ &= G_\sim(NotFV_{SVar}(\varphi), NotFV_{SVar}(\psi), \varphi, \psi). \end{aligned}$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$NotFV_{SVar}(Qx\varphi) = NotFV_{SVar}(\varphi) = G_Q(NotFV_{SVar}(\varphi), x, \varphi)$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$, we have that

$$\begin{aligned} NotFV_{SVar}(\overline{Q}X\varphi) &= SVar(\overline{Q}X\varphi) \setminus FV_{SVar}(\overline{Q}X\varphi) \\ &= \Big(SVar(\varphi) \cup \{X\}\Big) \setminus \Big(FV_{SVar}(\varphi) \setminus \{X\}\Big) \\ &\overset{(14)}{=} \Big(\big(SVar(\varphi) \cup \{X\}\big) \setminus FV_{SVar}(\varphi)\Big) \cup \{X\} \\ &\overset{(15)}{=} \big(SVar(\varphi) \setminus FV_{SVar}(\varphi)\big) \cup \{X\} = NotFV_{SVar}(\varphi) \cup \{X\} \\ &= G_{\overline{Q}}(NotFV_{SVar}(\varphi), X, \varphi). \end{aligned}$$

Thus, $NotFV_{SVar} : Pattern \to 2^{SVar}$ is the unique mapping given by Proposition 2.11. □

11 Substitution of free occurrences of set variables

Let $X \in SVar$ and φ, δ be patterns.

Definition 11.1. *We define $Subf_\delta^X\varphi$ to be the expression obtained from φ by replacing every free occurrence of X in φ with δ.*

Remark 11.2.

$$Subf_\delta^X\varphi = Repl_\delta^X(\varphi; FreeOccur_X(\varphi)).$$

Proposition 11.3 (Definition by recursion)**.**
The mapping

$$Subf_\delta^X : Pattern \to Expr, \quad Subf_\delta^X(\varphi) = Subf_\delta^X\varphi$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll} Subf_\delta^X(Z) &=& \begin{cases} \delta & \text{if } X = Z \\ Z & \text{if } X \neq Z \end{cases} & \text{if } Z \in SVar, \\ Subf_\delta^X(\varphi) &=& \varphi & \text{if } \varphi \in EVar \cup \Sigma \cup \mathcal{P}_C, \\ Subf_\delta^X(-\varphi) &=& -Subf_\delta^X(\varphi) & \text{for any } - \in \mathcal{P}_1, \\ Subf_\delta^X(\circ\varphi\psi) &=& \circ Subf_\delta^X(\varphi)Subf_\delta^X(\psi) & \text{for any } \circ \in \mathcal{P}_2, \\ Subf_\delta^X(\sim \varphi\psi) &=& \sim Subf_\delta^X(\varphi)Subf_\delta^X(\psi) & \text{for any } \sim\in Equal, \\ Subf_\delta^X(Qx\varphi) &=& QxSubf_\delta^X(\varphi) & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\ Subf_\delta^X(\overline{Q}Z\varphi) &=& \begin{cases} \overline{Q}Z\varphi & \text{if } X = Z \\ \overline{Q}ZSubf_\delta^X(\varphi) & \text{if } X \neq Z \end{cases} & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } Z \in SVar. \end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = Expr$ and

$$G_0(\varphi) = \begin{cases} \delta & \text{if } \varphi = X \\ \varphi & \text{if } \varphi \in (SVar \setminus \{X\}) \cup EVar \cup \Sigma \cup \mathcal{P}_C \end{cases}$$

$$\begin{array}{lcll} G_{-}(\theta,\varphi) & = & -\theta & \text{for any } - \in \mathcal{P}_1, \\ G_{\circ}(\theta,\tau,\varphi,\psi) & = & \circ\theta\tau & \text{for any } \circ \in \mathcal{P}_2, \\ G_{\sim}(\theta,\tau,\varphi,\psi) & = & \sim\theta\tau & \text{for any } \sim \in Equal, \\ G_{Q}(\theta,x,\varphi) & = & Qx\theta & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\ G_{\overline{Q}}(\theta,Z,\varphi) & = & \begin{cases} \overline{Q}Z\varphi & \text{if } X = Z \\ \overline{Q}Z\theta & \text{if } X \neq Z \end{cases} & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } Z \in SVar. \end{array}$$

Then

(i) If φ is an atomic pattern, we have the following cases:

  (a) $\varphi = X$. Then $Subf_\delta^X(\varphi) = Subf_\delta^X(X) = \delta = G_0(\varphi)$.

  (b) $\varphi \in (SVar \setminus \{X\}) \cup EVar \cup \Sigma \cup \mathcal{P}_C$. Then $Subf_\delta^X(\varphi) = \varphi = G_0(\varphi)$.

(ii) For $- \in \mathcal{P}_1$, we have that

$$Subf_\delta^X(-\varphi) = -Subf_\delta^X(\varphi) = G_{-}(Subf_\delta^X(\varphi), \varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$Subf_\delta^X(\circ\varphi\psi) = \circ Subf_\delta^X(\varphi)Subf_\delta^X(\psi) = G_{\circ}(Subf_\delta^X(\varphi), Subf_\delta^X(\psi), \varphi, \psi).$$

(iv) For $\sim \in Equal$, we have that

$$Subf_\delta^X(\sim\varphi\psi) = \sim Subf_\delta^X(\varphi)Subf_\delta^X(\psi) = G_{\sim}(Subf_\delta^X(\varphi), Subf_\delta^X(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$Subf_\delta^X(Qx\varphi) = QxSubf_\delta^X(\varphi) = G_Q(Subf_\delta^X(\varphi), x, \varphi)$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $Z \in SVar$ we have that

$$Subf_\delta^X(\overline{Q}Z\varphi) = \begin{cases} \overline{Q}Z\varphi & \text{if } X = Z \\ \overline{Q}ZSubf_\delta^X(\varphi) & \text{if } X \neq Z \end{cases} = G_{\overline{Q}}(Subf_\delta^X(\varphi), Z, \varphi).$$

Thus, $Subf_\delta^X$ is the unique mapping given by Proposition 2.11. □

Proposition 11.4. *$Subf_\delta^X\varphi$ is a pattern.*

Proof. The proof is immediate by induction on φ, using Proposition 11.3. □

Lemma 11.5. *$Subf_X^X\varphi = \varphi$.*

Proof. By Remark 11.2 and Lemma A.8.(i) with $\mathbf{a} := \varphi$, $\mathbf{b} := X$, $\mathbf{c} := X$, $I := FreeOccur_X(\varphi)$. □

Lemma 11.6. *If X does not occur (free) in φ, then $Subf_\delta^X\varphi = \varphi$.*

Proof. Apply Remark 11.2, the fact that $FreeOccur_X(\varphi) = \emptyset$ and Lemma A.8.(ii) with $\mathbf{a} := \varphi$, $\mathbf{b} := X$, $\mathbf{c} := \delta$. □

12 Bounded substitution of set variables

Let φ be a pattern and X, Y be set variables.

Definition 12.1. *We define $Subb^X_Y\varphi$ to be the expression obtained from φ by replacing every bound occurrence of X in φ with Y.*

Remark 12.2. *Then*

$$Subb^X_Y\varphi = Repl^X_Y(\varphi; BoundOccur_X(\varphi)).$$

Proposition 12.3 (Definition by recursion)**.**
If $X = Y$, then obviously $Subb^X_Y\varphi = \varphi$. Assume that $X \neq Y$. Then the mapping

$$Subb^X_Y : Pattern \to Expr, \quad Subb^X_Y(\varphi) = Subb^X_Y\varphi$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
Subb^X_Y(\varphi) & = & \varphi & \textit{if } \varphi \textit{ is an atomic pattern,}\\
Subb^X_Y(-\varphi) & = & -Subb^X_Y(\varphi) & \textit{for any } - \in \mathcal{P}_1,\\
Subb^X_Y(\circ\varphi\psi) & = & \circ Subb^X_Y(\varphi)Subb^X_Y(\psi) & \textit{for any } \circ \in \mathcal{P}_2,\\
Subb^X_Y(\sim\varphi\psi) & = & \sim Subb^X_Y(\varphi)Subb^X_Y(\psi) & \textit{for any } \sim\in Equal,\\
Subb^X_Y(Qx\varphi) & = & QxSubb^X_Y(\varphi) & \textit{for any } Q \in \mathcal{Q} \textit{ and } x \in EVar,\\
Subb^X_Y(\overline{Q}Z\varphi) & = & \begin{cases} \overline{Q}Y Subf^X_Y\big(Subb^X_Y(\varphi)\big) & \textit{if } X = Z\\ \overline{Q}ZSubb^X_Y(\varphi) & \textit{if } X \neq Z \end{cases} & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } Z \in SVar.
\end{array}$$

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = Expr$ and

$$\begin{array}{lcll}
G_0(\varphi) & = & \varphi & \text{if } \varphi \text{ is an atomic pattern,}\\
G_-(\theta,\varphi) & = & -\theta & \text{for any } - \in \mathcal{P}_1,\\
G_\circ(\theta,\tau,\varphi,\psi) & = & \circ\theta\tau & \text{for any } \circ \in \mathcal{P}_2,\\
G_\sim(\theta,\tau,\varphi,\psi) & = & \sim\theta\tau & \text{for any } \sim\in Equal,\\
G_Q(\theta,x,\varphi) & = & Qx\theta & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar,\\
G_{\overline{Q}}(\theta,Z,\varphi) & = & \begin{cases} \overline{Q}Y Subf^X_Y(\theta) & \text{if } X = Z\\ \overline{Q}Z\theta & \text{if } X \neq Z \end{cases} & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } Z \in SVar.
\end{array}$$

Then

(i) If φ is an atomic pattern, $Subb^X_Y(\varphi) = \varphi = G_0(\varphi)$.

(ii) For $- \in \mathcal{P}_1$, we have that

$$Subb^X_Y(-\varphi) = -Subb^X_Y(\varphi) = G_-(Subb^X_Y(\varphi),\varphi).$$

(iii) For $\circ \in \mathcal{P}_2$, we have that

$$Subb^X_Y(\circ\varphi\psi) = \circ Subb^X_Y(\varphi)Subb^X_Y(\psi) = G_\circ(Subb^X_Y(\varphi), Subb^X_Y(\psi), \varphi, \psi).$$

(iv) For $\sim\in Equal$, we have that

$$Subb^X_Y(\sim\varphi\psi) =\sim Subb^X_Y(\varphi)Subb^X_Y(\psi) = G_\sim(Subb^X_Y(\varphi), Subb^X_Y(\psi), \varphi, \psi).$$

(v) For $Q \in \mathcal{Q}$ and $x \in EVar$, we have that

$$Subb^X_Y(Qx\varphi) = QxSubb^X_Y(\varphi) = G_Q(Subb^X_Y(\varphi), x, \varphi).$$

(vi) For $\overline{Q} \in \overline{\mathcal{Q}}$ and $Z \in SVar$, we have that

$$Subb_Y^X(\overline{Q}Z\varphi) = \begin{cases} \overline{Q}Y\, Subf_Y^X\left(Subb_Y^X(\varphi)\right) & \text{if } X = Z \\ \overline{Q}Z\, Subb_Y^X(\varphi) & \text{if } X \neq Z \end{cases} = G_{\overline{Q}}(Subb_Y^X(\varphi), Z, \varphi).$$

Thus, $Subb_Y^X$ is the unique mapping given by Proposition 2.11. □

Proposition 12.4. *$Subb_Y^X\varphi$ is a pattern.*

Proof. The proof is immediate by induction on φ, using Proposition 12.3. □

Lemma 12.5. *$Subb_X^X\varphi = \varphi$.*

Proof. By Remark 12.2 and Lemma A.8.(i) with $\mathbf{a} := \varphi$, $\mathbf{b} := X$, $\mathbf{c} := X$, $I := BoundOccur_X(\varphi)$. □

Lemma 12.6. *If x does not occur (bound) in φ, then $Subb_Y^X\varphi = \varphi$.*

Proof. Apply Remark 12.2, the fact that $BoundOccur_X(\varphi) = \emptyset$ and Lemma A.8.(ii) with $\mathbf{a} := \varphi$, $\mathbf{b} := X$, $\mathbf{c} := Y$. □

13 Variable free for patterns

Let $xX \in EVar \cup SVar$ and δ, φ be patterns.

Definition 13.1. *We say that xX is **free for** δ in φ or that δ is **substitutable for** xX in φ if the following hold:*

(i) if z is an element variable occuring free in δ and Q is a quantifier on z in φ with scope θ, then xX does not occur free in θ.

(ii) if Z is a set variable occuring free in δ and $\overline{Q}$ is a binder on Z in φ with scope θ, then xX does not occur free in θ.

Definition 13.2. *Define the mappings*

$$FreeFor_\delta : Pattern \to 2^{EVar\cup SVar}, \quad FreeForE_\delta : Pattern \to 2^{EVar},$$
$$FreeForS_\delta : Pattern \to 2^{SVar}$$

as follows: for any pattern φ,

$$\begin{aligned} FreeFor_\delta(\varphi) &= \{xX \in EVar \cup SVar \mid xX \text{ is free for } \delta \text{ in } \varphi\}, \\ FreeForE_\delta(\varphi) &= FreeFor_\delta(\varphi) \cap EVar, \\ FreeForS_\delta(\varphi) &= FreeFor_\delta(\varphi) \cap SVar. \end{aligned}$$

As $EVar \cap SVar = \emptyset$, we have that $FreeForE_\delta(\varphi) \cap FreeForS_\delta(\varphi) = \emptyset$ and $FreeFor_\delta(\varphi) = FreeForE_\delta(\varphi) \cup FreeForS_\delta(\varphi)$.

Proposition 13.3 (Definition by recursion)**.**
The mapping

$$FreeFor_\delta : Pattern \to 2^{EVar\cup SVar}$$

can be defined by recursion on patterns as follows:

$$\begin{aligned} FreeFor_\delta(\varphi) &= EVar \cup SVar && \text{if } \varphi \text{ is an atomic pattern}, \\ FreeFor_\delta(-\psi) &= FreeFor_\delta(\psi) && \text{for any } - \in \mathcal{P}_1, \\ FreeFor_\delta(\ominus\psi\chi) &= FreeFor_\delta(\psi) \cap FreeFor_\delta(\chi) && \text{for any } \ominus \in \mathcal{P}_2 \cup Equal, \end{aligned}$$

$$FreeFor_\delta(Qx\psi) = \begin{cases} FreshF(\psi) \cup Fresh_{SVar}(\psi) & \text{if } x \text{ occurs in } \delta \\ FreeFor_\delta(\psi) & \text{if } x \text{ does not occur in } \delta \end{cases}$$
for any $Q \in \mathcal{Q}$ *and* $x \in EVar$,

$$FreeFor_\delta(\overline{Q}X\psi) = \begin{cases} FreshF(\psi) \cup Fresh_{SVar}(\psi) & \text{if } X \text{ occurs in } \delta \\ FreeFor_\delta(\psi) & \text{if } X \text{ does not occur in } \delta \end{cases}$$
for any $\overline{Q} \in \overline{\mathcal{Q}}$ *and* $X \in SVar$.

Lemma 13.4. *Let* φ *be a pattern.*

(i) $FreshF(\varphi) \cup Fresh_{SVar}(\varphi) \subseteq FreeFor_\delta(\varphi)$, *that is any element or set variable that*

either (1) does not occur in φ *or (2) occurs in* φ*, but does not occur free in* φ

is free for δ *in* φ.

(ii) *If* $EVar(\delta) \cap EVar(\varphi) = \emptyset$ *and* $SVar(\delta) \cap SVar(\varphi) = \emptyset$, *then* $FreeFor_\delta(\varphi) = EVar \cup SVar$, *that is if element/set variables of* δ *do not occur in* φ, *then any element/set variable is free for* δ *in* φ.

(iii) *If* $EVar(\delta) = SVar(\delta) = \emptyset$, *then* $FreeFor_\delta(\varphi) = EVar \cup SVar$.

(iv) *If* $EVar(\delta) \cap BV(\varphi) = \emptyset$ *and* $SVar(\delta) \cap BV_{SVar}(\varphi) = \emptyset$, *then* $FreeFor_\delta(\varphi) = EVar \cup SVar$, *that is if element/set variables of* δ *do not occur bound in* φ, *then any element/set variable is free for* δ *in* φ.

13.1 x free for y

Remark 13.5. *Let* $x, y \in EVar$. *The following are equivalent:*

(i) x *is free for* y *in* φ.

(ii) *For every quantifier* Q *on* y *in* φ *with scope* θ, *we have that* x *does not occur free in* θ.

(iii) *For every subpattern* $Qy\psi$ *of* φ, *we have that* x *does not occur free in* $Qy\psi$.

Definition 13.6. *Let* $y \in EVar$. *Define the mapping* $FreeForE_y : Pattern \to EVar$ *as follows: for any pattern* φ,

$$FreeForE_y(\varphi) = \{x \in EVar \mid x \text{ is free for } y \text{ in } \varphi\}.$$

Lemma 13.7. *Let* x, $y \in EVar$ *and* φ *be a pattern.*

(i) $y \in FreeForE_y(\varphi)$, *that is* y *is free for* y *in* φ.

(ii) *If* x *does not occur in* φ, *then* $x \in FreeForE_y(\varphi)$.

(iii) *If* x *occurs in* φ, *but* x *does not occur free in* φ, *then* $x \in FreeForE_y(\varphi)$.

(iv) $FreshF(\varphi) \subseteq FreeForE_y(\varphi)$.

(v) *If* y *does not occur in* φ, *then* $FreeForE_y(\varphi) = EVar$.

(vi) *If* y *occurs in* φ, *but* y *does not occur bound in* φ, *then* $FreeForE_y(\varphi) = EVar$.

(vii) *If* $y \in FreshB(\varphi)$, *then* $FreeForE_y(\varphi) = EVar$.

(viii) *If* $BV(\varphi) = \emptyset$, *then* $FreeForE_y(\varphi) = EVar$.

Proof. (i) Obviously. If $Qy\psi$ is a subpattern of φ, then y occurs bound in $Qy\psi$, hence y does not occur free in $Qy\psi$.

(ii) Obviously. Assume that x does not occur in φ. If $Qy\psi$ is a subpattern of φ, then x does not occur in $Qy\psi$. In particular, x does not occur free in $Qy\psi$.

(iii) Obviously. Assume that x occurs in φ, but x does not occur free in φ. If $Qy\psi$ is a subpattern of φ, then x does not occur free in $Qy\psi$.

(iv) Apply (ii), (iii) and Lemma 6.3(iv).

(v) Obviously. If y does not occur in φ, then there exists no subpattern $Qy\psi$ of φ.

(vi) Obviously. If y occurs in φ, but y does not occur bound in φ, then there exists no subpattern $Qy\psi$ of φ.

(vii) Apply (v), (vi) and Lemma 6.3(v).

(viii) By (vii).

□

Proposition 13.8 (Definition by recursion)**.**
Let $y \in EVar$. The mapping

$$FreeForE_y : Pattern \to EVar$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
FreeForE_y(\varphi) & = & EVar & \textit{if } \varphi \textit{ is an atomic pattern,}\\
FreeForE_y(-\psi) & = & FreeForE_y(\psi) & \textit{for any } - \in \mathcal{P}_1,\\
FreeForE_y(\ominus\psi\chi) & = & FreeForE_y(\psi) \cap FreeForE_y(\chi) & \textit{for any } \ominus \in \mathcal{P}_2 \cup Equal,\\
FreeForE_y(Qz\psi) & = & \begin{cases} FreshF(Qz\psi) & \textit{if } z = y \\ FreeForE_y(\psi) & \textit{if } z \neq y \end{cases} & \textit{for any } Q \in \mathcal{Q} \textit{ and } z \in EVar,\\
FreeForE_y(\overline{Q}X\psi) & = & FreeForE_y(\psi) & \textit{for any } \overline{Q} \in \overline{\mathcal{Q}} \textit{ and } X \in SVar.
\end{array}$$

Proof. Let us verify that $FreeForE_y$ as in Definition 13.6 satisfies the conditions.

(i) If φ is an atomic pattern, then $BV(\varphi) = \emptyset$. Apply Lemma 13.7(viii).

(ii) $\varphi = -\psi$ for $- \in \mathcal{P}_1$. For every $x \in EVar$, we have that

$$\begin{array}{lll}
x \in FreeForE_y(\varphi) & \text{iff} & \text{for every subpattern } Qy\delta \text{ of } \varphi,\ x \notin FV(Qy\delta)\\
 & \text{iff} & \text{for every subpattern } Qy\delta \text{ of } \psi,\ x \notin FV(Qy\delta),\\
 & & \text{by Proposition 2.14 and the fact that } Qy\delta \neq -\psi\\
 & \text{iff} & x \in FreeForE_y(\psi).
\end{array}$$

(iii) $\varphi = \ominus\psi\chi$ for $\ominus \in \mathcal{P}_2 \cup Equal$. For every $x \in EVar$, we have that

$$\begin{array}{lll}
x \in FreeForE_y(\varphi) & \text{iff} & \text{for every subpattern } Qy\delta \text{ of } \varphi,\ x \notin FV(Qy\delta)\\
 & \text{iff} & (\text{for every subpattern } Qy\delta \text{ of } \psi,\ x \notin FV(Qy\delta)) \text{ and}\\
 & & (\text{for every subpattern } Qy\delta \text{ of } \chi,\ x \notin FV(Qy\delta)),\\
 & & \text{by Proposition 2.14 and the fact that } Qy\delta \neq \ominus\psi\chi\\
 & \text{iff} & x \in FreeForE_y(\psi) \cap FreeForE_y(\chi).
\end{array}$$

(iv) $\varphi = Qz\psi$ for $Q \in \mathcal{Q}$ and $z \in EVar$. We have two cases:

(a) $z \neq y$. Then for every $x \in EVar$, we have that

$$\begin{aligned} x \in FreeForE_y(\varphi) \quad & \text{iff} \quad \text{for every subpattern } Qy\delta \text{ of } \varphi,\ x \notin FV(Qy\delta) \\ & \text{iff} \quad \text{for every subpattern } Qy\delta \text{ of } \psi,\ x \notin FV(Qy\delta), \\ & \qquad \text{by Proposition 2.14 and the fact that } Qy\delta \neq Qz\psi \\ & \text{iff} \quad x \in FreeForE_y(\psi). \end{aligned}$$

(b) $z = y$, hence $\varphi = Qy\psi$. We prove that $FreeForE_y(Qy\psi) = FreshF(Qy\psi)$ by double inclusion.

$\subseteq$ Let $x \in FreeForE_y(Qy\psi)$. As $Qy\psi$ is a subpattern of $Qy\psi$, we must have that x does not occur free in $Qy\psi$. Thus, $x \in FreshF(Qy\psi)$.

$\supseteq$ By Lemma 13.7(iv).

(v) $\varphi = \overline{Q}X\psi$ for $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$. For every $x \in EVar$, we have that

$$\begin{aligned} x \in FreeForE_y(\varphi) \quad & \text{iff} \quad \text{for every subpattern } Qy\delta \text{ of } \varphi,\ x \notin FV(Qy\delta) \\ & \text{iff} \quad \text{for every subpattern } Qy\delta \text{ of } \psi,\ x \notin FV(Qy\delta), \\ & \qquad \text{by Proposition 2.14 and the fact that } Qy\delta \neq \overline{Q}X\psi \\ & \text{iff} \quad x \in FreeForE_y(\psi). \end{aligned}$$

□

13.1.1 Useful lemmas for *Subb*

Proposition 13.9. *Assume that $x, y \in EVar$ are such that $x \neq y$ and y does not occur in φ. Then x is free for y in $Subb^x_y\varphi$.*

Proof. The proof is by induction on φ.

(i) φ is an atomic pattern. Then $Subb^x_y\varphi = \varphi$ and x is free for y in φ, as y does not occur in φ.

(ii) $\varphi = -\psi$ for $- \in \mathcal{P}_1$. As y does not occur in ψ, we can apply the induction hypothesis to get that x is free for y in $Subb^x_y\psi$. It follows that x is free for y in $Subb^x_y\varphi = -Subb^x_y\psi$.

(iii) $\varphi = \circ\psi\chi$ for $\circ \in \mathcal{P}_2$. As y does not occur in ψ, χ, we can apply the induction hypothesis to get that x is free for y in $Subb^x_y\psi$, $Subb^x_y\chi$. It follows that x is free for y in $Subb^x_y\varphi = \circ Subb^x_y(\psi) Subb^x_y(\chi)$.

(iv) $\varphi = \sim\psi\chi$ for $\sim \in Equal$. As y does not occur in ψ, χ, we can apply the induction hypothesis to get that x is free for y in $Subb^x_y\psi$, $Subb^x_y\chi$. It follows that x is free for y in $Subb^x_y\varphi = \sim Subb^x_y(\psi) Subb^x_y(\chi)$.

(v) $\varphi = Qz\psi$. We have two cases:

(a) $x = z$. Then $Subb^x_y\varphi = QySubf^x_y\big(Subb^x_y(\psi)\big)$. It is obvious that x does not occur in $Subb^x_y\varphi$, so x is free for y in $Subb^x_y\varphi$.

(b) $x \neq z$. Then $Subb^x_y\varphi = QzSubb^x_y(\psi)$. As y does not occur in ψ, we can apply the induction hypothesis to get that x is free for y in $Subb^x_y\psi$. As y does not occur in φ, we must have that $y \neq z$. Then, obviously x is free for y in $Subb^x_y\varphi$.

(vi) $\varphi = \overline{Q}X\psi$. As y does not occur in ψ, we can apply the induction hypothesis to get that x is free for y in $Subb^x_y\psi$. Then, obviously x is free for y in $Subb^x_y\varphi = \overline{Q}XSubb^x_y(\psi)$.

□

Lemma 13.10. *Let φ be a pattern and y, z be distinct variables. Then $FreeForE_y\left(Subb^y_z\varphi\right) = EVar$, that is any element variable is free for y in $Subb^y_z\varphi$.*

Proof. By Lemma 8.7(i), y does not occur bound in $Subb^y_z\varphi$, that is $y \notin BV\,(Subb^y_z\varphi))$, so $y \in FreshB\,(Subb^y_z\varphi))$. Apply Lemma 13.7(vii) to conclude that $FreeForE_y\,(Subb^y_z\varphi) = EVar$. □

Lemma 13.11. *Let φ be a pattern and x, y, z be variables. Then*

$$x \text{ is free for } y \text{ in } \varphi \quad \text{iff} \quad x \text{ is free for } y \text{ in } Subb^x_z\varphi.$$

Proof. If $x = z$, then $Subb^x_z\varphi = \varphi$, by Lemma 8.5. The conclusion is obvious. If $x = y$, then, by Lemma 13.7(i), both y is free for y in φ and y is free for y in $Subb^y_z\varphi$ hold.
Assume in the sequel that $x \neq y$ and $x \neq z$.
We prove by induction on φ, using the definition by recursion of $Subb^x_z\varphi$ (Proposition 8.3).

(i) φ is an atomic pattern. Then $\varphi = Subb^x_z\varphi$.The conclusion is obvious.

(ii) $\varphi = -\psi$ for $- \in \mathcal{P}_1$. We get that

$$\begin{array}{lll} x \in FreeForE_y(\varphi) & \text{iff} & x \in FreeForE_y(\psi) \\ & & \text{by Proposition 13.8} \\ & \text{iff} & x \in FreeForE_y(Subb^x_z\psi) \\ & & \text{by the induction hypothesis for } \psi \\ & \text{iff} & x \in FreeForE_y(-Subb^x_z\psi) \\ & & \text{by Proposition 13.8} \\ & \text{iff} & x \in FreeForE_y(Subb^x_z\varphi) \\ & & \text{by Proposition 8.3.} \end{array}$$

(iii) $\varphi = \ominus\psi\chi$ for $\ominus \in \mathcal{P}_2 \cup Equal$.
We get that

$$\begin{array}{lll} x \in FreeForE_y(\varphi) & \text{iff} & x \in FreeForE_y(\psi) \cap FreeForE_y(\chi) \\ & & \text{by Proposition 13.8} \\ & \text{iff} & x \in FreeForE_y(Subb^x_z\psi) \cap FreeForE_y(Subb^x_z\chi) \\ & & \text{by the induction hypothesis for } \psi,\ \chi \\ & \text{iff} & x \in FreeForE_y(\ominus Subb^x_z\psi Subb^x_z\chi) \\ & & \text{by Proposition 13.8} \\ & \text{iff} & x \in FreeForE_y(Subb^x_z\varphi) \\ & & \text{by Proposition 8.3.} \end{array}$$

(iv) $\varphi = \overline{Q}X\psi$ for $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$. We get that

$$\begin{array}{lll} x \in FreeForE_y(\varphi) & \text{iff} & x \in FreeForE_y(\psi) \\ & & \text{by Proposition 13.8} \\ & \text{iff} & x \in FreeForE_y(Subb^x_z\psi) \\ & & \text{by the induction hypothesis for } \psi \\ & \text{iff} & x \in FreeForE_y(\overline{Q}X Subb^x_z\psi) \\ & & \text{by Proposition 13.8} \\ & \text{iff} & x \in FreeForE_y(Subb^x_z\varphi) \\ & & \text{by Proposition 8.3.} \end{array}$$

(v) $\varphi = Qv\psi$ for $Q \in \mathcal{Q}$ and $v \in EVar$. We have the following cases:

(a) $v \neq y$ and $v \neq x$. Then

$$
\begin{aligned}
x \in FreeForE_y(\varphi) \quad & \text{iff} \quad x \in FreeForE_y(\psi) \\
& \qquad \text{by Proposition 13.8, as } v \neq y \\
& \text{iff} \quad x \in FreeForE_y(Subb^x_z\psi) \\
& \qquad \text{by the induction hypothesis for } \psi \\
& \text{iff} \quad x \in FreeForE_y(QvSubb^x_z\psi) \\
& \qquad \text{by Proposition 13.8, as } v \neq y \\
& \text{iff} \quad x \in FreeForE_y(Subb^x_z\varphi) \\
& \qquad \text{by Proposition 8.3, as } v \neq x.
\end{aligned}
$$

(b) $v = y$, hence $\varphi = Qy\psi$. Then

$$
\begin{aligned}
FreeForE_y(\varphi) &= FreeForE_y(Qy\psi) \\
&= FreshF(Qy\psi) \quad \text{by Proposition 13.8} \\
&= FreshF(\varphi)
\end{aligned}
$$

As $x \neq y$, we have, by Proposition 8.3, that $Subb^x_z\varphi = Subb^x_zQy\psi = QySubb^x_z\psi$. It follows that

$$
\begin{aligned}
FreeForE_y(Subb^x_z\varphi) &= FreeForE_y(Subb^x_zQy\psi) = FreeForE_y(QySubb^x_z\psi) \\
&= FreshF(QySubb^x_z\psi) \quad \text{by Proposition 13.8} \\
&= FreshF(Subb^x_z\varphi)
\end{aligned}
$$

As $x \neq z$, we have, by Lemma 8.7(i) that x does not occur bound in $Subb^x_z\varphi$, that is $x \notin BV(Subb^x_z\varphi)$. Apply Lemma 6.3(iii) to get that $x \notin NotFV(Subb^x_z\varphi)$. It follows that

$$
\begin{aligned}
x \in FreshF(Subb^x_z\varphi) \quad & \text{iff} \quad x \text{ does not occur in } Subb^x_z\varphi \quad && \text{by Lemma 6.3(iv)} \\
& \text{iff} \quad x \in FreshF(\varphi) && \text{by Lemma 8.7(iii)}
\end{aligned}
$$

Thus, $x \in FreeForE_y(\varphi)$ iff $x \in FreeForE_y(Subb^x_z\varphi)$.

(c) $v = x$, hence $\varphi = Qx\psi$. By Lemma 13.7(iii), we have that $x \in FreeForE_y(\varphi)$
Apply Lemma 8.7(viii) and the fact that $x \neq z$ to get that x does not occur in $Subb^x_z\varphi$.
By Lemma 13.7(ii), $x \in FreeForE_y(Subb^x_z\varphi)$.
Thus, both $x \in FreeForE_y(\varphi)$ and $x \in FreeForE_y(Subb^x_z\varphi)$.

□

13.1.2 Useful lemmas for $Subf$

Lemma 13.12. *Let φ be a pattern and x, y, z be variables such that z does not occur in φ. Then*

$$x \text{ is free for } y \text{ in } \varphi \quad \text{iff} \quad z \text{ is free for } y \text{ in } Subf^x_z\varphi.$$

Proof. If $x = z$, then $Subf^x_z\varphi = \varphi$, by Lemma 7.5. The conclusion is obvious.
Assume in the sequel that $x \neq y$ and $x \neq z$.
We prove by induction on φ, using the definition by recursion of $Subf^x_z\varphi$ (Proposition 7.3).

(i) φ is an atomic pattern. Then $Subf^x_z\varphi = \varphi$. The conclusion is obvious.

(ii) $\varphi = -\psi$ for $- \in \mathcal{P}_1$. We get that

$$
\begin{aligned}
x \in FreeForE_y(\varphi) \quad & \text{iff} \quad x \in FreeForE_y(\psi) \\
& \qquad \text{by Proposition 13.8} \\
& \text{iff} \quad z \in FreeForE_y(Subf^x_z\psi) \\
& \qquad \text{by the induction hypothesis for } \psi \\
& \text{iff} \quad z \in FreeForE_y(-Subf^x_z\psi) \\
& \qquad \text{by Proposition 13.8} \\
& \text{iff} \quad z \in FreeForE_y(Subf^x_z\varphi) \\
& \qquad \text{by Proposition 7.3.}
\end{aligned}
$$

(iii) $\varphi = \ominus\psi\chi$ for $\ominus \in \mathcal{P}_2 \cup Equal$.
We get that

$$\begin{aligned} x \in FreeForE_y(\varphi) \quad &\text{iff} \quad x \in FreeForE_y(\psi) \cap FreeForE_y(\chi) \\ &\quad \text{by Proposition 13.8} \\ &\text{iff} \quad z \in FreeForE_y(Subf_z^x\psi) \cap FreeForE_y(Subf_z^x\chi) \\ &\quad \text{by the induction hypothesis for } \psi, \chi \\ &\text{iff} \quad z \in FreeForE_y(\ominus Subb_z^x\psi Subf_z^x\chi) \\ &\quad \text{by Proposition 13.8} \\ &\text{iff} \quad z \in FreeForE_y(Subf_z^x\varphi) \\ &\quad \text{by Proposition 7.3.} \end{aligned}$$

(iv) $\varphi = \overline{Q}X\psi$ for $\overline{Q} \in \overline{\mathcal{Q}}$ and $X \in SVar$. We get that

$$\begin{aligned} x \in FreeForE_y(\varphi) \quad &\text{iff} \quad x \in FreeForE_y(\psi) \\ &\quad \text{by Proposition 13.8} \\ &\text{iff} \quad z \in FreeForE_y(Subf_z^x\psi) \\ &\quad \text{by the induction hypothesis for } \psi \\ &\text{iff} \quad z \in FreeForE_y(\overline{Q}XSubf_z^x\psi) \\ &\quad \text{by Proposition 13.8} \\ &\text{iff} \quad z \in FreeForE_y(Subf_z^x\varphi) \\ &\quad \text{by Proposition 7.3.} \end{aligned}$$

(v) $\varphi = Qv\psi$ for $Q \in \mathcal{Q}$ and $v \in EVar$. We have the following cases:

(a) $v \neq x$ and $v \neq y$. Then

$$\begin{aligned} x \in FreeForE_y(\varphi) \quad &\text{iff} \quad x \in FreeForE_y(\psi) \\ &\quad \text{by Proposition 13.8, as } v \neq y \\ &\text{iff} \quad z \in FreeForE_y(Subf_z^x\psi) \\ &\quad \text{by the induction hypothesis for } \psi \\ &\text{iff} \quad z \in FreeForE_y(QvSubf_z^x\psi) \\ &\quad \text{by Proposition 13.8, as } v \neq y \\ &\text{iff} \quad z \in FreeForE_y(Subf_z^x\varphi) \\ &\quad \text{by Proposition 7.3, as } v \neq x. \end{aligned}$$

(b) $v = y$ and $v \neq x$, hence $\varphi = Qy\psi$ and $x \neq y$. Then

$$\begin{aligned} FreeForE_y(\varphi) &= FreeForE_y(Qy\psi) \\ &= FreshF(Qy\psi) \quad \text{by Proposition 13.8} \\ &= FreshF(\varphi) \end{aligned}$$

As $x \neq y$, by Proposition 7.3, we have that $Subf_z^x\varphi = Subf_z^xQy\psi = QySubf_z^x\psi$. It follows that

$$\begin{aligned} FreeForE_y(Subf_z^x\varphi) &= FreeForE_y(Subf_z^xQy\psi) = FreeForE_y(QySubf_z^x\psi) \\ &= FreshF(QySubf_z^x\psi) \quad \text{by Proposition 13.8} \\ &= FreshF(Subf_z^x\varphi) \end{aligned}$$

We have that

$$\begin{aligned} x \in FreeForE_y(\varphi) \quad &\text{iff} \quad x \in FreshF(\varphi) \\ &\text{iff} \quad FreeOccur_x(\varphi) = \emptyset && \text{by Lemma 6.5(ii)} \\ &\text{iff} \quad FreeOccur_z(Subf_z^x\varphi) = \emptyset && \text{by Lemma 7.9(iii),} \\ & && \text{as } z \text{ does not occur in } \varphi \\ &\text{iff} \quad z \in FreshF(Subf_z^x\varphi) && \text{by Lemma 6.5(ii)} \\ &\text{iff} \quad z \in FreeForE_y(Subf_z^x\varphi) \end{aligned}$$

(c) $v = x$, hence $\varphi = Qx\psi$. Then $Subf_z^x \varphi = \varphi$.
By Lemma 13.7(iii), we have that x is free for y in φ.
Furthermore, by Lemma 13.7(ii) and the fact that z does not occur in φ, we have that z is free for y in φ.

□

13.2 X free for Y

Remark 13.13. *Let $X, Y \in SVar$.*

(i) X is free for Y in φ if the following holds: for every binder $\overline{Q}$ on Y in φ with scope θ, we have that X does not occur free in θ.

(ii) X is not free for Y in φ if the following holds: there exists a binder $\overline{Q}$ on Y in φ with scope θ such that X occurs free in θ.

Definition 13.14. *Let $Y \in SVar$. Define the mapping $FreeForS_Y : Pattern \to SVar$ as follows: for any pattern φ and $Y \in SVar$,*

$$FreeForS_Y(\varphi) = \{X \in SVar \mid X \text{ is free for } Y \text{ in } \varphi\}.$$

Proposition 13.15 (Definition by recursion)**.**
Let $Y \in SVar$. The mapping

$$FreeForS_Y : Pattern \to SVar$$

can be defined by recursion on patterns as follows:

$$\begin{array}{lcll}
FreeForS_Y(\varphi) & = & SVar & \text{if } \varphi \text{ is an atomic pattern,} \\
FreeForS_Y(-\psi) & = & FreeForS_Y(\psi) & \text{for any } - \in \mathcal{P}_1, \\
FreeForS_Y(\ominus\psi\chi) & = & FreeForS_Y(\psi) \cap FreeForS_Y(\chi) & \text{for any } \ominus \in \mathcal{P}_2 \cup Equal, \\
FreeForS_Y(Qx\psi) & = & FreeForS_Y(\psi) & \text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar, \\
FreeForS_Y(\overline{Q}X\psi) & = & \begin{cases} Fresh_{SVar}(\varphi) & \text{if } X = Y \\ FreeForS_Y(\psi) & \text{if } X \neq Y \end{cases} & \text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } X \in SVar.
\end{array}$$

Lemma 13.16. *Let $Y \in SVar$ and φ be a pattern.*

(i) $Y \in FreeForS_Y(\varphi)$, that is Y is free for Y in φ.

(ii) $Fresh_{SVar}(\varphi) \subseteq FreeForS_Y(\varphi)$, that is any set variable that

either (1) does not occur in φ or (2) occurs in φ, but does not occur free in φ

is free for Y in φ.

(iii) If $Y \notin SVar(\varphi)$, then $FreeForS_Y(\varphi) = SVar$, that is if Y does not occur in φ, then any set variable is free for Y in φ.

(iv) If $Y \notin BV_{SVar}(\varphi)$, then $FreeForS_Y(\varphi) = SVar$, that is if Y occurs in φ, but Y does not occur bound in φ, then any set variable is free for Y in φ.

Proposition 13.17. *Assume that $X, Y \in SVar$ are such that $X \neq Y$ and Y does not occur in φ. Then X is free for Y in $Subb_Y^X \varphi$.*

Proof. The proof is by induction on φ.

(i) φ is an atomic pattern. Then $Subb^X_Y \varphi = \varphi$ and X is free for Y in φ, as Y does not occur in φ.

(ii) $\varphi = -\psi$ for $- \in \mathcal{P}_1$. As Y does not occur in ψ, we can apply the induction hypothesis to get that X is free for Y in $Subb^X_Y \psi$. It follows that X is free for Y in $Subb^X_Y \varphi = -Subb^X_Y \psi$.

(iii) $\varphi = \circ\psi\chi$ for $\circ \in \mathcal{P}_2$. As Y does not occur in ψ, χ, we can apply the induction hypothesis to get that X is free for Y in $Subb^X_Y \psi$, $Subb^X_Y \chi$. It follows that X is free for Y in $Subb^X_Y \varphi = \circ Subb^X_Y(\psi) Subb^X_Y(\chi)$.

(iv) $\varphi = \sim \psi\chi$ for $\sim \in Equal$. As Y does not occur in ψ, χ, we can apply the induction hypothesis to get that X is free for Y in $Subb^X_Y \psi$, $Subb^X_Y \chi$. It follows that X is free for Y in $Subb^X_Y \varphi = \sim Subb^X_Y(\psi) Subb^X_Y(\chi)$.

(v) $\varphi = Qx\psi$. As Y does not occur in ψ, we can apply the induction hypothesis to get that X is free for Y in $Subb^X_Y \psi$. Then, obviously X is free for Y in $Subb^X_Y \varphi = Qx Subb^X_Y(\psi)$.

(vi) $\varphi = \overline{Q} Z \psi$. We have two cases:

  (a) $X = Z$. Then $Subb^X_Y \varphi = \overline{Q} Y Subf^X_Y \big(Subb^X_Y(\psi)\big)$. It is obvious that X does not occur in $Subb^X_Y \varphi$, so X is free for Y in $Subb^X_Y \varphi$.

  (b) $X \neq Z$. Then $Subb^X_Y \varphi = \overline{Q} Z Subb^X_Y(\psi)$. As Y does not occur in ψ, we can apply the induction hypothesis to get that X is free for Y in $Subb^X_Y \psi$. As Y does not occur in φ, we must have that $Y \neq Z$. Then, obviously X is free for Y in $Subb^X_Y \varphi$.

□

14 Positive and negative occurences of set variables

Assume that $\to \in \mathcal{P}_2$ and that $SVar \neq \emptyset$. Let X be a set variable.

Definition 14.1. *Let $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ be a pattern.*

*(i) We say that $\to$ is **an implication at the ith place of φ with left scope ψ and right scope χ** if $\varphi_i = \to$ and $\psi = \varphi_{i+1} \dots \varphi_j$, $\chi = \varphi_{j+1} \dots \varphi_l$ are the unique patterns given by Proposition 2.10.*

*(ii) X **occurs left at the kth place of** φ if X occurs free at the kth place of φ and there exist $0 \leq i < k \leq j \leq n-1$ such that $\psi = \varphi_{i+1} \dots \varphi_j$ is the left scope of an implication $\to$ at the ith place of φ.*

Definition 14.2. *We define the mapping*

$$N_{X,L} : Pattern \to Fun(\mathbb{N}, \mathbb{N})$$

by recursion on patterns as follows:

(i) φ is an atomic pattern. Then $N_{X,L}(\varphi)(k) = 0$ for every $k \in \mathbb{N}$.

(ii) $\varphi = -\psi$, where $- \in \mathcal{P}_1$. Thus, $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ with $\varphi_0 = -$, $\psi = \varphi_1 \dots \varphi_{n-1}$. We have the following cases:

  (a) $k = 0$ or $k \geq n$. Then $N_{X,L}(\varphi)(k) = 0$.

  (b) $1 \leq k \leq n-1$. Then $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-1)$.

(iii) $\varphi = \ominus\psi\chi$, where $\ominus \in Equal \cup (\mathcal{P}_2 \setminus \{\to\})$. Thus, $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ with $\varphi_0 = \ominus$, $\psi = \varphi_1 \dots \varphi_j$ and $\chi = \varphi_{j+1} \dots \varphi_{n-1}$ for some $1 \leq j < n-1$. We have the following cases:

  (a) $k = 0$ or $k \geq n$. Then $N_{X,L}(\varphi)(k) = 0$.

(b) $1 \leq k \leq j$. *Then* $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-1)$.

(c) $j+1 \leq k \leq n-1$. *Then* $N_{X,L}(\varphi)(k) = N_{X,L}(\chi)(k-j-1)$.

(iv) $\varphi = \rightarrow \psi\chi$. *Thus,* $\varphi = \varphi_0\varphi_1 \dots \varphi_{n-1}$ *with* $\varphi_0 = \rightarrow$, $\psi = \varphi_1 \dots \varphi_j$ *and* $\chi = \varphi_{j+1} \dots \varphi_{n-1}$ *for some* $1 \leq j < n-1$. *We have the following cases:*

(a) $k = 0$ *or* $k \geq n$. *Then* $N_{X,L}(\varphi)(k) = 0$.

(b) $1 \leq k \leq j$. *Then* $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-1) + 1$.

(c) $j+1 \leq k \leq n-1$. *Then* $N_{X,L}(\varphi)(k) = N_{X,L}(\chi)(k-j-1)$.

(v) $\varphi = Qx\psi$, *where* $Q \in \mathcal{Q}$ *and* $x \in EVar$. *Thus,* $\varphi_0\varphi_1 = Qx$ *and* $\psi = \varphi_2 \dots \varphi_{n-1}$. *We have the following cases:*

(a) $k \in \{0,1\}$ *or* $k \geq n$. *Then* $N_{X,L}(\varphi)(k) = 0$.

(b) $2 \leq k \leq n-1$. *Then* $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-2)$.

(vi) $\varphi = \overline{Q}Z\psi$, *where* $Q \in \overline{\mathcal{Q}}$ *and* $Z \in SVar$. *If* $X = Z$, *then* $N_{X,L}(\mu Z\psi)(k) = 0$ *for all* $k \in \mathbb{N}$. *If* $X \neq Z$, *then we have the following cases:*

(a) $k \in \{0,1\}$ *or* $k \geq n$. *Then* $N_{X,L}(\varphi)(k) = 0$.

(b) $2 \leq k \leq n-1$. *Then* $N_{X,L}(\varphi)(k) = N_{X,L}(\psi)(k-2)$.

Proof. Apply Recursion principle on patterns (Proposition 2.11) with $D = Fun(\mathbb{N}, \mathbb{N})$ and

$$G_0(\varphi)(k) \quad = \quad 0 \quad \text{if } \varphi \text{ is an atomic pattern,}$$

$$G_{\ominus}(f,g,\psi,\chi)(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq n, \\ f(k-1) & \text{if } 1 \leq k \leq j, \\ g(k-j-1) & \text{if } j+1 \leq k \leq n-1 \end{cases} \quad \text{for any } \ominus \in Equal \cup (\mathcal{P}_2 \setminus \{\rightarrow\}),$$

$$G_{\rightarrow}(f,g,\psi,\chi)(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq n, \\ f(k-1)+1 & \text{if } 1 \leq k \leq j, \\ g(k-j-1) & \text{if } j+1 \leq k \leq n-1 \end{cases},$$

$$G_Q(f,x,\psi)(k) = \begin{cases} 0 & \text{if } k \in \{0,1\} \text{ or } k \geq n, \\ f(k-2) & \text{if } 2 \leq k \leq n-1 \end{cases}$$
$$\text{for any } Q \in \mathcal{Q} \text{ and } x \in EVar,$$

$$G_{\overline{Q}}(f,Z,\psi)(k) = \begin{cases} 0 & \text{if } Z = X, \\ 0 & \text{if } Z \neq X \text{ and } (k \in \{0,1\} \text{ or } k \geq n), \\ f(k-2) & \text{if } Z \neq X \text{ and } 2 \leq k \leq n-1 \end{cases}$$
$$\text{for any } \overline{Q} \in \overline{\mathcal{Q}} \text{ and } Z \in SVar.$$

Then

(i) $N_{X,L}(\varphi)(k) = 0 = G_0(\varphi)(k)$ for every $k \in \mathbb{N}$ if φ is an atomic pattern.

(ii) $\varphi = \ominus\psi\chi$. Then

$$N_{X,L}(\ominus\psi\chi)(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq n, \\ N_{X,L}(\psi)(k-1) & \text{if } 1 \leq k \leq j, \\ N_{X,L}(\chi)(k-j-1) & \text{if } j+1 \leq k \leq n-1 \end{cases}$$
$$= G_{\ominus}(N_{X,L}(\psi), N_{X,L}(\chi), \psi, \chi)(k).$$

(iii) $\varphi = \rightarrow \psi\chi$. Then

$$N_{X,L}(\rightarrow \psi\chi)(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \geq n, \\ N_{X,L}(\psi)(k-1) + 1 & \text{if } 1 \leq k \leq j, \\ N_{X,L}(\chi)(k-j-1) & \text{if } j+1 \leq k \leq n-1 \end{cases}$$
$$= G_{\rightarrow}(N_{X,L}(\psi), N_{X,L}(\chi), \psi, \chi)(k).$$

(iv) $\varphi = Qx\psi$. Then

$$N_{X,L}(Qx\psi)(k) = \begin{cases} 0 & \text{if } k \in \{0,1\} \text{ or } k \geq n, \\ N_{X,L}(\psi)(k-2) & \text{if } 2 \leq k \leq n-1 \end{cases}$$
$$= G_Q(N_{X,L}(\psi), x, \psi)(k).$$

(v) $\varphi = \overline{Q}Z\psi$. If $Z = X$, then $N_{X,L}(\overline{Q}X\psi)(k) = 0 = G_{\overline{Q}}(N_{X,L}(\psi), Z, \psi)(k)$. Assume that $Z \neq X$. Then

$$N_{X,L}(\overline{Q}Z\psi)(k) = \begin{cases} 0 & \text{if } k \in \{0,1\} \text{ or } k \geq n, \\ N_{X,L}(\psi)(k-2) & \text{if } 2 \leq k \leq n-1 \end{cases}$$
$$= G_{\overline{Q}}(N_{X,L}(\psi), Z, \psi)(k).$$

□

Notation 14.3. *Let $k \in \mathbb{N}$ and φ be a pattern. We denote $N_{X,L}(\varphi)(k)$ with $N_L(\varphi, X, k)$.*

Definition 14.4. *Let X $k \in \mathbb{N}$ and φ be a pattern such that X occurs free at the kth place of φ.*

*(i) We say that X **occurs positively at the kth place of** φ (or that X **has a positive occurence at the kth place of** φ) if $N_L(\varphi, X, k) = 0$ or $N_L(\varphi, X, k)$ is an even natural number.*

*(ii) We say that X **occurs negatively at the kth place of** φ (or that X **has a negative occurence at the kth place of** φ) if $N_L(\varphi, X, k)$ is an odd natural number.*

Definition 14.5. *We say that φ is **positive in** X if one of the following is true:*

(i) X does not occur free in φ.

(ii) For every $k \in \mathbb{N}$, if X occurs free at the kth place of φ, then X occurs positively at the kth place of φ.

Definition 14.6. *We say that φ is **negative in** X if one of the following is true:*

(i) X does not occur free in φ.

(ii) For every $k \in \mathbb{N}$, if X occurs free at the kth place of φ, then X occurs negatively at the kth place of φ.

Remark 14.7 (Alternative definition)**.** *The property that φ is positive in X can be defined by recursion on patterns as follows:*

(i) If φ is atomic, then

(a) φ is positive in X;

(b) φ is negative in X iff $\varphi \neq X$.

(ii) If $\varphi = -\psi$, where $- \in \mathcal{P}_1$, then

(a) φ *is positive in* X *iff* ψ *is positive in* X;

(b) φ *is negative in* X *iff* ψ *is negative in* X.

(iii) *If* $\varphi = \ominus\psi\chi$, *where* $\ominus \in Equal \cup (\mathcal{P}_2 \setminus \{\rightarrow\})$, *then*

(a) φ *is positive in* X *iff both* ψ, χ *are positive in* X;

(b) φ *is negative in* X *iff both* ψ, χ *are negative in* X.

(iv) *If* $\varphi = \rightarrow \psi\chi$, *then*

(a) φ *is positive in* X *iff* ψ *is negative in* X *and* χ *is positive in* X;

(b) φ *is negative in* X *iff* ψ *is positive in* X *and* χ *is negative in* X.

(v) *If* $\varphi = Qx\psi$, *where* $Q \in \mathcal{Q}$ *and* $x \in EVar$, *then*

(a) φ *is positive in* X *iff* ψ *is positive in* X.

(b) φ *is negative in* X *iff* ψ *is negative in* X.

(vi) *If* $\varphi = \overline{Q}X\psi$, *where* $\overline{Q} \in \overline{\mathcal{Q}}$, *then*

(a) φ *is positive in* X;

(b) φ *is negative in* X.

(vii) *If* $\varphi = \overline{Q}Z\psi$, *where* $\overline{Q} \in \overline{\mathcal{Q}}$ *and* $Z \in EVar \setminus \{X\}$, *then*

(a) φ *is positive in* X *iff* ψ *is positive in* X.

(b) φ *is negative in* X *iff* ψ *is negative in* X.

15 Proof systems

Let $\mathcal{L}$ be a language for abstract matching logic.

Definition 15.1. *An* $\mathcal{L}$***-proof system*** *is a pair* $\mathcal{P} = (Axm, DedRules)$, *where*

(i) $Axm \subseteq Pattern_{\mathcal{L}}$ *is a set of* ***axioms***.

(ii) *DedRules is a set of* ***deduction rules*** *(or* ***inference rules****). A deduction rule has one of the following forms:*

$$(I)\ \frac{\varphi_1 \quad \varphi_2 \quad \dots \quad \varphi_n}{\psi}, \qquad (II)\ \frac{\varphi_1 \quad \varphi_2 \quad \dots \quad \varphi_n}{\psi}(C)$$

where $n \geq 1$, $\varphi_1, \varphi_2 \dots \varphi_n, \psi \in Pattern_{\mathcal{L}}$ *and* C *is a condition.*

$\varphi_1, \varphi_2 \dots \varphi_n$ are said to be the **premises** of the rule and ψ is the **condition** of the rule. For a deduction rule of form (II), C is said to be the **conclusion** of the rule.
A deduction rule of form (I) is read as: from $\varphi_1, \varphi_2, \dots, \varphi_n$ deduce/infer ψ.
A deduction rule of form (II) is read as: if condition C holds, from $\varphi_1, \varphi_2, \dots, \varphi_n$ deduce/infer ψ.
We denote deduction rules by $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}_1$, $\mathcal{D}_2$, etc.
Let $\mathcal{P} = (Axm, DedRules)$ be an $\mathcal{L}$-proof system.

Definition 15.2. *A set* Γ *of* $\mathcal{L}$*-patterns is said to be closed to DedRules if the following hold:*

(i) *For every deduction rule* $\mathcal{D} \in DedRules$ *of form (I), if* Γ *contains the premises of* $\mathcal{D}$, *then the conclusion of* $\mathcal{D}$ *is also in* Γ.

(ii) *For every deduction rule* $\mathcal{D} \in DedRules$ *of form (II), if the condition* C *of* $\mathcal{D}$ *holds and* Γ *contains the premises of* $\mathcal{D}$, *then the conclusion of* $\mathcal{D}$ *is also in* Γ.

Let Γ be a set of $\mathcal{L}$-patterns.

Definition 15.3. *The set of Γ-$\mathcal{P}$-**theorems** is the intersection of all sets Δ of $\mathcal{L}$-patterns that have the following properties:*

(i) $Axm \subseteq \Delta$.

(ii) $\Gamma \subseteq \Delta$.

(iii) *Δ is closed to DedRules.*

The set of Γ-$\mathcal{P}$-theorems is denoted by $\Gamma - Thm_{\mathcal{P}}$. If φ is a Γ-$\mathcal{P}$-theorem, then we also say that φ is **deduced from the hypotheses** Γ.
As an immediate consequence of Definition 15.3, we get the induction principle for Γ-$\mathcal{P}$-theorems.

Proposition 15.4. *[Induction principle on Γ-$\mathcal{P}$-theorems]*
Let Δ be a set of $\mathcal{L}$-patterns satisfying the following properties:

(i) $Axm \subseteq \Delta$.

(ii) $\Gamma \subseteq \Delta$.

(iii) *Δ is closed to DedRules.*

Then $\Gamma - Thm_{\mathcal{P}} \subseteq \Delta$.

Proof. By hypothesis, $\Delta \subseteq Pattern_{\mathcal{L}}$. By Definition 15.3, we get that $\Gamma - Thm_{\mathcal{P}} \subseteq \Delta$. □

Definition 15.5. *The set $Thm_{\mathcal{P}}$ of $\mathcal{P}$-**theorems** is defined by $Thm_{\mathcal{P}} = \emptyset - Thm_{\mathcal{P}}$.*

Notation 15.6. *Let Γ, Δ be sets of $\mathcal{L}$-patterns and φ be an $\mathcal{L}$-pattern. We use the following notations*

$\Gamma \vdash_{\mathcal{P}} \varphi \quad := \quad \varphi$ *is a Γ-$\mathcal{P}$-theorem,*

$\vdash_{\mathcal{P}} \varphi \quad := \quad \varphi$ *is a $\mathcal{P}$-theorem,*

$\Gamma \vdash_{\mathcal{P}} \Delta \quad \Leftrightarrow \quad \Gamma \vdash_{\mathcal{P}} \varphi$ *for any* $\varphi \in \Delta$.

Proposition 15.7. *Let Γ, Δ be sets of $\mathcal{L}$-patterns.*

(i) Assume that $\Delta \subseteq \Gamma$. Then for every $\mathcal{L}$-pattern φ, $\Delta - Thm_{\mathcal{P}} \subseteq \Gamma - Thm_{\mathcal{P}}$, that is

$$\Delta \vdash_{\mathcal{P}} \varphi \text{ implies } \Gamma \vdash_{\mathcal{P}} \varphi.$$

(ii) For every $\mathcal{L}$-pattern φ, $Thm_{\mathcal{P}} \subseteq \Gamma - Thm_{\mathcal{P}}$, that is

$$\vdash_{\mathcal{P}} \varphi \text{ implies } \Gamma \vdash_{\mathcal{P}} \varphi.$$

(iii) Assume that $\Gamma \vdash_{\mathcal{P}} \Delta$. Then for every $\mathcal{L}$-pattern φ, $\Delta - Thm_{\mathcal{P}} \subseteq \Gamma - Thm_{\mathcal{P}}$, that is

$$\Delta \vdash_{\mathcal{P}} \varphi \text{ implies } \Gamma \vdash_{\mathcal{P}} \varphi.$$

(iv) For every $\mathcal{L}$-pattern φ, $(\Gamma - Thm_{\mathcal{P}}) - Thm_{\mathcal{P}} = \Gamma - Thm_{\mathcal{P}}$, that is

$$\Gamma - Thm_{\mathcal{P}} \vdash_{\mathcal{P}} \varphi \text{ iff } \Gamma \vdash_{\mathcal{P}} \varphi.$$

Proof. (i) As $\Delta \subseteq \Gamma$, one proves immediately by induction on Δ-theorems that $\Delta - Thm_{\mathcal{P}} \subseteq \Gamma - Thm_{\mathcal{P}}$.

(ii) Apply (i) with $\Delta = \emptyset$.

(iii) As, by hypothesis, $\Delta \subseteq Thm_{\mathcal{L}}(\Gamma)$, one proves immediately by induction on Δ-theorems that $\Delta - Thm_{\mathcal{P}} \subseteq \Gamma - Thm_{\mathcal{P}}$.

(iv) $\Leftarrow$ As, by definition, $\Gamma \subseteq \Gamma - Thm_{\mathcal{P}}$, we can apply (i) to get that $\Gamma - Thm_{\mathcal{P}} \subseteq (\Gamma - Thm_{\mathcal{P}}) - Thm_{\mathcal{P}}$.

$\Rightarrow$ We have that $\Gamma \vdash_{\mathcal{P}} \Gamma - Thm_{\mathcal{P}}$, so we can apply (iii) with $\Delta = \Gamma - Thm_{\mathcal{P}}$ to get that $(\Gamma - Thm_{\mathcal{P}}) - Thm_{\mathcal{P}} \subseteq \Gamma - Thm_{\mathcal{P}}$.

□

15.1 Γ-$\mathcal{P}$-proof

Let $\mathcal{P} = (Axm, DedRules)$ be an $\mathcal{L}$-proof system and Γ be a set of $\mathcal{L}$-patterns.

Definition 15.8. *A Γ-$\mathcal{P}$-**proof** is a sequence of $\mathcal{L}$-patterns $\theta_1, \dots, \theta_n$ such that for all $i \in \{1, \dots, n\}$, one of the following holds:*

(i) $\theta_i \in Axm$.

(ii) $\theta_i \in \Gamma$.

(iii) θ_i is the conclusion of a deduction rule $\mathcal{D} \in DedRules$ of form (I) and the premises of $\mathcal{D}$ are previous $\mathcal{L}$-patterns.

(iv) θ_i is the conclusion of a deduction rule $\mathcal{D} \in DedRules$ of form (II), the premises of $\mathcal{D}$ are previous $\mathcal{L}$-patterns and the condition C of $\mathcal{D}$ holds.

*An $\emptyset$-$\mathcal{P}$-proof is called simply a $\mathcal{P}$-**proof**.*

Definition 15.9. *Let φ be an $\mathcal{L}$-pattern. A* **Γ-$\mathcal{P}$-*proof of*** *φ is a Γ-$\mathcal{P}$-proof $\theta_1, \dots, \theta_n$ such that $\theta_n = \varphi$.*

Proposition 15.10. *For any $\mathcal{L}$-pattern φ,*

$$\Gamma \vdash_{\mathcal{P}} \varphi \quad \text{iff} \quad \text{there exists a } \Gamma\text{-}\mathcal{P}\text{-proof of } \varphi.$$

Proof.

$$\Theta = \{\varphi \in Pattern_{\mathcal{L}} \mid \text{there exists a } \Gamma\text{-}\mathcal{P}\text{-proof of } \varphi\}.$$

$\Rightarrow$ We prove by induction on Γ-$\mathcal{P}$-theorems that $\Gamma - Thm_{\mathcal{P}} \subseteq \Theta$:
If φ is an axiom or a member of Γ, then $\theta_0 = \varphi$ is a Γ-$\mathcal{P}$-proof of φ. Hence, $\varphi \in \Theta$.
Let us prove that Θ is closed to $DedRules$.

(i) Let $\mathcal{D} = \frac{\varphi_1 \quad \varphi_2 \quad \dots \quad \varphi_n}{\psi}$ be a deduction rule of form (I) such that $\varphi_1, \varphi_2, \dots, \varphi_n \in \Theta$. Then for every $i = 1, \dots, n$ there exists a Γ-$\mathcal{P}$-proof $\delta_1^i, \delta_2^i \dots, \delta_{k_i}^i = \varphi_i$ of φ_i. It follows that

$$\delta_1^1, \delta_2^1 \dots, \delta_{k_1}^1 = \varphi_1, \delta_1^2, \delta_2^2 \dots, \delta_{k_2}^2 = \varphi_2, \dots, \delta_1^n, \delta_2^n \dots, \delta_{k_n}^n = \varphi_n, \psi$$

is a Γ-$\mathcal{P}$-proof of ψ. Thus, $\psi \in \Theta$.

(ii) Let $\mathcal{D} = \frac{\varphi_1 \quad \varphi_2 \quad \dots \quad \varphi_n}{\psi}(C)$ be a deduction rule of form (II) such that the condition C of $\mathcal{D}$ holds and $\varphi_1, \varphi_2, \dots, \varphi_n \in \Theta$. Then for every $i = 1, \dots, n$ there exists a Γ-$\mathcal{P}$-proof $\delta_1^i, \delta_2^i \dots, \delta_{k_i}^i = \varphi_i$ of φ_i. It follows that

$$\delta_1^1, \delta_2^1 \dots, \delta_{k_1}^1 = \varphi_1, \delta_1^2, \delta_2^2 \dots, \delta_{k_2}^2 = \varphi_2, \dots, \delta_1^n, \delta_2^n \dots, \delta_{k_n}^n = \varphi_n, \psi$$

is a Γ-$\mathcal{P}$-proof of ψ. Thus, $\psi \in \Theta$.

$\Leftarrow$ Assume that φ has a Γ-$\mathcal{P}$-proof $\theta_1, \dots, \theta_n = \varphi$. We prove by induction on i that for all $i = 1, \dots, n$, $\Gamma \vdash_{\mathcal{P}} \theta_i$. As a consequence, $\Gamma \vdash_{\mathcal{P}} \theta_n = \varphi$.
If $i = 1$, then θ_1 must be an axiom or a member of Γ. Then obviously $\Gamma \vdash_{\mathcal{P}} \theta_1$.
Assume that the induction hypothesis is true for all $j = 1, \dots, i$. We have the following cases for θ_{i+1}:

(i) θ_{i+1} is an axiom or a member of Γ. Then obviously $\Gamma \vdash_{\mathcal{P}} \theta_{i+1}$.

(ii) θ_{i+1} is the conclusion of a deduction rule $\mathcal{D} = \frac{\varphi_1 \quad \varphi_2 \quad \ldots \quad \varphi_n}{\theta_{i+1}}$ of form (I) and φ_1, φ_2, $\ldots \varphi_n$ are previous $\mathcal{L}$-patterns. Then by the induction hypothesis we have that $\Gamma \vdash_{\mathcal{P}} \varphi_k$ for all $k = 1, \ldots, n$. By the definition of Γ-$\mathcal{P}$-theorems, it follows that $\Gamma \vdash_{\mathcal{P}} \theta_{i+1}$.

(iii) θ_{i+1} is the conclusion of a deduction rule $\mathcal{D} = \frac{\varphi_1 \quad \varphi_2 \quad \ldots \quad \varphi_n}{\theta_{i+1}}(C)$ of form (II) and φ_1, φ_2, $\ldots \varphi_n$ are previous $\mathcal{L}$-patterns and the condition C of $\mathcal{D}$ holds. Then by the induction hypothesis we have that $\Gamma \vdash_{\mathcal{P}} \varphi_k$ for all $k = 1, \ldots, n$. By the definition of Γ-$\mathcal{P}$-theorems, it follows that $\Gamma \vdash_{\mathcal{P}} \theta_{i+1}$.

□

15.2 Comparison of $\mathcal{L}$-proof systems

Let $\mathcal{P}_1 = (Axm_1, DedRules_1)$, $\mathcal{P}_2 = (Axm_2, DedRules_2)$ be two $\mathcal{L}$-proof systems.

Definition 15.11. *$\mathcal{P}_1$ is said to be **weaker** than $\mathcal{P}_2$ (we write $\mathcal{P}_1 \lesssim \mathcal{P}_2$) if the following holds: for every $\mathcal{L}$-pattern φ and for every set Γ of $\mathcal{L}$-patterns,*

$$\Gamma \vdash_{\mathcal{P}_1} \varphi \text{ implies } \Gamma \vdash_{\mathcal{P}_2} \varphi.$$

Thus, $\mathcal{P}_1 \lesssim \mathcal{P}_2$ iff $\Gamma - Thm_{\mathcal{P}_1} \subseteq \Gamma - Thm_{\mathcal{P}_2}$ for every set Γ of $\mathcal{L}$-patterns.

Proposition 15.12. *Assume that*

(i) $\vdash_{\mathcal{P}_2} \varphi$ for every axiom $\varphi \in Axm_1$,

(ii) For every set Γ of $\mathcal{L}$-patterns, $\Gamma - Thm_{\mathcal{P}_2}$ is closed to $DedRules_1$.

Then $\mathcal{P}_1 \lesssim \mathcal{P}_2$.

Proof. Let Γ be a set of $\mathcal{L}$-patterns. We have that

(i) $Axm_1 \subseteq Thm_{\mathcal{P}_2} \subseteq \Gamma - Thm_{\mathcal{P}_2}$.

(ii) $\Gamma \subseteq \Gamma - Thm_{\mathcal{P}_2}$.

(iii) $\Gamma - Thm_{\mathcal{P}_2}$ is closed to $DedRules_1$.

By induction on Γ-$\mathcal{P}_1$-theorems (Proposition 15.4), we get that $\Gamma - Thm_{\mathcal{P}_1} \subseteq \Gamma - Thm_{\mathcal{P}_2}$. Thus, $\mathcal{P}_1 \lesssim \mathcal{P}_2$. □

Definition 15.13. *The $\mathcal{L}$-proof systems $\mathcal{P}_1$, $\mathcal{P}_2$ are said to be **equivalent** (we write $\mathcal{P}_1 \sim \mathcal{P}_2$) if for every set Γ of $\mathcal{L}$-patterns,*

$$\Gamma \vdash_{\mathcal{P}_1} \varphi \text{ implies } \Gamma \vdash_{\mathcal{P}_2} \varphi.$$

Proposition 15.14. *The following are equivalent:*

(i) $\mathcal{P}_1 \sim \mathcal{P}_2$,

(ii) $\mathcal{P}_1 \lesssim \mathcal{P}_2$ and $\mathcal{P}_2 \lesssim \mathcal{P}_1$,

(iii) $\Gamma - Thm_{\mathcal{P}_1} = \Gamma - Thm_{\mathcal{P}_2}$ for every set Γ of $\mathcal{L}$-patterns.

15.3 Abstract matching logics

Definition 15.15. *An **abstract matching logic** is a pair $\mathcal{AML} = (\mathcal{L}, \mathcal{P})$, where*

*(i) $\mathcal{L}$ is a language for abstract matching logic. We say that $\mathcal{L}$ is the **language** of $\mathcal{AML}$.*

*(ii) $\mathcal{P}$ is an $\mathcal{L}$-proof system. We say that $\mathcal{P}$ is the **proof system** for $\mathcal{AML}$.*

We also write $\mathcal{L}_{\mathcal{AML}}$ instead of $\mathcal{L}$ and $\mathcal{P}_{\mathcal{AML}}$ instead of $\mathcal{P}$.
For every set Γ of $\mathcal{L}$-patterns, the set $\Gamma - Thm_{\mathcal{P}}$ is called the set of **Γ-theorems** of $\mathcal{AML}$. The set $Thm_{\mathcal{P}}$ is called the set of **theorems** of $\mathcal{AML}$.

A Expressions over a set

Let A be a nonempty set whose elements will be called **symbols**.

Notation A.1. *For every $m, n \in \mathbb{N}$ be such that $m \leq n$. We denote*

$$[m, n] := \{m, m+1, \dots, n-1, n\}.$$

Notation A.2. *If $I \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, then*

$$I + k := \{n + k \mid n \in I\}.$$

An **expression over** A or simply **expression** is a finite sequence of symbols from A. We denote an expression over A of **length** $n \in \mathbb{N}^*$ by $a_0 a_1 \dots a_{n-1}$, where $a_i \in A$ for every $i = 0, \dots, n-1$. The empty expression (of length 0) is denoted by λ. The length of an expression $\mathbf{a}$ is denoted by $\ell(\mathbf{a})$.
The concatenation of expressions over A is defined as follows: if $\mathbf{a} = a_0 a_1 \dots a_{n-1}$ and $\mathbf{b} = b_0 \dots b_{k-1}$, then $\mathbf{ab} = a_0 \dots a_{n-1} b_0 \dots b_{k-1}$.

Definition A.3. *Let $\mathbf{a} = a_0 a_1 \dots a_{n-1}$ be an expression.*

*(i) If $i, j \in [0, n-1]$ are such that $i \leq j$, then the expression $a_i \dots a_j$ is called the (i,j)-**subexpression** of $\mathbf{a}$.*

*(ii) A **proper initial segment** of $\mathbf{a}$ is an $(0, j)$-subexpression of $\mathbf{a}$, that is an expression $a_0 a_1 \dots a_i$, where $i \in [0, n-2]$.*

Definition A.4. *Let $\mathbf{a} = a_0 a_1 \dots a_{n-1}$ and $\mathbf{b}$ be expressions.*

*(i) Let $i \in [0, n-1]$. We say that $\mathbf{b}$ **occurs at place** i **in** $\mathbf{a}$ if there exists $j \in [i, n-1]$ such that $\mathbf{b}$ is the (i,j)-subexpression of $\mathbf{a}$.*

*(ii) We say that $\mathbf{b}$ **occurs in** $\mathbf{a}$ if there exists $i \in [0, n-1]$ such that $\mathbf{b}$ occurs at place i in $\mathbf{a}$.*

*(iii) An **occurrence** of $\mathbf{b}$ in $\mathbf{a}$ is an $i \in [0, n-1]$ such that $\mathbf{b}$ occurs at place i in $\mathbf{a}$.*

We denote the set of all occurrences of $\mathbf{b}$ in $\mathbf{a}$ by $Occur_{\mathbf{b}}(\mathbf{a})$.

Notation A.5. *Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be expressions and $I \subseteq Occur_{\mathbf{b}}(\mathbf{a})$. We denote by $Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I)$ the expression obtained by replacing $\mathbf{b}$ with $\mathbf{c}$ in $\mathbf{a}$ at every place $i \in I$.*
If $I = \{i_1, \dots, i_k\}$, we also write $Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; i_1, \dots, i_k)$ instead of $Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I)$.
We shall write, for simplicity, $Replall_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a})$ instead of $Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; Occur_{\mathbf{b}}(\mathbf{a}))$.

Lemma A.6. *Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be expressions and $I \subseteq Occur_{\mathbf{b}}(\mathbf{a})$. Assume that $\ell(\mathbf{a}) = n$, $\ell(\mathbf{b}) = k$ and $\ell(\mathbf{c}) = p$. Then*

(i) $|I| k \leq n$.

(ii) $\ell(Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I)) = n + |I|(p - k)$.

Lemma A.7. *Let $\mathbf{a}$, $\mathbf{b}$ be expressions such that $\mathbf{b}$ occurs in $\mathbf{a}$ at $0 \leq i_1 < i_2 \leq \ell(\mathbf{a})$. If $\mathbf{b}$ is a not constant expression, then*

$$i_2 \geq i_1 + \ell(\mathbf{b}).$$

Proof. If $\ell(\mathbf{b}) = 1$, then obviously $i_2 \geq i_1 + 1$.
Otherwise,

$$p := \ell(\mathbf{b}) \geq 2 \quad \text{and} \quad \mathbf{b} = b_0 \dots b_{p-1}.$$

Assume by contradiction that $i_2 < i_1 + \ell(\mathbf{b})$. As $\mathbf{b}$ is a not constant expression, there are $i < j \in [0, p-1]$ such that $b_i \neq b_j$, so

$$I := \{i \in [0, p-1] \mid b_i \neq b_j \text{ for some } j < i\} \neq \emptyset.$$

Let $i_0 := \min\{i \in [0, p-1] \mid b_i \neq b_j \text{ for some } j < i\}$. □

A.1 Useful results

Lemma A.8. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be expressions and* $I \subseteq Occur_{\mathbf{b}}(\mathbf{a})$.

(i) $Repl^{\mathbf{b}}_{\mathbf{b}}(\mathbf{a}; I) = \mathbf{a}$.

(ii) $Repl^{\mathbf{b}}_{\mathbf{c}}(\mathbf{a}; \emptyset) = \mathbf{a}$.

Proof. Obviously. □

Lemma A.9. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be expressions. If* $\mathbf{a}$ *does not occur in* $\mathbf{b}$, $\mathbf{c}$, *then* $\mathbf{a}$ *does not occur in* $\mathbf{bc}$.

Lemma A.10. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *be expressions. Then*

$$Replall^{\mathbf{a}}_{\mathbf{d}}(\mathbf{bc}) = Replall^{\mathbf{a}}_{\mathbf{d}}(\mathbf{b})Replall^{\mathbf{a}}_{\mathbf{d}}(\mathbf{c}).$$

Lemma A.11. *Let* $\mathbf{a} = a_0 a_1 \ldots a_{n-1}$, $\mathbf{b} = b_0 b_1 \ldots b_{l-1}$, $\mathbf{c}$, *be expressions and* k *be such that* $a_k = b_k = d$. *Assume that*

$$Replall^{d}_{\mathbf{c}}(\mathbf{a})\{k\} = Replall^{d}_{\mathbf{c}}(\mathbf{b})\{k\}.$$

Then $\mathbf{a} = \mathbf{b}$.

Proof. Let $\mathbf{c} = c_0 c_1 \ldots c_{p-1}$. Denote $\mathbf{a}^1 = Repl^{d}_{\mathbf{c}}(\mathbf{a}; \{k\})$ and $\mathbf{b}^1 = Repl^{d}_{\mathbf{c}}(\mathbf{b}; \{k\})$. Then

$$\ell(\mathbf{a}^1) = n - 1 + p,\ \mathbf{a}^1 = a^1_0 a^1_1 \ldots a^1_{n-2+p}, \quad \ell(\mathbf{b}^1) = l - 1 + p,\ \mathbf{b}^1 = b^1_0 b^1_1 \ldots b^1_{l-2+p}$$

and

$$a^1_i = \begin{cases} a_i & \text{for } i \in [0, k-1] \\ c_{k-i} & \text{for } i \in [k, k+p-1] \\ a_{i-p+1} & \text{for } i \in [k+p, n-2+p] \end{cases} \qquad b^1_i = \begin{cases} b_i & \text{for } i \in [0, k-1] \\ c_{k-i} & \text{for } i \in [k, k+p-1] \\ b_{i-p+1} & \text{for } i \in [k+p, n-2+p] \end{cases}$$

As $\mathbf{a}^1 = \mathbf{b}^1$, we must have that $n = l$, $a_i = b_i$ for all $i \in [0, k-1]$ and $a_{i-p+1} = b_{i-p+1}$ for all $i \in [k+p, n-2+p]$, that is $a_j = b_j$ for all $j \in [k+1, n-1]$. As $a_k = b_k$ by hypothesis, it follows that $\mathbf{a} = \mathbf{b}$. □

Lemma A.12. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *be expressions and* $k \in Occur_{\mathbf{b}}(\mathbf{a})$. *Then*

$$Repl^{\mathbf{b}}_{\mathbf{d}}(\mathbf{a}; \{k\}) = Repl^{\mathbf{c}}_{\mathbf{d}}(Repl^{\mathbf{b}}_{\mathbf{c}}(\mathbf{a}; \{k\}); \{k\}).$$

Lemma A.13. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}^*$ *be expressions and* $k \in Occur_{\mathbf{b}}(\mathbf{a})$ *such that* $\mathbf{a}^* = Repl^{\mathbf{b}}_{\mathbf{c}}(\mathbf{a}; \{k\})$. *Then* $\mathbf{a} = Repl^{\mathbf{c}}_{\mathbf{b}}(\mathbf{a}^*; \{k\})$.

Proof. Apply Lemma A.12 with $\mathbf{d} := \mathbf{b}$. □

Lemma A.14. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{c}_1$, $\mathbf{b}_1$ *be expressions. Assume that*

(i) $\mathbf{a}$ *does not occur in* $\mathbf{b}$, $\mathbf{c}_1$;

(ii) $\mathbf{b}_1$ *is obtained from* $\mathbf{b}$ *by replacing zero or more occurrences of* $\mathbf{c}$ *with* $\mathbf{c}_1$.

Then $\mathbf{a}$ *does not occur in* $\mathbf{b}_1$.

Lemma A.15. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *be expressions and* $k \in [0, \ell(\mathbf{a}) - 1]$ *such that*

(i) $\mathbf{b}$ *occurs in* $\mathbf{a}$ *at place* k;

(ii) $\mathbf{c}$ *contains symbols that are not in* $\mathbf{a}$;

(iii) $\mathbf{d} = Repl^{\mathbf{b}}_{\mathbf{c}}(\varphi; \{k\})$.

Then $Occur_{\mathbf{c}}(\mathbf{d}) = \{k\}$. *Thus,* $\mathbf{c}$ *occurs uniquely in* $\mathbf{d}$ *at place* k.

A.2 $\ell(\mathbf{b}) = \ell(\mathbf{c})$

Lemma A.16. *Let* $\mathbf{b}$, $\mathbf{c}$ *be expressions such that* $\ell(\mathbf{b}) = \ell(\mathbf{c})$.

(i) Assume that $\mathbf{a}$ *is an expression and* $I \subseteq Occur_{\mathbf{b}}(\mathbf{a})$. *Then* $I \subseteq Occur_{\mathbf{c}}(Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I))$.

(ii) Assume that $\mathbf{a}$ *is an expression and* $I \subseteq Occur_{\mathbf{b}}(\mathbf{a})$. *Then*

$$\ell(\mathbf{a}) = \ell(Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I)).$$

(iii) Assume that $\mathbf{a}$ *is an expression and* $I, J \subseteq Occur_{\mathbf{b}}(\mathbf{a})$ *are such that* $I \cap J = \emptyset$. *Then*

$$Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I \cup J) = Repl_{\mathbf{c}}^{\mathbf{b}}(Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I); J) = Repl_{\mathbf{c}}^{\mathbf{b}}(Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; J); I).$$

(iv) Assume that $\mathbf{a}$, $\mathbf{d}$ *are expressions,* $I \subseteq Occur_{\mathbf{b}}(\mathbf{a})$ *and* $\ell(\mathbf{d}) = \ell(\mathbf{b}) = \ell(\mathbf{c})$. *Then*

$$Repl_{\mathbf{d}}^{\mathbf{b}}(\mathbf{a}; I) = Repl_{\mathbf{d}}^{\mathbf{c}}(Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I); I).$$

(v) Assume that $\mathbf{a}$, $\mathbf{a}^*$ *are expressions,* $I \subseteq Occur_{\mathbf{b}}(\mathbf{a})$ *and* $\mathbf{a}^* = Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I)$. *Then*

$$\mathbf{a} = Repl_{\mathbf{b}}^{\mathbf{c}}(\mathbf{a}^*; I).$$

Proof. Let $l = \ell(\mathbf{b}) = \ell(\mathbf{c})$ and $\mathbf{b} = b_0 b_1 \ldots b_{l-1}$, $\mathbf{c} = c_0 c_1 \ldots c_{l-1}$, $l \geq 1$

(i) Let $n \geq 1$, $\mathbf{a} = a_0 a_1 \ldots a_{n-1}$. If $I = \emptyset$, then the conclusion is obvious. Assume that $I = \{i_1, i_2, \ldots, i_k\} \neq \emptyset$, where $k \geq 1$ and $0 \leq i_1 < i_2 < \ldots < i_k \leq n-1$. Thus, $i_2 \geq i_1$. Let

$$\mathbf{a}^0 = a_0 a_1 \ldots a_{i_1 - 1},\ \mathbf{a}^1 = a_{i_1} a_{i_1+1} \ldots a_{i_2-1},\ \ldots,\ \mathbf{a}^k = a_{i_k} a_{i_k+1} \ldots a_{n-1}.$$

Then

$$\mathbf{a} = \mathbf{a}^0 \mathbf{b} \mathbf{a}^1 \mathbf{b} \ldots \mathbf{a}^{k-1} \mathbf{b} \mathbf{a}^k$$

and

$$\mathbf{a} = \mathbf{a}^0 \mathbf{b} \mathbf{a}^1 \mathbf{b} \ldots \mathbf{a}^{k-1} \mathbf{b} \mathbf{a}^k$$

(ii) TO WRITE

(iii) Apply (iv) with $\mathbf{d} := \mathbf{b}$. We have that $Repl_{\mathbf{b}}^{\mathbf{b}}(\mathbf{a}; I) = \mathbf{a}$ and $Repl_{\mathbf{b}}^{\mathbf{c}}(Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; I); I) = Repl_{\mathbf{b}}^{\mathbf{c}}(\mathbf{a}^*; I)$.

□

Lemma A.17. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *be expressions and* $k \in Occur_{\mathbf{b}}(\mathbf{a})$ *such that* $\ell(\mathbf{b}) = \ell(\mathbf{c}) = p$ *and* $\mathbf{d} = Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}; \{k\})$. *Assume that* $i \leq k < k + p - 1 \leq l \leq n - 1$.

(i) Let $\mathbf{d}^*$ *be the* (i, l)*-subexpression of* $\mathbf{d}$. *Then* $k - i \in Occur_{\mathbf{b}}(\mathbf{d}^*)$ *and* $\mathbf{a}^* = Repl_{\mathbf{b}}^{\mathbf{c}}(\mathbf{d}^*; \{k - i\})$ *is the* (i, l)*-subexpression of* $\mathbf{a}$.

(ii) Let $\mathbf{a}^*$ *be the* (i, l)*-subexpression of* $\mathbf{a}$. *Then* $k - i \in Occur_{\mathbf{b}}(\mathbf{a}^*)$ *and* $\mathbf{d}^* = Repl_{\mathbf{c}}^{\mathbf{b}}(\mathbf{a}^*; \{k - i\})$ *is the* (i, l)*-subexpression of* $\mathbf{d}$.

A.3 Replacement of symbols

Lemma A.18. *Let x, y be symbols,* $\mathbf{a}$ *be an expression and* $I \subseteq Occur_x(\mathbf{a})$*. Then* $\ell(\mathbf{a}) = \ell(Repl^x_y(\mathbf{a}; I))$*.*

Proof. Apply Lemma A.16 (ii) with $\mathbf{b} := x$ and $\mathbf{c} := y$ □

Lemma A.19. *Let x, y be symbols,* $\mathbf{a}$ *be an expression and* $I \subseteq Occur_x(\mathbf{a})$*.*
Then $I \subseteq Occur_y(Repl^x_y(\mathbf{a}; I))$*.*

Proof. Apply Lemma A.16(i) with $\mathbf{b} := x$ and $\mathbf{c} := y$ □

Lemma A.20. *Let x, y be symbols,* $\mathbf{a}$ *be an expression and* $I, J \subseteq Occur_x(\mathbf{a})$ *be such that* $I \cap J = \emptyset$*. Then*

$$Repl^x_y(\mathbf{a}; I \cup J) = Repl^x_y(Repl^x_y(\mathbf{a}; I); J) = Repl^x_y(Repl^x_y(\mathbf{a}; J); I).$$

Proof. Apply Lemma A.16(iii) with $\mathbf{b} := x$ and $\mathbf{c} := y$ □

Lemma A.21. *Let x, y, z be symbols,* $\mathbf{a}$ *be an expression and* $I \subseteq Occur_x(\mathbf{a})$*. Then*

$$Repl^x_z(\mathbf{a}; I) = Repl^y_z(Repl^x_y(\mathbf{a}; I); I).$$

Proof. Apply Lemma A.16(iv) with $\mathbf{b} := x$, $\mathbf{c} := y$ and $\mathbf{d} := z$. □

Lemma A.22. *Let x, y be symbols,* $\mathbf{a}$, $\mathbf{a}^*$ *be expressions and* $I \subseteq Occur_x(\mathbf{a})$*. Assume that* $\mathbf{a}^* = Repl^x_y(\mathbf{a}; I)$*. Then* $\mathbf{a} = Repl^y_x(\mathbf{a}^*; I)$*.*

Proof. Apply Lemma A.16(v) with $\mathbf{b} := x$, $\mathbf{c} := y$ and $\mathbf{d} := y$ □

Lemma A.23. *Let x, y be symbols and* $\mathbf{a}$ *be an expression such that y does not occur in* $\mathbf{a}$*. Then*

$$Occur_x(\mathbf{a}) = Occur_y(Replall^x_y(\mathbf{a})).$$

Proof. Let $\mathbf{a} = a_0 a_1 \dots a_{n-1}$ and denote $\mathbf{d} := Replall^x_y(\mathbf{a})$. Then, by Lemma A.18, we have that $\ell(\mathbf{a}) = \ell(\mathbf{d}) = n$, hence $\mathbf{d} = d_0 d_1 \dots d_{n-1}$.
$\subseteq$ By Lemma A.19.
$\supseteq$ Let $i \in [0, n-1] \setminus I$. Then $d_i = a_i \neq y$, as y does not occur in $\mathbf{a}$. □

Lemma A.24. *Let x be symbol and* $\mathbf{a}$, $\mathbf{b}$ *be expressions. Then*

$$Occur_x(\mathbf{ab}) = Occur_x(\mathbf{a}) \cup (Occur_x(\mathbf{b}) + \ell(\mathbf{a})).$$

Lemma A.25. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *be expressions and x, y be symbols such that*

(i) $\mathbf{b}$ *occurs uniquely in* $\mathbf{a}$ *at place k;*

(ii) x, y do not occur in $\mathbf{b}$*;*

(iii) $\mathbf{d}$ *is obtained from* $\mathbf{a}$ *by replacing zero or more occurrences of x with y.*

Then $\mathbf{b}$ *occurs uniquely in* $\mathbf{d}$ *at place k.*

Lemma A.26. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *be expressions,* $k \in [0, \ell(\mathbf{a}) - 1]$ *and x, z be symbols such that*

(i) $\mathbf{b}$ *occurs in* $\mathbf{a}$ *at k;*

(ii) z does not occur in $\mathbf{b}$*;*

(iii) $\mathbf{c} = Replall^x_z(\mathbf{b})$*;*

(iv) $\mathbf{d} = Repl^{\mathbf{b}}_{\mathbf{c}}(\mathbf{a}; \{k\})$*.*

Then

$$\mathbf{d} = Repl_z^x(\mathbf{a}; Occur_x(\mathbf{b}) + k).$$

Lemma A.27. *Let* $\mathbf{a}$ *be an expression and* $x, y, z \in A$. *Assume that* x *occurs in* $\mathbf{a}$ *and* $I \subseteq Occur_x(\mathbf{a})$, *where* $n \geq 1$. *Let*

$$\mathbf{b} := Repl_y^x(\mathbf{a}; I), \quad \mathbf{c} := Repl_z^x(\mathbf{a}; I).$$

Then

(i) $I \subseteq Occur_z(\mathbf{c})$;

(ii) $\mathbf{b} = Repl_y^z(\mathbf{c}; I)$.

Lemma A.28. *Let* x, y, z *be symbols,* $\mathbf{a}$ *be an expression and* $I, J \subseteq Occur_x(\mathbf{a})$ *be such that* $I \cup J = Occur_x(\mathbf{a})$ *and* $I \cap J = \emptyset$.
Then x *does not occur in* $Repl_y^x(Repl_z^x(\mathbf{a}; J); I)$.

Lemma A.29. *Let* x, y, z *be symbols such that* $x \neq y, z$, $\mathbf{a}$ *be an expression and* $I, J \subseteq Occur_x(\mathbf{a})$ *be such that* $I \cap J = \emptyset$. *Then*

$$Repl_y^x(Repl_z^x(\mathbf{a}; J); I) = Repl_z^x(Repl_y^x(\mathbf{a}; I); J).$$

B Set theory

Let A, B be sets. We use the following notations:

(i) $A \cup B$ for the union of A and B.

(ii) $A \cap B$ for the intersection of A and B.

(iii) $A \setminus B$ for the difference between A and B.

(iv) $A \Delta B$ for the symmetric difference of A and B.

(v) 2^A for the powerset of A.

(vi) $C_A B$ for the complementary of B, when $B \subseteq A$.

B.1 Set-theoretic properties used in the lecture notes

Proposition B.1. *Let* A, B, C *be sets. Then*

(i) If $B, C \subseteq A$, *then*

$$A \setminus (B \setminus C) = (A \setminus B) \cup C. \tag{14}$$

(ii) If $B \subseteq A$, *then*

$$\big((A \cup C) \setminus B\big) \cup C = (A \setminus B) \cup C. \tag{15}$$

(iii) If $B \subseteq A$ *and* $C \cap A = \emptyset$, *then*

$$(A \cup C) \setminus B = (A \setminus B) \cup C. \tag{16}$$

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