SNSB
Winter Term 2010/2011
Ergodic Ramsey Theory
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07.12.2010

Homework 4

(H4.1) Let $A \subseteq Z$. Prove the following

(i) If $m \notin A$ for some $m \in \mathbb{Z}_+$, then $\sigma(A) \leq 1 - \frac{1}{m} < 1$.

(ii) $\sigma(A) = 1$ if and only if A contains $\mathbb{Z}_+$.

(iii) If A is finite, then $\sigma(A) = 0$.

(iv) If $A \subseteq \mathbb{Z}_+$ is the set of even integers, then $\sigma(A) = 0$.

(v) If $A \subseteq \mathbb{Z}_+$ is the set of odd integers, then $\sigma(A) = \dfrac{1}{2}$.

(H4.2) Prove the following.

(i) If $A \subseteq Z$ contains $\mathbb{Z}_+$, then $d(A) = 1$.

(ii) Finite sets have density 0.

(iii) If $A = \{a + id \mid i \geq 0\}, a, d \in \mathbb{Z}_+$ is an arithmetic progession, then $d(A) = \frac{1}{d}$. In particular, the set of positive even (resp. odd) integers has density $\frac{1}{2}$.

(iv) The set $A \subseteq \mathbb{Z}_+$ of prime numbers has density 0.

(v) Let A be the set of positive integers with leading digit equal to 1 in the expansion to base 10. Then $\underline{d}(A) \neq \overline{d}(A)$.

(vi) If $A, B \subseteq \mathbb{Z}$, then $\overline{d}(A \cup B) \leq \overline{d}(A) + \overline{d}(B)$.

(vii) If $\overline{d}(A) > 0$ and $A = \bigcup_{i=1}^{r} C_i$, then there exists $i \in [1, r]$ such that $\overline{d}(C_i) \geq \dfrac{\overline{d}(A)}{r} > 0$.

(H4.3) Prove the following.

(i) If $A \subseteq \mathbb{Z}$ contains $\mathbb{Z}_+$, then $Bd^\star(A) = 1$.

(ii) Finite sets have Banach density 0.

(iii) If $A, B \subseteq \mathbb{Z}$, then $Bd^\star(A \cup B) \leq Bd^\star(A) + Bd^\star(B)$.

(iv) If $A = \{a + id \mid i \geq 0\}, a, d \in \mathbb{Z}_+$ is an arithmetic progession, then $Bd(A) = \frac{1}{d}$. In particular, the set of positive even (resp. odd) integers has Banach density $\frac{1}{2}$.

(v) If $Bd^\star(A) > 0$ and $A = \bigcup_{i=1}^r C_i$, then there exists $i \in [1, r]$ such that $Bd^\star(C_i) \geq \dfrac{Bd^\star(A)}{r} > 0$.

(H4.4) Let $A = \displaystyle\bigcup_{n=1}^{\infty} [n^3, n^3 + n]$. Prove that $Bd^\star(A) > 0$ and $d(A) = 0$.