**SNSB**
**Winter Term 2010/2011**
**Ergodic Ramsey Theory**
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# Homework 5

**(H5.1)** Let $H$ be a Hilbert space. Prove that

$$\eta : (0,2] \to (0,1], \quad \eta(\varepsilon) := \frac{\varepsilon^2}{8} \tag{D.8}$$

is a modulus of uniform convexity for $H$.

**(H5.2)** Let $X$ be a uniformly convex Banach space and $\eta$ be a modulus of uniform convexity. Define $u_\eta : (0,2] \to (0,1], \quad u_\eta(\varepsilon) = \frac{\varepsilon}{2} \cdot \eta(\varepsilon)$. Then for all $\varepsilon > 0$ and for all $x, y \in X$

$$\|x\| \leq \|y\| \leq 1 \text{ and } \|x-y\| \geq \varepsilon \quad \text{imply} \quad \left\|\frac{1}{2}(x+y)\right\| \leq \|y\| - u_\eta(\varepsilon). \tag{D.9}$$

**(H5.3)** Let $X$ be a Banach space, $T : X \to X$ be linear and $(x_n)$ be the Cesaro average starting with $x$.

(i) For all $n, k \in \mathbb{N}$,

$$x_{n+k} = \frac{n}{n+k}x_n + \frac{1}{n+k}\sum_{i=0}^{k-1} T^{n+i}x, \tag{D.10}$$

$$x_{kn} = \frac{1}{k}\sum_{i=0}^{k-1} T^{in}x_n, \tag{D.11}$$

$$x_{2kn} = \frac{1}{k}\sum_{i=0}^{k-1} \frac{1}{2}T^{in}\left(x_n + T^{kn}x_n\right). \tag{D.12}$$

(ii) Assume moreover that $T$ satisfies $\|Tx\| \leq \|x\|$ for all $x \in X$. Then for all $n, k \in \mathbb{N}$,

$$\|x_{n+k} - x_n\| \leq \frac{2k\|x\|}{n+k}, \tag{D.13}$$

$$\|x_{kn} - x_n\| \leq \max_{i=0,\ldots,k-1} \|T^{in}x_n - x_n\| \tag{D.14}$$