SNSB
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Ergodic Ramsey Theory
Laurenţiu Leuştean

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Seminar 5

(S5.1) Let D be set and let $\mathcal{A}$ be a subset of $\mathcal{P}(D)$ which has the finite intersection property. Then there is an ultrafilter p on D such that $\mathcal{A} \subseteq p$.

Proof. See [47, Theorem 3.8, p.50]. □

(S5.2) Let $\mathcal{A} = \{A \subseteq \mathbb{Z}_+ \mid \mathbb{Z}_+ \setminus A \text{ is finite}\}$. Prove that there exists a non-principal ultrafilter $\mathcal{U}$ on $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{U}$.

Proof. Clearly $\mathcal{A}$ has the finite intersection property. Apply Proposition 2.2.5 to get an ultrafilter $\mathcal{U}$ on $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{U}$. By Proposition 2.2.4.(iii), $\mathcal{U}$ is non-principal. □

(S5.3) Let D be set, let $\mathcal{F}$ be a filter on D, and let $A \subseteq D$. Then $A \notin \mathcal{F}$ if and only if there is some ultrafilter $\mathcal{U}$ with $\mathcal{F} \cup \{D \setminus A\} \subseteq \mathcal{U}$.

Proof. See [47, Corollary 3.9, p.50]. □

(S5.4) Let D be a set and let $\mathcal{G} \subseteq \mathcal{P}(D)$. The following are equivalent.

(i) Whenever $r \geq 1$ and $D = \bigcup_{i=1}^r C_i$, there exists $i \in [1, r]$ and $G \in \mathcal{G}$ such that $G \subseteq C_i$.

(ii) There is an ultrafilter $\mathcal{U}$ on d such that for every member A of $\mathcal{U}$, there exists $G \in \mathcal{G}$ with $G \subseteq A$.

Proof. See [47, Theorem 5.7, p.92]. □

(S5.5) Verify that Hilbert theorem 1.6.13 is a special case of the Finite Sums theorem.

Proof. Let $r \in \mathbb{Z}_+$, $\mathbb{Z}_+ = \bigcup_{i=1}^{r} C_i$, and $l \geq 1$. Apple the Finite Sums theorem 2.2.20 to get $i \in [1, r]$ and a sequence $(x_n)_{n\geq 1}$ in $\mathbb{Z}_+$ such that $FS\big((x_n)_{n\geq 1}\big) \subseteq C_i$. Take $n_1, n_2, \ldots, n_l$ to be $x_1, \ldots, x_l$ in increasing order, and $B := \{x_{l+1} + \ldots x_{l+p} \mid p \in \mathbb{Z}_+\}$.

□

(S5.6) Let X be a Hausdorff topological space and $(x_n)_{n\geq 1}$ be a sequence in X.

(i) For every $p \in \beta\mathbb{Z}_+$, the following are satisfied:

(a) The p-limit of (x_n), if exists, is unique.

(b) If X is compact, then $p\text{-}\lim x_n$ exists.

(c) If $f : X \to Y$ is continuous and $p\text{-}\lim x_n = x$, then $p\text{-}\lim f(x_n) = f(x)$.

(ii) $\lim_{n\to\infty} x_n = x$ implies $p\text{-}\lim x_n = x$ for every non-principal ultrafilter p.

Proof. (i) Let $p \in \beta\mathbb{Z}_+$.

(a) Assume that $x \neq y \in X$ are such that $p - \lim x_n = x$ and $p - \lim x_n = y$. Since X is Hausdorff, there are disjoint open neighboorhoods U of x and V of y. Then $A_U := \{n \in \mathbb{Z}_+ \mid x_n \in U\} \in p$ and $A_V := \{n \in \mathbb{Z}_+ \mid x_n \in V\} \in p$, hence $A_U \cap A_V \in p$. On the other hand, $A_U \cap A_V = \emptyset$. We have got a contradiction with the fact that p is a filter.

(b) Assume by contradiction that there exist a sequence (x_n) and an ultrafilter p such that $p\text{-}\lim x_n$ does not exist. Then for each $x \in X$ there exists an open neighborhood U_x of x such that $A_x := \{n \in \mathbb{Z}_+ \mid x_n \in U_x\} \notin p$. It follows that the family $(U_x)_{x\in X}$ is an open cover of X, so from compactness, we get a finite subset $F \subseteq X$ with $X = \bigcup_{x\in F} U_x$. We get that

$$\bigcup_{x\in F} A_x = \mathbb{Z}_+ \in p,$$

hence we must have $A_x \in p$ for some $x \in F$. We have got a contradiction.

(c) Let $x = p\text{-}\lim x_n$ and let V be an open neighborhood of $f(x)$. Since f is continuous, we have that $f(U) \subseteq V$ for some open neighborhood U of x. Let $A := \{n \in \mathbb{Z}_+ \mid x_n \in U\}$ and $B := \{n \in \mathbb{Z}_+ \mid f(x_n) \in V\}$. Then $A \in p$ and $A \subseteq B$, hence $B \in p$.

(ii) Assume that $\lim_{n\to\infty} x_n = x$ and let p be a nonprincipal ultrafilter on $\mathbb{Z}_+$. If U is an open neighborhood of x, then we get $N \geq 1$ such that $x_n \in U$ for all $n > N$. Let $n = 1, \ldots N$. Since $p \neq e(n)$, there exists $A_n \in p$ such that $n \notin A_n$. It follows that $A_1 \cap \ldots A_N \subseteq \mathbb{Z}_+ \setminus [1, N] = \{n \in \mathbb{Z}_+ \mid n > N\} \subseteq \{n \in \mathbb{Z}_+ \mid x_n \in U\}$. Since $A_1 \cap \ldots A_N \in p$ and p is a filter, we conclude that $\{n \in \mathbb{Z}_+ \mid x_n \in U\} \in p$. Thus, $p\text{-}\lim x_n = x$.

□

(S5.7) Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ be bounded sequences in $\mathbb{R}$, and p be a non-principal ultrafilter on $\mathbb{Z}_+$.

(i) (x_n) has a unique p-limit. If $a \le x_n \le b$, then $a \le p-\lim x_n \le b$.

(ii) For any $c \in \mathbb{R}$, $p-\lim cx_n = c \cdot p-\lim x_n$.

(iii) $p-\lim(x_n + y_n) = p-\lim x_n + p-\lim y_n$.

Proof. (i) Apply Proposition 2.2.26.(ib) for the compact space $X = [a, b]$.

(ii) Apply Proposition 2.2.26.(ic) for the continuous mapping $f : \mathbb{R} \to \mathbb{R}$, $f(x) = cx$.

(iii) Let $x = p-\lim x_n$ and $y = p-\lim y_n$. For any $\varepsilon > 0$, $A_\varepsilon := \{n \in \mathbb{Z}_+ \mid x_n \in (x - \varepsilon/2, x + \varepsilon/2)\} \in p$ and $B_\varepsilon := \{n \in \mathbb{Z}_+ \mid y_n \in (y - \varepsilon/2, y + \varepsilon/2)\} \in p$, hence $A_\varepsilon \cap B_\varepsilon \in p$. On the other hand

$$A_\varepsilon \cap B_\varepsilon \subseteq C_\varepsilon := \{n \in \mathbb{Z}_+ \mid x_n + y_n \in (x + y - \varepsilon, x + y + \varepsilon)\},$$

so $C_\varepsilon \in p$. Hence, $p-\lim(x_n + y_n) = x + y$.

□

(S5.8) Let $\mathcal{U} \subseteq \mathcal{P}(D)$. The following are equivalent:

(i) $\mathcal{U}$ is an ultrafilter on D.

(ii) $\mathcal{U}$ has the finite intersection property and for each $A \in \mathcal{P}(D) \setminus \mathcal{U}$ there is some $B \in \mathcal{U}$ such that $A \cap B = \emptyset$.

(iii) $\mathcal{U}$ is maximal with respect to the finite intersection property. (That is, $\mathcal{U}$ is a maximal member of $\{\mathcal{V} \subseteq \mathcal{P}(D) \mid \mathcal{V}$ has the finite intersection property$\}$.)

(iv) $\mathcal{U}$ is a filter on D and for any collection $C_1, \ldots, C_n$ of subsets of D, if $\bigcup_{i=1}^{n} C_i \in \mathcal{U}$, then $C_j \in \mathcal{U}$ for some $j = 1, \ldots n$.

(v) $\mathcal{U}$ is a filter on D and for all $A \subseteq D$ either $A \in \mathcal{U}$ or $D \setminus A \in \mathcal{U}$.

Proof. See [47, Theorem 3.6, p.49]. □