**SNSB**
**Winter Term 2010/2011**
**Ergodic Ramsey Theory**
**Laurenţiu Leuştean**

21.12.2010

# Seminar 6

**(S6.1)** Let $(X, \mathcal{B}, \mu, T)$ be a MDS. The following are equivalent

(i) $T$ is ergodic.

(ii) Whenever $f$ is measurable and $U_T f = f$ a.e., then $f$ is constant a.e.

**(S6.2)** Let $f \in \mathcal{M}_{\mathbb{C}}(X, \mathcal{B})$ and $n \geq 1$.

(i) If $f$ is $T$-invariant (a.e.), then $S_n f = f$ (a.e.).

(ii) $S_n f \in \mathcal{M}_{\mathbb{C}}(X, \mathcal{B})$.

(iii) $S_n f = \dfrac{1}{n} \sum_{k=0}^{n-1} U_{T^k} f$.

(iv) For any $p \geq 1$, $f \in L^p(X, \mathcal{B}, \mu)$ (resp. $L^p_{\mathbb{R}}(X, \mathcal{B}, \mu)$) implies $S_n f \in L^p(X, \mathcal{B}, \mu)$ (resp. $L^p_{\mathbb{R}}(X, \mathcal{B}, \mu)$).

(v) For all $x \in X$, $\dfrac{n+1}{n} S_{n+1}(x) - S_n f(Tx) = \dfrac{1}{n} f(x)$.

(vi) If $f \in \mathcal{M}_{\mathbb{R}}(X, \mathcal{B})$, then $\underline{f} \circ T = \underline{f}$ and $\overline{f} \circ T = \overline{f}$.

(vii) $\displaystyle\int_X S_n f \, d\mu = \int_X f \, d\mu$.

(viii) If $f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$ is nonnegative, then $S_n f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$ is nonnegative and $\|S_n f\|_1 = \|f\|_1$.

**(S6.3)** Let $A, B \in \mathcal{B}$ and $n \geq 1$.

(i) $S_n \chi_A = \dfrac{1}{n} \sum_{k=0}^{n-1} \chi_{T^{-k}(A)}$ and $\chi_B \cdot S_n \chi_A = \dfrac{1}{n} \sum_{k=0}^{n-1} \chi_{T^{-k}(A) \cap B}$.

(ii) $\displaystyle\int_X S_n\chi_A = \mu(A).$

(iii) $\displaystyle\int_X \chi_B \cdot S_n\chi_A \, d\mu = \frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}(A) \cap B).$

**(S6.4)**

(i) Let $X$ be a nonempty set, $(E_n)_{n\geq 1}$ be a sequence of subsets of $X$ and $f : X \to \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \chi_{\cup_{i=1}^n E_i} f = \chi_{\cup_{i\geq 1} E_i} f. \tag{D.2}$$

(ii) Let $(X, \mathcal{B}, \mu)$ be a probability space, $f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$, $(E_n)_{n\geq 1}$ be an increasing sequence of measurable sets, and $E = \displaystyle\bigcup_{n\geq 1} E_n$. Prove that

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_{E_n} f \, d\mu. \tag{D.3}$$