SNSB
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Ergodic Ramsey Theory
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Seminar 7

(S7.1) Let $(x_n)_{n\geq 1}$ be a sequence in a metric space (X, d). Prove that the following are equivalent:

(i) (x_n) is Cauchy.

(ii) $\forall \varepsilon > 0 \; \exists N \geq 1 \; \forall p \in \mathbb{N} \;\; d(x_{N+p}, x_N) < \varepsilon$.

(iii) $\forall \varepsilon > 0 \; \exists N \geq 1 \; \forall p \in \mathbb{N} \; \forall i, j \in [N, N+p] \;\; d(x_i, x_j) < \varepsilon$.

(iv) $\forall \varepsilon > 0 \; \forall g : \mathbb{Z}_+ \to \mathbb{Z}_+ \; \exists N \geq 1 \; \forall i, j \in [N, N+g(N)] \;\; d(x_i, x_j) < \varepsilon$.

Proof. It is easy to see that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and $(iii) \Rightarrow (iv)$.
$(iv) \Rightarrow (iii)$ If (iii) would be false, then for some $\varepsilon > 0$,

$$\forall n \in \mathbb{N} \; \exists M_n \geq 1 \; \exists i, j \in [n, n+M_n] \quad d(x_i, x_j) \geq \varepsilon.$$

Define $g(n) := M_n$ and apply (iv) to this g to get a contradiction. □

(S7.2) Let $(a_n)_{n\geq 1}$ be a sequence of nonnegative real numbers.

(i) Prove that for all $\varepsilon > 0$ there exists $N \geq 1$ such that $a_N \leq a_m + \varepsilon$ for all $m \geq 1$.

(ii) Prove the following for all $\varepsilon > 0$, all $g : \mathbb{Z}_+ \to \mathbb{Z}_+$, and all $b \geq a_1$:

(a) There exists $N \leq \Theta(b, \varepsilon, g)$ such that $a_N \leq a_{g(N)} + \varepsilon$, where

$$\Theta(b, \varepsilon, g) := \max_{i \leq K} g^i(1), \qquad K := \left\lceil \frac{b}{\varepsilon} \right\rceil.$$

Moreover, $N = g^i(1)$ for some $i < K$.

(b) There exists $N \leq h^K(1)$ such that $a_N \leq a_m + \varepsilon$ for all $m \leq g(N)$, where

$$h(n) := \max_{i \leq n} g(i), \quad K \text{ is as above.}$$

Proof. (i) Since $a_n \geq 0$ for all n, there exists $\alpha := \inf_{n\geq 1} a_n$. Given $\varepsilon > 0$, there exists $N \geq 1$ such that $a_N \leq \alpha + \varepsilon \leq a_m + \varepsilon$ for all $m \geq 1$.

(ii) (a) Assume by contradiction that $a_{g^i(1)} - a_{g^{i+1}(1)} > \varepsilon$ for all $i = 0, \ldots, K-1$. By adding these inequalities, we get that

$$a_1 - a_{g^K(1)} > K\varepsilon = \left\lceil \frac{b}{\varepsilon} \right\rceil \cdot \varepsilon \geq \frac{a_1}{\varepsilon} \cdot \varepsilon = a_1,$$

hence $a_1 - a_{g^M(1)} > a_1$, which is a contradiction, since $a_{g^M(1)} \geq 0$.

(b) Define

$$\tilde{g} : \mathbb{Z}_+ \to \mathbb{Z}_+, \quad \tilde{g}(n) := \text{ the least } i \leq g(n) \text{ satisfying } a_i = \min\{a_j \mid j \leq g(n)\}.$$

Then, for all $n \geq 1$ and for all $m \leq g(n)$, we have that $a_m \geq a_{\tilde{g}(n)}$. Applying now (i) for ε and $\tilde{g}$, we get that there exists $N \leq \Theta(b, \varepsilon, \tilde{g})$ such that $a_N \leq a_{\tilde{g}(N)} + \varepsilon \leq a_m + \varepsilon$ for all $m \leq g(N)$. Let us now define

$$h : \mathbb{Z}_+ \to \mathbb{Z}_+, \quad h(n) = \max_{i\leq n} g(i).$$

Then h is increasing and $h(n) \geq g(n) \geq \tilde{g}(n)$ for all $n \geq 1$. Furthermore, $h^i(n) \geq \tilde{g}^i(n)$ and $h^i(1) \geq h^{i-1}(1)$ for all $i, n \geq 1$. Hence,

$$h^K(1) = \max_{i\leq K} h^i(1) \geq \max_{i\leq K} \tilde{g}^i(1) = \Theta(b, \varepsilon, \tilde{g}) \geq N.$$

□

(S7.3) Let $f : \mathbb{N} \to \mathbb{N}$ be a function. Show that there exists $N \in \mathbb{N}$ such that

$$f(N) \leq f(m)$$

for all $m \in \mathbb{N}$.

Proof. Since $\emptyset \neq f(\mathbb{N}) \subseteq \mathbb{N}$, it follows that $f(\mathbb{N})$ has a minimum k. Let $N \in \mathbb{N}$ be such that $f(N) = k$. □

(S7.4) Let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary function. Prove that there exists $n \in \mathbb{N}$ such that

$$f(f(n) + 1) \neq n.$$

Proof. Assume by contradiction that $f(f(n) + 1) = n$ for all $n \in \mathbb{N}$. Then by taking $n := f(1)$ we get

$$f(1) = f(f(f(1)) + 1),$$

and $1 < 2 \leq f(f(1)) + 1$. Hence, if we take $n_1 = f(f(1)) + 1$, we get $n_1 \neq 1$, but $f(n_1) = f(1)$. It follows that

$$1 = f(f(1) + 1) = f(f(n_1) + 1) = n_1,$$

that is a contradiction. □