SNSB
Winter Term 2010/2011
Ergodic Ramsey Theory
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Seminar 7

(S7.1) Let $(x_n)_{n\geq 1}$ be a sequence in a metric space (X, d). Prove that the following are equivalent:

(i) (x_n) is Cauchy.

(ii) $\forall \varepsilon > 0\ \exists N \geq 1\ \forall p \in \mathbb{N}\ \ d(x_{N+p}, x_N) < \varepsilon$.

(iii) $\forall \varepsilon > 0\ \exists N \geq 1\ \forall p \in \mathbb{N}\ \forall i, j \in [N, N+p]\ \ d(x_i, x_j) < \varepsilon$.

(iv) $\forall \varepsilon > 0\ \forall g : \mathbb{Z}_+ \to \mathbb{Z}_+\ \exists N \geq 1\ \forall i, j \in [N, N+g(N)]\ \ d(x_i, x_j) < \varepsilon$.

(S7.2) Let $(a_n)_{n\geq 1}$ be a sequence of nonnegative real numbers.

(i) Prove that for all $\varepsilon > 0$ there exists $N \geq 1$ such that $a_N \leq a_m + \varepsilon$ for all $m \geq 1$.

(ii) Prove the following for all $\varepsilon > 0$, all $g : \mathbb{Z}_+ \to \mathbb{Z}_+$, and all $b \geq a_1$:

(a) There exists $N \leq \Theta(b, \varepsilon, g)$ such that $a_N \leq a_{g(N)} + \varepsilon$, where

$$\Theta(b, \varepsilon, g) := \max_{i\leq K} g^i(1), \qquad K := \left\lceil \frac{b}{\varepsilon} \right\rceil.$$

Moreover, $N = g^i(1)$ for some $i < K$.

(b) There exists $N \leq h^K(1)$ such that $a_N \leq a_m + \varepsilon$ for all $m \leq g(N)$, where

$$h(n) := \max_{i\leq n} g(i), \quad K \text{ is as above.}$$

(S7.3) Let $f : \mathbb{N} \to \mathbb{N}$ be a function. Show that there exists $N \in \mathbb{N}$ such that

$$f(N) \leq f(m)$$

for all $m \in \mathbb{N}$.

(S7.4) Let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary function. Prove that there exists $n \in \mathbb{N}$ such that

$$f(f(n) + 1) \neq n.$$