SNSB
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Ergodic Ramsey Theory
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Seminar 9

(S9.1) Prove that the Bernoulli shift is strong mixing.

Proof. Let $(W^{\mathbb{Z}}, \mathcal{B}, \mu, T)$ be the Bernoullis shift, and let $\mathcal{C}$ be the set of all cylinders. By Proposition 3.2.1, the set $\mathcal{S} = \mathcal{C} \cup \{\emptyset\}$ is a semialgebra that generates $\mathcal{B}$. Applying Proposition 6.0.3.(ii), it is enough to prove that $\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$ for $A, B \in \mathcal{C}$. Let $A = C^{w_{i_1},\ldots,w_{i_t}}_{n_1,\ldots,n_t}$ and $B = C^{w_{j_1},\ldots,w_{j_l}}_{m_1,\ldots,m_l}$ be two cylinders. For all $n \geq 0$, we have that $T^{-n}(A) = C^{w_{i_1},\ldots,w_{i_t}}_{n+n_1,\ldots,n+n_t}$. Let $N \in \mathbb{N}$ be such that

$$\min\{N + n_1, \ldots, N + n_t\} > \max\{m_1, \ldots, m_l\}.$$

Then for all $n \geq N$, we have that

$$T^{-n}(A) \cap B = C^{w_{j_1},\ldots,w_{j_l},w_{i_1},\ldots,w_{i_t}}_{m_1,\ldots,m_l,n+n_1,\ldots,n+n_t},$$

hence for all $n \geq N$,

$$\mu(T^{-n}(A) \cap B) \;=\; p_{j_1} \cdot \ldots \cdot p_{j_l} \cdot p_{i_1} \cdot \ldots \cdot p_{i_t} = \mu(A)\mu(B).$$

Obviously $\lim_{n\to\infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$. □

(S9.2) Any syndetic set $A \subseteq \mathbb{Z}_+$ has positive lower density.

Proof. Since A is syndetic, there exists $N \geq 1$ such that $[k, k+N] \bigcap A \neq \emptyset$ for any $k \in \mathbb{Z}_+$. Let $n \in \mathbb{Z}_+$ be arbitrary and $q \in \mathbb{Z}_+$, $0 \leq r < N+1$ be such that $n = q(N+1) + r$. Then for all $k = 0, \ldots, q-1$, we have that $[k(N+1)+1, (k+1)(N+1)] \subseteq [1, N]$ and $\big|A \cap [k(N+1)+1, (k+1)(N+1)]\big| \geq 1$. It follows that for all $n \geq N+1$ we have that $q \geq 1$ and

$$\frac{|A \cap [1,n]|}{n} \;\geq\; \frac{q}{n} = \frac{q}{q(N+1)+r} \geq \frac{1}{N+1}.$$

As a consequence,

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n} \geq \frac{1}{N+1} > 0.$$

□

Definition . *Let (X,d) be a metric space, $A \subseteq X$ and $\varepsilon > 0$. We say that $x, y \in A$ are ε-**separated** if $d(x,y) \geq \varepsilon$. A is said to be ε-**separated** if every two points of A are ε-separated.*

(S9.3) Let (X,d) be a metric space and $A \subseteq X$ be totally bounded with $|A| \geq 2$. Define for all $\varepsilon > 0$, the set

$$S_\varepsilon = \{m \geq 2 \mid \text{ there exist } m \ \varepsilon-\text{separated points in } A\} \tag{E.5}$$

Prove that if S_ε is nonempty, then it is bounded from above.

Proof. Firstly, let us remark that since there are distinct points $x, y \in A$, by taking $\varepsilon \leq d(x,y)$ we have that $2 \in A_\varepsilon$. Let $\varepsilon > 0$ be such that $S_\varepsilon \neq \emptyset$. Consider the following set

$$T = \{n \geq 1 \mid \text{ there exists a cover of } A \text{ by } m \ \varepsilon/2-\text{balls}\}.$$

Since A is totally bounded, we have that $T \neq \emptyset$. Hence, T has a minimum α. Thus, there are $x_1, \ldots, x_\alpha \in A$ such that $A = (B_{\varepsilon/2}(x_1) \cap A) \cup \ldots \cup (B_{\varepsilon/2}(x_\alpha) \cap A)$. We shall prove that α is an upper bound for S_ε. For $m \in S_\varepsilon$, then there are $y_1, \ldots, y_m \in A$ such that $d(y_i, y_j) \geq \varepsilon$ for all $1 \leq i \neq j \leq m$. If $m > \alpha$, then two y_i, y_j must be in the same $\varepsilon/2$-ball, so $d(y_i, y_j) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, which is a contradiction.

Thus, we must have $m \leq \alpha$ for all $m \in S_\varepsilon$.

□

(S9.4) Let $f \in AP(X)$. Prove that for any $\varepsilon > 0$ the set

$$A_\varepsilon := \{n \geq 1 \mid \|U_T^n f - f\|_2 < \varepsilon\} \tag{E.6}$$

is syndetic.

Proof. Let $\varepsilon > 0$. Since $\{U_T^n f \mid n \geq 0\}$ is totally bounded, we can find a finite subset $\{U_T^{n_1} f, U_T^{n_2} f, \ldots, U_T^{n_r} f\}$ which is ε-separated and which has the maximum cardinality r of such a subset (see S9.3). Let $N := \max\{n_1, \ldots, n_r\}$. We shall prove that $[n, n+N] \bigcap A_\varepsilon \neq \emptyset$ for all $n \geq 1$. We have that for all $i, j = 1, \ldots, r$

$$\|U_T^{n+n_i} f - U_T^{n+n_j} f\| = \|U_T^n (U_T^{n_i} f - U_T^{n_j} f)\| = \|U_T^{n_i} f - U_T^{n_j} f\| \geq \varepsilon.$$

Since the set $\{f, U_T^{n+n_1} f, U_T^{n+n_2} f, \ldots, U_T^{n+n_r} f\}$ has $r+1$ elements, it cannot be ε-separated. Thus, we must have $\|U_T^{n+n_i} f - f\| < \varepsilon$ for some $i = 1, \ldots, r$. Hence, $n + n_i \in A_\varepsilon \cap [n, n + N]$. □