SNSB
Summer Term 2013
Ergodic Theory and Additive Combinatorics
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Seminar 1

(S1.1) Define an equivalence relation on $\mathbb{R}$ by

$$x \sim y \text{ if and only if } x - y \in \mathbb{Z}, \tag{2.15}$$

let $\mathbb{R}/\mathbb{Z}$ be the set of equivalence classes $[x]$, and $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the natural projection. Endow $\mathbb{R}/\mathbb{Z}$ with the quotient topology and for every $\alpha \in [0,1)$ define

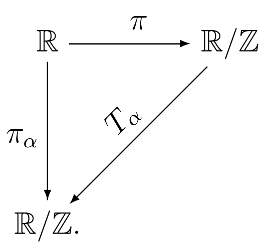
$$T_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad T_\alpha([x]) = [x+\alpha].$$

Prove that $(\mathbb{R}/\mathbb{Z}, T_\alpha)$ is a TDS isomorphic with $(\mathbb{S}^1, R_a)$, where $\alpha \in [0,1)$ and $a = e^{2\pi i\alpha}$.

Proof. It is easy to see that $\sim$ is indeed an equivalence relation. For every $\alpha \in [0,1)$, let us define

$$\pi_\alpha : \mathbb{R} \to \mathbb{R}/Z, \quad \pi_\alpha(x) = [x+\alpha].$$

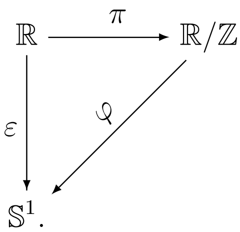
Then π_α is a quotient map, and for every $x, y \in \mathbb{R}$, $\pi_\alpha(x) = \pi_\alpha(y)$ if and only if $\pi(x) = \pi(y)$. Thus, we can apply B.8.5 to conclude that T_α is the unique homeomorphism making the following diagram commutative:



Let us consider the map

$$\varepsilon : \mathbb{R} \to \mathbb{S}^1, \quad \varepsilon(t) = e^{2\pi i t}.$$

Then ε is a quotient map, by B.12.4. Furthermore, for every $x, y \in \mathbb{R}$, $\pi(x) = \pi(y)$ if and only if $x - y \in \mathbb{Z}$ if and only if $\varepsilon(x) = \varepsilon(y)$. Thus, we can apply again B.8.5 to conclude that there exists a unique homemorphism $\varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ such that the following diagram is commutative:



Thus, $\varphi([x]) = \varepsilon(x) = e^{2\pi i x}$. Furthermore, since $\mathbb{S}^1$ is compact Hausdorff, it follows that $\mathbb{R}/\mathbb{Z}$ is also compact Hausdorff. Hence, $(\mathbb{R}/\mathbb{Z}, T_\alpha)$ is an invertible TDS.

Finally, for $a = e^{2\pi i\alpha}$,

$$\begin{aligned}(\varphi \circ T_\alpha)([x]) &= \varphi([x+\alpha]) = e^{2\pi i(x+\alpha)} = e^{2\pi i\alpha} \cdot e^{2\pi i x} = a \cdot \varphi([x]) \\ &= (R_a \circ \varphi)([x]).\end{aligned}$$

Hence, $\varphi : (\mathbb{R}/\mathbb{Z}, T_\alpha) \to (\mathbb{S}^1, R_a)$ is an isomorphism of TDSs. □

Let $\mathcal{F}$ be a collection of blocks over W, which we will think of as being the **forbidden blocks**. For any such $\mathcal{F}$, define $X_{\mathcal{F}}$ to be the set of sequences which do not contain any block in $\mathcal{F}$.

Definition 2.3.10. *A* **shift space** *(or simply* **shift***) is a subset X of a full shift $W^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks over W.*

Note that the empty space is a shift space, since putting $\mathcal{F} = W^{\mathbb{Z}}$ rules out every point. Furthermore, the full shift $W^{\mathbb{Z}}$ is a shift space; we can simply take $\mathcal{F} = \emptyset$, reflecting the fact that there are no constraints, so that $W^{\mathbb{Z}} = X_{\mathcal{F}}$.

The collection $\mathcal{F}$ may be finite or infinite. In any case it is at most countable since its elements can be arranged in a list (just write down its blocks of length 1 first, then those of length 2, and so on).

Definition 2.3.11. *Let X be a subset of the full shift $W^{\mathbb{Z}}$, and let $\mathcal{B}_n(X)$ denote the set of all n-blocks that occur in points of X. The* **language of** *X is the collection*

$$\mathcal{B}(X) = \bigcup_{n \geq 0} \mathcal{B}_n(X). \tag{2.16}$$

For a block $u \in \mathcal{B}(X)$, we say also that u **occurs in** X or x **appears in** X or x **is allowed in** X.

(S1.2) Let $X \subseteq W^{\mathbb{Z}}$ be a nonempty subset of $W^{\mathbb{Z}}$.

(i) $X \subseteq X_{\mathcal{B}(X)^c}$.

(ii) If X is a shift space, then $X = X_{\mathcal{B}(X)^c}$. Thus, the language of a shift space determines the shift space.

Proof. (i) Let $\mathbf{x} \in X$. If u is a block in $\mathcal{B}(X)^c$, then u does not occur in X; in particular, u does not occur in $\mathbf{x}$.

(ii) We have that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks. Let $\mathbf{x} \in X_{\mathcal{B}(X)^c}$. If u is a block in $\mathcal{F}$, then u does not occur in X, hence $u \in \mathcal{B}(X)^c$, so u does not occur in $\mathbf{x}$.

□

(S1.3) Let $X \subseteq W^{\mathbb{Z}}$ be a nonempty subset of $W^{\mathbb{Z}}$. The following are equivalent

(i) X is a shift space.

(ii) For every $\mathbf{x} \in W^{\mathbb{Z}}$, if $\mathbf{x}_{[i,j]} \in \mathcal{B}(X)$ for all $i \geq j \in \mathbb{Z}$, then $\mathbf{x} \in X$.

(iii) X is a closed strongly T-invariant subset of $W^{\mathbb{Z}}$.

Proof. $(i)(\Leftrightarrow)(ii)$ It is easy to see that (ii) is equivalent with $X_{\mathcal{B}(X)^c} \subseteq X$. Apply now S1.6.(ii).
$(ii)(\Rightarrow)(iii)$ Let $\mathbf{x} \in X$ and $\mathbf{y} := T^{-1}(\mathbf{x})$. For all $i \geq j \in \mathbb{Z}$,

$$(T\mathbf{x})_{[i,j]} = \mathbf{x}_{[i+1,j+1]} \in \mathcal{B}(X), \quad \mathbf{y}_{[i,j]} = \mathbf{x}_{[i-1,j-1]} \in \mathcal{B}(X).$$

Apply (ii) to conclude that $\mathbf{x}, \mathbf{y} \in X$. Thus, $T(X) = X$, so X is strongly T-invariant. An inspection of the proof the sequential compactness of the full shift $W^{\mathbb{Z}}$ (see Theorem 1.2.6), shows that in fact it holds for any subset X satisfying (ii).

We get that X is T-invariant and compact, hence closed, since $W^{\mathbb{Z}}$ is Hausdorff.
$(iii) \Rightarrow (ii)$ We prove the contrapositive of (ii). Assume that $\mathbf{x} \in W^{\mathbb{Z}} \setminus X$. Since $W^{\mathbb{Z}} \setminus X$ is open, there exists $k \geq 0$ such that $B_{2^{-k+1}}(\mathbf{x}) \subseteq W^{\mathbb{Z}} \setminus X$. Let $u := \mathbf{x}_{[-k,k]}$. If $u \in \mathcal{B}(X)$, then $u = \mathbf{y}_{[i,i+2k]}$ for some $\mathbf{y} \in X$ and $i \in \mathbb{Z}$. Let $l := i + k$. Since X is strongly T-invariant, we have that $T^l\mathbf{y} \in X$. On the other hand, $(T^l\mathbf{y})_{[-k,k]} = \mathbf{y}_{[i,i+2k]} = \mathbf{x}_{[-k,k]}$, so $T^l\mathbf{y} \in B_{2^{-k+1}}(\mathbf{x})$, hence $T^l\mathbf{y} \notin X$. We have got a contradiction. Thus, $\mathbf{x}_{[-k,k]} \notin \mathcal{B}(X)$.

□

(S1.4) Determine whether the following sets are shift spaces or not:

(i) X is the set of all binary sequences with no two 1's next to each other.

(ii) X is the set of all binary sequences so that between any two 1's there are an even number of 0's.

(iii) X is the set of points each of which contains exactly one symbol 1 and the rest 0's.

Proof. (i) X is a shift space: $X = X_{\mathcal{F}}$ with $\mathcal{F} = \{11\}$.

(ii) Take $\mathcal{F} = \{10^{2n+1}1 \mid n \geq 0\}$. Then $X = X_{\mathcal{F}}$, hence X is a shift space.

(iii) X is not a shift space. We have that $0^\infty \notin X$, while any block of 0's occurs in X, so (ii) from the above exercise is contradicted.

□