SNSB
Summer Term 2013
Ergodic Theory and Additive Combinatorics
Laurenţiu Leuştean

01.05.2013

Seminar 2

(S2.1) Let (X, T) be a TDS.

(i) Any strongly T-invariant set is also T-invariant.

(ii) The complement of a strongly T-invariant set is strongly T-invariant.

(iii) The closure of a T-invariant set is also T-invariant.

(iv) The union of any family of (strongly) T-invariant sets is (strongly) T-invariant.

(v) The intersection of any family of (strongly) T-invariant sets is (strongly) T-invariant.

(vi) If A is T-invariant, then $T^n(A) \subseteq A$ and $T^n(A)$ is T-invariant for all $n \geq 0$.

(vii) If A is strongly T-invariant, then $T^n(A) \subseteq A$ and $T^{-n}(A) = A$ for all $n \geq 0$; in particular, $T^{-n}(A)$ is strongly T-invariantfor all $n \geq 0$.

(viii) For any $x \in X$, the forward orbit $\mathcal{O}_+(x)$ of x is the smallest T-invariant set containing x and $\overline{\mathcal{O}}_+(x)$ is the smallest T-invariant closed set containing x.

Proof. (i) By A.0.13.(v).

(ii) If $T^{-1}(A) = A$, then $T^{-1}(X \setminus A) = X \setminus T^{-1}(A) = X \setminus A$.

(iii) If $T(A) \subseteq A$, then $T(\overline{A}) \subseteq \overline{T(A)} \subseteq \overline{A}$, by B.4.2.

(iv) Let $(A_i)_{i \in I}$ be a family of subsets of X. If $T(A_i) \subseteq A_i$ for all $i \in I$, then

$$T(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} T(A_i) \subseteq \bigcup_{i \in I} A_i.$$

If $T^{-1}(A_i) = A_i$ for all $i \in I$, then

$$T^{-1}(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} T^{-1}(A_i) = \bigcup_{i \in I} A_i.$$

(v) Let $(A_i)_{i\in I}$ be a family of subsets of X. If $T(A_i) \subseteq A_i$ for all $i \in I$, then

$$T(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} T(A_i) \subseteq \bigcap_{i\in I} A_i.$$

If $T^{-1}(A_i) = A_i$ for all $i \in I$, then

$$T^{-1}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} T^{-1}(A_i) = \bigcap_{i\in I} A_i.$$

(vi) By A.0.13.(i).

(vii) By (i), A is T-invariant, hence we can apply (vi) to conclude that $T^n(A) \subseteq A$ for all $n \geq 0$. Apply A.0.13.(vi) to obtain that $T^{-n}(A) = A$ for all $n \geq 0$.

(viii) By Lemma 1.0.3, We have that $T(\mathcal{O}_+(x)) = \mathcal{O}_{>0}(x) \subseteq \mathcal{O}_+(x)$, hence $\mathcal{O}_+(x)$ is T-invariant . If B is a T-invariant set containing x, then $T^n x \in T^n(B) \subseteq B$ for all $n \geq 1$. Thus, $\mathcal{O}_+(x) \subseteq B$.

By (iii), $\overline{\mathcal{O}}_+(x)$ is also T-invariant. Furthermore, if B is a closed T-invariant set containing x, then $\mathcal{O}_+(x) \subseteq B$ and, since B is closed, $\overline{\mathcal{O}}_+(x) \subseteq B$.

□

(S2.2) Let (X, T) be an invertible TDS.

(i) $A \subseteq X$ is strongly T-invariant if and only if $T(A) = A$ if and only if A is strongly T^{-1}-invariant.

(ii) The closure of a strongly T-invariant set is also strongly T-invariant.

(iii) If $A \subseteq X$ is strongly T-invariant, then $T^n(A) = A$ for all $n \in \mathbb{Z}$; in particular, $T^n(A)$ is strongly T-invariantfor all $n \in \mathbb{Z}$.

(iv) For any $x \in X$, the orbit $\mathcal{O}(x)$ of x is the smallest strongly T-invariant set containing x and $\overline{\mathcal{O}}(x)$ is the smallest strongly T-invariant closed set containing x.

(v) For any nonempty open set U of X, $\bigcup_{n\in\mathbb{Z}} T^n(U)$ is a nonempty open strongly T-invariant set and $X \setminus \bigcup_{n\in\mathbb{Z}} T^n(U)$ is a closed strongly T-invariant proper subset of X.

Proof. (i) Using the fact that T is a homeomorphism, we get that $A \subseteq X$ is strongly T-invariantif and only if $T^{-1}(A) = A$ if and only if $T(T^{-1}(A)) = T(A)$ if and only if $A = T(A)$.

(ii) Let A be strongly T-invariant. By (i) and B.4.6, we get that $T(\overline{A}) = \overline{T(A)} = \overline{A}$, hence $\overline{A}$ is also strongly T-invariant, by (i).

(iii) Apply (i) and A.0.14.(ii).

(iv)
$$T(\mathcal{O}(x)) = T\left(\bigcup_{n\in\mathbb{Z}} T^n x\right) = \bigcup_{n\in\mathbb{Z}} T^{n+1}x = \mathcal{O}(x),$$

so $\mathcal{O}(x)$ is strongly T-invariant. If B is a strongly T-invariant set containing x, then for all $n \in \mathbb{Z}$, $T^n x \in T^n(B) = B$, by (iii). Thus, $\mathcal{O}(x) \subseteq B$.

By (ii), $\overline{\mathcal{O}}(x)$ is also strongly T-invariant. Furthermore, if B is a closed strongly T-invariantset containing x, then $\mathcal{O}(x) \subseteq B$ and, since B is closed, $\overline{\mathcal{O}}(x) \subseteq B$.

(v) Let $A := \bigcup_{n\in\mathbb{Z}} T^n(U)$. Then A is open, since T^n is an open mapping for all $n \in \mathbb{Z}$, and A is nonempty, since $\emptyset \neq U = T^0(U) \subseteq A$. Furthermore,

$$T(A) \;=\; T\left(\bigcup_{n\in\mathbb{Z}} T^n(U)\right) = \bigcup_{n\in\mathbb{Z}} T^{n+1}(U) = A.$$

Finally, $X \setminus A \neq X$ is closed and strongly T-invariant, as a complement of an open strongly T-invariant set).

□

(S2.3) Let (X, T) be a TDS and $x \in X$. Then

(i) x is a forward transitive point if and only if $x \in \bigcup_{n\geq 0} T^{-n}(U)$ for every nonempty open subset U of X.

(ii) Assume that (X, T) is invertible. Then x is a transitive point if and only if $x \in \bigcup_{n\in\mathbb{Z}} T^n(U)$ for every nonempty open subset U of X.

Proof. (i) Applying B.1.5.(ii) and Lemma 1.0.3.(ii), we get that x is forward transitive if and only if $\mathcal{O}_+(x) \cap U \neq \emptyset$ for any nonempty open set U iff $x \in \bigcup_{n\geq 0} T^{-n}(U)$ for any nonempty open set U.

(ii) Similarly, using Lemma 1.0.3.(iii).

□

(S2.4) Let (X, T) be a TDS with X metrizable and $(U_n)_{n\geq 1}$ be a countable basis of X. Then

(i) $\{x \in X \mid \overline{\mathcal{O}}_+(x) = X\} = \bigcap_{n\geq 1} \bigcup_{k\geq 0} T^{-k}(U_n)$.

(ii) If (X, T) is invertible, then $\{x \in X \mid \overline{\mathcal{O}}(x) = X\} = \bigcap_{n\geq 1} \bigcup_{k\in\mathbb{Z}} T^k(U_n)$.

Proof. As the proof of the above lemma, using B.1.5.(iii). ☐

(S2.5) Let (X, T) be a TDS. The following are equivalent:

(i) If U is a nonempty open subset of X such that $T(U) = U$, then U is dense.

(ii) If $E \neq X$ is a proper closed subset of X such that $T(E) = E$, then E is nowhere dense.

Proof. Take $U := X \setminus E$. Then U is nonempty iff E is proper, U is open iff E is closed, U is dense in X iff E is nowhere dense, by B.1.5.(iv). Furthermore, since T is bijective, $T(U) = T(X \setminus E) = X \setminus T(E)$, hence, $T(U) = U$ iff $T(E) = E$. ☐