SNSB
Summer Term 2013
Ergodic Theory and Additive Combinatorics
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22.05.2013

Seminar 4

(S4.1) Consider the full shift $W^{\mathbb{Z}}$. The following are equivalent:

(i) $\mathbf{x} \in W^{\mathbb{Z}}$ is recurrent.

(ii) Every nonempty block of $\mathbf{x}$ occurs a second time.

(iii) Every nonempty block of $\mathbf{x}$ occurs infinitely often.

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathbf{x}$ is recurrent, and let $u := \mathbf{x}_{[i,j]}$ be a nonempty block of $\mathbf{x}$. Take $k := \max\{|i|, |j|\}$, so that $\mathbf{x}_{[i,j]}$ is a subblock of $\mathbf{x}_{[-k,k]}$. Apply the fact that $\mathbf{x}$ is recurrent to get $n \geq 1$ such that $T^n\mathbf{x} \in B_{2^{-k+1}}(\mathbf{x})$, that is $\mathbf{x}_{[-k,k]} = \mathbf{x}_{[n-k,n+k]}$. It follows that $\mathbf{x}_{[i,j]} = \mathbf{x}_{[n+i,n+j]}$ and $n + i > i$.
$(ii) \Rightarrow (i)$ It is enough to prove that for all $k \geq 0$ there exists $n \geq 1$ such that $T^n x \in B_{2^{-k+1}}(\mathbf{x})$, i.e. $\mathbf{x}_{[-k,k]} = \mathbf{x}_{[n-k,n+k]}$. Apply (ii) for the central block $\mathbf{x}_{[-k,k]}$.
$(iii) \Rightarrow (ii)$ Obviously.
$(ii) \Rightarrow (iii)$ Apply (ii) repeatedly. □

(S4.2) Let $x \in X$ and $\mathbf{x} = (x, \ldots, x) \in X^l_\Delta$ (see the notations from Section 1.7). The following are equivalent:

(i) x is multiply recurrent for $T_1, \ldots, T_l$.

(ii) $\mathbf{x}$ is a recurrent point in $(X^l, \tilde{T})$.

(iii) For all $\varepsilon > 0$ there exists $N \geq 1$ such that $d(\mathbf{x}, \tilde{T}^N\mathbf{x}) < \varepsilon$.

(iv) For all $\varepsilon > 0$ there exists $N \geq 1$ such that $d(x, T_i^N x) < \varepsilon$ for all $i = 1, \ldots, l$.

Proof. $(i) \Leftrightarrow (ii)$ Apply Lemma 1.6.8 and the fact that, by the definition of the metric d_l, we have that $\lim\limits_{k\to\infty} \tilde{T}^{n_k}\mathbf{x} = \mathbf{x}$ if and only if for all $i = 1, \ldots, k$, $\lim\limits_{k\to\infty} T_i^{n_k} x = x$.
$(ii) \Leftrightarrow (iii)$ By definition, $\mathbf{x}$ is recurrent if and only for every open neighborhood U of $\mathbf{x}$ there exists $N \geq 1$ such that $\tilde{T}^N\mathbf{x} \in U$ if and only for every $\varepsilon > 0$ there exists $N \geq 1$ such that $\tilde{T}^N\mathbf{x} \in B(\mathbf{x}, \varepsilon)$.
$(iii) \Leftrightarrow (iv)$ Obviously. □

(S4.3) Let X be a compact Hausdorff topological space, $l \geq 1$, and $T_1, \ldots, T_l : X \to X$ be commuting homeomorphisms. Then

(i) X contains a subset X_0 which is minimal with the property that it is nonempty closed and strongly T_i-invariant for all $i = 1, \ldots, l$.

(ii) For every nonempty open subset U of X_0, there are $M \geq 1$ and $n_{ij} \in \mathbb{Z}, i = 1, \ldots, l,\ j = 1, \ldots, M$ such that $X_0 = \bigcup_{j=1}^{M} \left(T_1^{n_{1j}} \circ \ldots \circ T_l^{n_{lj}}\right)(U)$.

(iii) $(X_0)^l_\Delta$ is strongly $\tilde{T}_i$-invariant for all $i = 1, \ldots, l$.

Proof. (i) Let $\mathcal{M}$ be the family of all nonempty closed subsets of X that are strongly T_i-invariant for all$i = 1, \ldots, l$, with the partial ordering by inclusion. Then, of course, $X \in \mathcal{M}$, so $\mathcal{M}$ is non-empty. Let $(A_i)_{i \in I}$ be a chain in $\mathcal{M}$ and take $A := \bigcap_{i \in I} A_i$. Then $A \in \mathcal{M}$, since A is nonempty (by B.10.4), A is closed, and A is strongly T_i-invariant for all $i = 1, \ldots, l$ (by Proposition 1.3.4.(v)). Thus, by Zorn's Lemma A.0.12 there exists a minimal element $X_0 \in \mathcal{M}$.

(ii) Let $A := \bigcup_{n_1 \in \mathbb{Z}} \ldots \bigcup_{n_l \in \mathbb{Z}} \left(T_1^{n_1} \circ \ldots \circ T_l^{n_l}\right)(U)$. Then A is nonempty, open and strongly T_i-invariant for all $i = 1, \ldots, l$. Thus, $X_0 \setminus A$ is a proper subset of X_0 which is closed and strongly T_i-invariant for all $i = 1, \ldots, l$. From the minimality of X_0, we must have $X_0 = A$. Since X_0 is compact, as a closed subset of the compact space X, we can choose a finite subcover.

(iii) We have that $\tilde{T}_i((X_0)^l_\Delta) = (T_i(X_0))^l_\Delta = (X_0)^l_\Delta$.

□