SNSB
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Ergodic Theory and Additive Combinatorics
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Seminar 5

(S5.1) Let us consider the following statements

(vdW1) Let $r \in \mathbb{Z}_+$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. For any $k \geq 1$ there exists $i \in [1, r]$ such that C_i contains an arithmetic progression of length k.

(vdW2) Let $r \in \mathbb{Z}_+$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. There exists $i \in [1, r]$ such that C_i contains arithmetic progression of arbitrary finite length.

(vdW3) Let $r \in \mathbb{Z}_+$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. For any finite set $F \subseteq \mathbb{N}$ there exists $i \in [1, r]$ such that C_i contains affine images of F.

(vdW4) Let $r \in \mathbb{Z}_+$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. There exists $i \in [1, r]$ such that C_i contains affine images of every finite set $F \subseteq \mathbb{N}$.

Let (**vdWi**$*$), $i = 1, 2, 3, 4$ be the statements obtained from (**vdWi**), $i = 1, 2, 3, 4$ by changing $\mathbb{N}$ to $\mathbb{Z}$ in their formulations.

Prove that (**vdWi**), (**vdWi**$*$), $i = 1, 2, 3, 4$ are all equivalent.

Proof. (**vdW2**) $\Rightarrow$ (**vdW1**), (**vdW4**) $\Rightarrow$ (**vdW3**) are obvious.
(**vdW1**) $\Rightarrow$ (**vdW2**) By **(vdW1)** we know that for every $k \in \mathbb{N}$ there exists $i \in [1, r]$ such that C_i contains an arithmetic progression of length k. Since $[1, r]$ is finite, it follows that one of C_i's will occur for infinitely many k. That is, there exists $i \in [1, r]$ such that C_i contains arithmetic progressions of length k for every $k \in K \subseteq \mathbb{N}$, where K is infinite.

It follows easily that this C_i is the desired one. For $l \geq 1$, there exists $k \in K$ such that $l \leq k$, since K is infinite. We get that C_i contains an arithmetic progression $\{a, a + d, \ldots a + (k-1)d\}$ of length k. Since $l \leq k$,

$$\{a, a + d, \ldots a + (l-1)d\} \subseteq \{a, a + d, \ldots a + (k-1)d\} \subseteq C_i,$$

hence C_i contains an arithmetic progression of length l.
(**vdW2**) $\Rightarrow$ (**vdW4**) Let i be as in **(vdW2)**. If $F \subseteq \mathbb{N}$ is a finite set, then $F \subseteq \{0, \ldots, k-$

1} for some $k \geq 1$. By **(vdW2)**, C_i contains an arithmetic progression of length k. so there are $a, d \in \mathbb{N}$ such that $\{a, a+d, \dots a+(k-1)d\} \subseteq C_i$. It follows that C_i contains the affine image $a + dF$ of F.

(**vdW3**) $\Rightarrow$ (**vdW1**) is immediate since any arithmetic progression $\{a, a+d, \dots a+(k-1)d\}$ of length k is an affine image of the set $F = \{0, \dots, k-1\}$.

(**vdW1**) $\Rightarrow$ (**vdW1**$*$) Let $r, k \in \mathbb{Z}_+$ and $\mathbb{Z} = \bigcup_{i=1}^{r} C_i$. Then $\mathbb{N} = \bigcup_{i=1}^{r} (C_i \cap \mathbb{N})$, and by taking the nonempty $C_i \cap \mathbb{N}$'s, we get a finite partition of $\mathbb{N}$. Apply **(vdW1)** to get i such that $C_i \cap \mathbb{N}$, hence C_i, contains an arithmetic progression of length k.

(**vdW1**$*$) $\Rightarrow$ (**vdW1**) Let $r \in \mathbb{Z}_+$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. By taking $D_i := C_i \cup (-C_i)$, we get a partition $\mathbb{Z} = \bigcup_{i=1}^{r} D_i$. By **(vdW1*)**, there exists $i \in [1, r]$ with the property that D_i contains an arithmetic progression of length $2k - 1$. Hence, either C_i or $-C_i$ contains an arithmetic progression of length k. Remark now that $\{a, a+d, \dots, a+(k-1)d\} \subseteq -C_i$ iff $\{-a, -a-d, \dots, -a+(k-1)(-d)\} \subseteq C_i$.

The remaining implications follow similarly. □

(S5.2) Let us consider the following statement

(*) Let (X, T) be a TDS and $(U_i)_{i \in I}$ be an open cover of X. Then there exists an open set U_{i_0} in this cover such that $U_{i_0} \cap T^{-n}(U_{i_0}) \neq \emptyset$ for infinitely many n.

(i) Prove **(*)** in two ways:

(a) applying Birkhoff Recurrence Theorem.

(b) using the Infinite Pigeonhole Principle (IPP): Whenever $\mathbb{N}$ is coloured into finitely many colours, one of the colour classes is infinite.

(ii) Deduce IPP from **(*)**.

Proof. (i) For every $i \in I$, let $C_i = \{n \geq 1 \mid U_i \cap T^{-n}(U_i) \neq \emptyset\}$.

(a) Apply Birkhoff Recurrence Theorem 1.6.10 to get an almost periodic point $x \in X$. Since $X = \bigcup_{i \in I} U_i$, we have that $x \in U_{i_0}$ for some $i_0 \in I$. If $n \in rt(x, U_{i_0})$, then $x \in U_{i_0} \cap T^{-n}(U_{i_0})$, hence $n \in C_{i_0}$. Thus, $rt(x, U_0) \subseteq C_{i_0}$. Since $rt(x, U_{i_0})$ is syndetic, hence infinite, we conclude that C_{i_0} is infinite too.

(b) As X is compact, there is a finite subcover $X = \bigcup_{k=1}^{r} U_{i_k}$ of X. Let $x \in X$ be arbitrary and define $D_k := \{n \geq 0 \mid T^n x \in U_{i_k}\}$ for all $k = 1, \dots, r$. Then $\mathbb{N} = \bigcup_{k=1}^{r} D_k$, so we can apply (IPP) to get the existence of K such that D_K is infinite. Let $N := \min D_K$ and $y := T^N x$. We get that $y \in U_{i_K}$ and for all $n \in D_K \setminus \{N\}$, we have that $n - N \geq 1$, and $T^{n-N} y = T^n x \in U_{i_K}$. Hence, $n \in D_K \setminus \{N\}$ implies $n - N \in C_{i_K}$, so C_{i_K} is infinite.

(ii) Let $r \geq 1$ and let $\mathbb{N} = \bigcup_{i=1}^{r} D_i$ be a finite partition of $\mathbb{N}$. Set $W = \{1, 2, \ldots, r\}$ and consider the full shift $(W^{\mathbb{Z}}, T)$. Let $\gamma \in W^Z$ be defined by:

$$\gamma_n = \begin{cases} i & \text{if } n \geq 0 \text{ and } n \in D_i \\ \text{arbitrarily} & \text{if } n < 0. \end{cases}$$

Let $X := \overline{\{T^n \gamma \mid n \geq 0\}}$ be the orbit closure of γ and consider the subsystem (X, T_X). Consider the elementary cylinders $C_0^i,\ i \in W$. Then $W^{\mathbb{Z}} = \bigcup_{i \in W} C_0^i$, so we get an open cover $X = \bigcup_{i \in W} (C_0^i \cap X)$ of X. Apply now **(*)** to get $i_0 \in W$ such that

$$A = \{n \geq 1 \mid C_0^{i_0} \cap X \cap T^{-n}(C_0^{i_0} \cap X) \neq \emptyset\}$$

is infinite.

For every $n \in A$, there exists $\mathbf{x} \in X$ such that $x_0 = i_0$ and $x_n = (T^n \mathbf{x})_0 = i_0$. Let $k = n + 1$. Since $\mathbf{x} \in X$, there exists $M_n \in \mathbb{N}$ such that

$$d(\mathbf{x}, T^{M_n} \gamma) < 2^{-k}, \quad \text{hence, } \mathbf{x}_{[-n,n]} = (T^{M_n} \gamma)_{[-n,n]}.$$

As a consequence, $\gamma_{M_n} = (T^{M_n} \gamma)_0 = x_0 = i_0$, and $\gamma_{M_n + n} = (T^{M_n} \gamma)_n = x_n = i_0$. Thus,

$$B := \{M_n \mid n \in A\} \cup \{M_n + n \mid n \in A\} \subseteq D_{i_0}.$$

If $\{M_n \mid n \in A\}$ is infinite, then B is infinite. If $\{M_n \mid n \in A\}$ is finite, then there exists $N \in A$ such that for all $p \in A, p \geq N$, we have that $M_p = M_N$. It follows that the set $\{M_p + p \mid p \in A, p \geq N\} = M_N + \{p \in A \mid p \geq N\} = M_N + (A \setminus [0, N-1])$ is infinite. We get again that B is infinite.

It follows that D_{i_0} is infinite too.

$\square$