

Seminar 6

(S6.1) Verify that Hilbert theorem 1.6.13 is a special case of the Finite Sums theorem.

Proof. Let $r \in \mathbb{Z}_+$, $\mathbb{Z}_+ = \bigcup_{i=1}^r C_i$, and $l \geq 1$. Apply the Finite Sums theorem 2.4.20 to get $i \in [1, r]$ and a sequence $(x_n)_{n \geq 1}$ in \mathbb{Z}_+ such that $FS((x_n)_{n \geq 1}) \subseteq C_i$. Take n_1, n_2, \dots, n_l to be x_1, \dots, x_l in increasing order, and $B := \{x_{l+1} + \dots + x_{l+p} \mid p \in \mathbb{Z}_+\}$. \square

(S6.2) Let X be a Hausdorff topological space and $(x_n)_{n \geq 1}$ be a sequence in X .

(i) For every $p \in \beta\mathbb{Z}_+$, the following are satisfied:

- (a) The p -limit of (x_n) , if exists, is unique.
- (b) If X is compact, then p - $\lim x_n$ exists.
- (c) If $f : X \rightarrow Y$ is continuous and p - $\lim x_n = x$, then p - $\lim f(x_n) = f(x)$.

(ii) $\lim_{n \rightarrow \infty} x_n = x$ implies p - $\lim x_n = x$ for every non-principal ultrafilter p .

Proof. (i) Let $p \in \beta\mathbb{Z}_+$.

- (a) Assume that $x \neq y \in X$ are such that p - $\lim x_n = x$ and p - $\lim x_n = y$. Since X is Hausdorff, there are disjoint open neighborhoods U of x and V of y . Then $A_U := \{n \in \mathbb{Z}_+ \mid x_n \in U\} \in p$ and $A_V := \{n \in \mathbb{Z}_+ \mid x_n \in V\} \in p$, hence $A_U \cap A_V \in p$. On the other hand, $A_U \cap A_V = \emptyset$. We have got a contradiction with the fact that p is a filter.
- (b) Assume by contradiction that there exist a sequence (x_n) and an ultrafilter p such that p - $\lim x_n$ does not exist. Then for each $x \in X$ there exists an open neighborhood U_x of x such that $A_x := \{n \in \mathbb{Z}_+ \mid x_n \in U_x\} \notin p$. It follows that

the family $(U_x)_{x \in X}$ is an open cover of X , so from compactness, we get a finite subset $F \subseteq X$ with $X = \bigcup_{x \in F} U_x$. We get that

$$\bigcup_{x \in F} A_x = \mathbb{Z}_+ \in p,$$

hence we must have $A_x \in p$ for some $x \in F$. We have got a contradiction.

(c) Let $x = p\text{-}\lim x_n$ and let V be an open neighborhood of $f(x)$. Since f is continuous, we have that $f(U) \subseteq V$ for some open neighborhood U of x . Let $A := \{n \in \mathbb{Z}_+ \mid x_n \in U\}$ and $B := \{n \in \mathbb{Z}_+ \mid f(x_n) \in V\}$. Then $A \in p$ and $A \subseteq B$, hence $B \in p$.

(ii) Assume that $\lim_{n \rightarrow \infty} x_n = x$ and let p be a nonprincipal ultrafilter on \mathbb{Z}_+ . If U is an open neighborhood of x , then we get $N \geq 1$ such that $x_n \in U$ for all $n > N$. Let $n = 1, \dots, N$. Since $p \neq e(n)$, there exists $A_n \in p$ such that $n \notin A_n$. It follows that $A_1 \cap \dots \cap A_N \subseteq \mathbb{Z}_+ \setminus [1, N] = \{n \in \mathbb{Z}_+ \mid n > N\} \subseteq \{n \in \mathbb{Z}_+ \mid x_n \in U\}$. Since $A_1 \cap \dots \cap A_N \in p$ and p is a filter, we conclude that $\{n \in \mathbb{Z}_+ \mid x_n \in U\} \in p$. Thus, $p\text{-}\lim x_n = x$. □

(S6.3) Let $(x_n)_{n \geq 1}, (y_n)_{n \geq 1}$ be bounded sequences in \mathbb{R} , and p be a non-principal ultrafilter on \mathbb{Z}_+ .

(i) (x_n) has a unique p -limit. If $a \leq x_n \leq b$, then $a \leq p\text{-}\lim x_n \leq b$.

(ii) For any $c \in \mathbb{R}$, $p\text{-}\lim cx_n = c \cdot p\text{-}\lim x_n$.

(iii) $p\text{-}\lim(x_n + y_n) = p\text{-}\lim x_n + p\text{-}\lim y_n$.

Proof. (i) Apply Proposition 2.4.26.(ib) for the compact space $X = [a, b]$.

(ii) Apply Proposition 2.4.26.(ic) for the continuous mapping $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = cx$.

(iii) Let $x = p\text{-}\lim x_n$ and $y = p\text{-}\lim y_n$. For any $\varepsilon > 0$, $A_\varepsilon := \{n \in \mathbb{Z}_+ \mid x_n \in (x - \varepsilon/2, x + \varepsilon/2)\} \in p$ and $B_\varepsilon := \{n \in \mathbb{Z}_+ \mid y_n \in (y - \varepsilon/2, y + \varepsilon/2)\} \in p$, hence $A_\varepsilon \cap B_\varepsilon \in p$. On the other hand

$$A_\varepsilon \cap B_\varepsilon \subseteq C_\varepsilon := \{n \in \mathbb{Z}_+ \mid x_n + y_n \in (x + y - \varepsilon, x + y + \varepsilon)\},$$

so $C_\varepsilon \in p$. Hence, $p\text{-}\lim(x_n + y_n) = x + y$. □

(S6.4) Let D be set and let \mathcal{A} be a subset of $\mathcal{P}(D)$ which has the finite intersection property. Then there is an ultrafilter p on D such that $\mathcal{A} \subseteq p$.

Proof. See [56, Theorem 3.8, p.50]. □

(S6.5) Let $\mathcal{A} = \{A \subseteq \mathbb{Z}_+ \mid \mathbb{Z}_+ \setminus A \text{ is finite}\}$. Prove that there exists a non-principal ultrafilter \mathcal{U} on \mathcal{D} such that $\mathcal{A} \subseteq \mathcal{U}$.

Proof. Clearly \mathcal{A} has the finite intersection property. Apply Proposition 2.4.5 to get an ultrafilter \mathcal{U} on \mathcal{D} such that $\mathcal{A} \subseteq \mathcal{U}$. By Proposition 2.4.4.(iii), \mathcal{U} is non-principal. □

(S6.6) Let D be set, let \mathcal{F} be a filter on D , and let $A \subseteq D$. Then $A \notin \mathcal{F}$ if and only if there is some ultrafilter \mathcal{U} with $\mathcal{F} \cup \{D \setminus A\} \subseteq \mathcal{U}$.

Proof. See [56, Corollary 3.9, p.50]. □

(S6.7) Let D be a set and let $\mathcal{G} \subseteq \mathcal{P}(D)$. The following are equivalent.

- (i) Whenever $r \geq 1$ and $D = \bigcup_{i=1}^r C_i$, there exists $i \in [1, r]$ and $G \in \mathcal{G}$ such that $G \subseteq C_i$.
- (ii) There is an ultrafilter \mathcal{U} on d such that for every member A of \mathcal{U} , there exists $G \in \mathcal{G}$ with $G \subseteq A$.

Proof. See [56, Theorem 5.7, p.92]. □

(S6.8) Let $\mathcal{U} \subseteq \mathcal{P}(D)$. The following are equivalent:

- (i) \mathcal{U} is an ultrafilter on D .
- (ii) \mathcal{U} has the finite intersection property and for each $A \in \mathcal{P}(D) \setminus \mathcal{U}$ there is some $B \in \mathcal{U}$ such that $A \cap B = \emptyset$.
- (iii) \mathcal{U} is maximal with respect to the finite intersection property. (That is, \mathcal{U} is a maximal member of $\{\mathcal{V} \subseteq \mathcal{P}(D) \mid \mathcal{V} \text{ has the finite intersection property}\}$.)
- (iv) \mathcal{U} is a filter on D and for any collection C_1, \dots, C_n of subsets of D , if $\bigcup_{i=1}^n C_i \in \mathcal{U}$, then $C_j \in \mathcal{U}$ for some $j = 1, \dots, n$.
- (v) \mathcal{U} is a filter on D and for all $A \subseteq D$ either $A \in \mathcal{U}$ or $D \setminus A \in \mathcal{U}$.

Proof. See [56, Theorem 3.6, p.49]. □