SNSB
Summer Term 2013
Ergodic Theory and Additive Combinatorics
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Seminar 6

(S6.1) Verify that Hilbert theorem 1.6.13 is a special case of the Finite Sums theorem.

(S6.2) Let X be a Hausdorff topological space and $(x_n)_{n\geq 1}$ be a sequence in X.

(i) For every $p \in \beta\mathbb{Z}_+$, the following are satisfied:

(a) The p-limit of (x_n), if exists, is unique.

(b) If X is compact, then $p-\lim x_n$ exists.

(c) If $f : X \to Y$ is continuous and $p-\lim x_n = x$, then $p-\lim f(x_n) = f(x)$.

(ii) $\lim_{n\to\infty} x_n = x$ implies $p-\lim x_n = x$ for every non-principal ultrafilter p.

(S6.3) Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}$ be bounded sequences in $\mathbb{R}$, and p be a non-principal ultrafilter on $\mathbb{Z}_+$.

(i) (x_n) has a unique p-limit. If $a \leq x_n \leq b$, then $a \leq p-\lim x_n \leq b$.

(ii) For any $c \in \mathbb{R}$, $p-\lim cx_n = c \cdot p-\lim x_n$.

(iii) $p-\lim(x_n + y_n) = p-\lim x_n + p-\lim y_n$.

(S6.4) Let D be set and let $\mathcal{A}$ be a subset of $\mathcal{P}(D)$ which has the finite intersection property. Then there is an ultrafilter p on D such that $\mathcal{A} \subseteq p$.

(S6.5) Let $\mathcal{A} = \{A \subseteq \mathbb{Z}_+ \mid \mathbb{Z}_+ \setminus A \text{ is finite}\}$. Prove that there exists a non-principal ultrafilter $\mathcal{U}$ on $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{U}$.

(S6.6) Let D be set, let $\mathcal{F}$ be a filter on D, and let $A \subseteq D$. Then $A \notin \mathcal{F}$ if and only if there is some ultrafilter $\mathcal{U}$ with $\mathcal{F} \cup \{D \setminus A\} \subseteq \mathcal{U}$.

(S6.7) Let D be a set and let $\mathcal{G} \subseteq \mathcal{P}(D)$. The following are equivalent.

(i) Whenever $r \geq 1$ and $D = \bigcup_{i=1}^{r} C_i$, there exists $i \in [1, r]$ and $G \in \mathcal{G}$ such that $G \subseteq C_i$.

(ii) There is an ultrafilter $\mathcal{U}$ on d such that for every member A of $\mathcal{U}$, there exists $G \in \mathcal{G}$ with $G \subseteq A$.

(S6.8) Let $\mathcal{U} \subseteq \mathcal{P}(D)$. The following are equivalent:

(i) $\mathcal{U}$ is an ultrafilter on D.

(ii) $\mathcal{U}$ has the finite intersection property and for each $A \in \mathcal{P}(D) \setminus \mathcal{U}$ there is some $B \in \mathcal{U}$ such that $A \cap B = \emptyset$.

(iii) $\mathcal{U}$ is maximal with respect to the finite intersection property. (That is, $\mathcal{U}$ is a maximal member of $\{\mathcal{V} \subseteq \mathcal{P}(D) \mid \mathcal{V}$ has the finite intersection property$\}$.)

(iv) $\mathcal{U}$ is a filter on D and for any collection $C_1, \ldots, C_n$ of subsets of D, if $\bigcup_{i=1}^{n} C_i \in \mathcal{U}$, then $C_j \in \mathcal{U}$ for some $j = 1, \ldots n$.

(v) $\mathcal{U}$ is a filter on D and for all $A \subseteq D$ either $A \in \mathcal{U}$ or $D \setminus A \in \mathcal{U}$.