SNSB
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Ergodic Theory and Additive Combinatorics
Laurenţiu Leuştean

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Seminar 7

(S7.1)

(i) $1_X : X \to X$, the identity on $(X, \mathcal{B}, \mu)$, is an invertible measure-preserving transformation.

(ii) The composition of two measure-preserving transformations is a measure-preserving transformation.

(iii) If $(X, \mathcal{B}, \mu, T)$ is a MPS, then $\mu(T^{-n}(A)) = \mu(A)$ for all $A \in \mathcal{B}$ and all $n \geq 1$.

(iv) If $(X, \mathcal{B}, \mu, T)$ is invertible, then $\mu(T^n(A)) = \mu(A)$ for all $A \in \mathcal{B}$ and all $n \in \mathbb{Z}$.

(S7.2) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be probability spaces and $T : X \to Y$ be bijective such that both T and T^{-1} are measurable. The following are equivalent

(i) T is measure-preserving.

(ii) $\mu(B) = \nu(T(B)))$ for all $B \in \mathcal{B}$.

(iii) T^{-1} is measure-preserving.

(S7.3) Let $(X, \mathcal{B}), (Y, \mathcal{C}), (Z, \mathcal{D})$ be measurable spaces, $T : X \to Y, S : Y \to Z$ be measurable transformations.

(i) $U_{S \circ T} = U_T \circ U_S$.

(ii) U_T is linear and $U_T(f \cdot g) = (U_T f) \cdot (U_T g)$ for all $f, g \in \mathcal{M}_{\mathbb{C}}(Y, \mathcal{C})$.

(iii) If $f : Y \to \mathbb{C}, f(y) = c$ is a constant function, then $U_T(f)(x) = c$ for every $x \in X$.

(iv) $U_T(\mathcal{M}_{\mathbb{R}}(Y, \mathcal{C})) \subseteq \mathcal{M}_{\mathbb{R}}(X, \mathcal{B})$.

(v) If $f \in \mathcal{M}_{\mathbb{R}}(Y, \mathcal{C})$ is nonnegative, then $U_T f$ is nonnegative too, hence U_T is a positive operator.

(vi) For all $C \in \mathcal{C}$, $U_T(\chi_C) = \chi_{T^{-1}(C)}$.

(vii) If f is a simple function in $\mathcal{M}_{\mathbb{C}}(Y, \mathcal{C})$, $f = \sum_{i=1}^{n} c_i \chi_{C_i}$, $c_i \in \mathbb{C}, C_i \in \mathcal{C}$, then $U_T f$ is a simple function in $\mathcal{M}_{\mathbb{C}}(X, \mathcal{B})$, $U_T f = \sum_{i=1}^{n} c_i \chi_{T^{-1}(C_i)}$.

(S7.4) Let $(X, \mathcal{B})$ be a measurable space and $T : X \to X$ be measurable.

(i) $U_{1_X} = 1_{\mathcal{M}_{\mathbb{C}}(X,\mathcal{B})}$

(ii) $U_{T^n} = (U_T)^n$ for all $n \in \mathbb{N}$.

(iii) If $T : X \to X$ is bijective and both T and T^{-1} are measurable, then U_T is invertible and its inverse is $U_{T^{-1}}$. Furthermore, $U_{T^n} = (U_T)^n$ for all $n \in \mathbb{Z}$.

(S7.5) Let (X, d) be a compact metric space and $T : X \to X$ be a continuous mapping. For all $l \geq 1$, there exists a multiply recurrent point for $T, T^2, \ldots, T^l$.

(S7.6) For any $A \in \mathcal{B}$, let us recall that

$$\limsup_{n\to\infty} T^{-n}(A) = \bigcap_{n\geq 1} \bigcup_{i\geq n} T^{-i}(A).$$

Then

(i) $\limsup_{n\to\infty} T^{-n}(A)$ is T-invariant.

(ii) $\mu(A \Delta \limsup_{n\to\infty} T^{-n}(A)) \leq \sum_{k=1}^{\infty} k\mu(A\Delta T^{-1}(A))$. In particular, $\mu(A\Delta T^{-1}(A)) = 0$ implies $\mu(A\Delta \limsup_{n\to\infty} T^{-n}(A)) = 0$.