SNSB
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Ergodic Theory and Additive Combinatorics
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Seminar 8

(S8.1)

(i) $\mathcal{S} = \mathcal{C} \cup \{\emptyset\}$ is a semialgebra on $W^{\mathbb{Z}}$.

(ii) $\mathcal{B} = \sigma(\mathcal{S}) = \sigma(\mathcal{C}_e)$.

(iii) $\mathcal{B}$ coincides with the Borel σ-algebra on $W^{\mathbb{Z}}$.

Proof. (i) We have that $\emptyset \in \mathcal{S}$ and that $\mathcal{S}$ is closed under finite intersections as an immediate consequence of Lemma 1.2.8.(ii). Furthermore,

$$W^{\mathbb{Z}} \setminus C^{w_{i_1},\ldots,w_{i_t}}_{n_1,\ldots,n_t} = \bigcup_{u_1 \neq w_{i_1}} C^{u_1}_{n_1} \cup \bigcup_{u_2 \neq w_{i_2}} C^{w_{i_1},u_2}_{n_1,n_2} \cup \ldots \cup \bigcup_{u_t \neq w_{i_t}} C^{w_{i_1},\ldots,w_{i_{t-1}},u_t}_{n_1,\ldots,n_{t-1},n_t}$$

is a finite union of pairwise disjoint cylinders.

(ii) $\mathcal{B}$ is the σ-algebra generated by the set $\mathcal{R}$ of measurable rectangles. By (3.9), we have that $\mathcal{C}_e \subseteq \mathcal{R} \subseteq \mathcal{A}(\mathcal{C}_e)$, hence $\sigma(\mathcal{C}_e) \subseteq \mathcal{B} = \sigma(\mathcal{R}) \subseteq \sigma(\mathcal{A}(\mathcal{C}_e)) = \sigma(\mathcal{C}_e)$. Thus, $\mathcal{B} = \sigma(\mathcal{C}_e)$. Since $\mathcal{C}_e \subseteq \mathcal{S} \subseteq \mathcal{R}$, we also get that $\sigma(\mathcal{S}) = \mathcal{B}$.

(iii) Let $\mathcal{B}(W^{\mathbb{Z}})$ be the Borel σ-algebra on $W^{\mathbb{Z}}$. We have to prove that $\mathcal{B} = \mathcal{B}(W^{\mathbb{Z}})$.
"$\subseteq$" follows from the fact that the elementary cylinders are open sets in $W^{\mathbb{Z}}$.
"$\supseteq$" The set $\mathcal{C}$ of cylinders is countable, since W is finite. Since $\mathcal{C}$ is a basis for the product topology on $W^{\mathbb{Z}}$, any open set U of $W^{\mathbb{Z}}$ is a union of sets in $\mathcal{C}$, hence U is an at most countable union of sets in $\mathcal{C}$. Thus, any open set is in $\sigma(\mathcal{C}) = \sigma(\mathcal{S}) = \mathcal{B}$. $\square$

(S8.2) Let $A \in \mathcal{B}$.

(i) $A \setminus A_{ret}$ is wandering.

(ii) $A \setminus A_{inf} = A \cap \bigcup_{n \geq 0} T^{-n}(A \setminus A_{ret})$.

Proof. (i) Remark that for every $n \geq 0$, $T^{-n}(A \setminus A_{ret})$ consists of all points which are in A at moment n, but then leave A for ever.

(ii)

$$\begin{aligned} A \setminus A_{inf} &= A \setminus \bigcap_{n\geq 1} T^{-n}(A^\star) = A \cap \left(X \setminus \bigcap_{n\geq 1} T^{-n}(A^\star) \right) = A \cap \bigcup_{n\geq 1} \left(X \setminus T^{-n}(A^\star)\right) \\ &= A \cap \Big[(X \setminus T^{-1}(A^\star)) \cup \ldots \cup (X \setminus T^{-n}(A^\star)) \cup \ldots \Big] = A \cap \bigcup_{n\geq 1}^{\infty} C_n, \end{aligned}$$

where $C_n := X \setminus T^{-n}(A^\star)$ for all $n \geq 1$. Remark that (C_n) is an increasing sequence, since $T^{-1}(A^\star) = A^+ \subseteq A^\star$. By defining $D_0 := C_1$ and $D_n := C_{n+1} - C_n$ for all $n \geq 1$, we get that $D_0, D_1, \ldots$ are disjoint and $\bigcup_{n\geq 1} C_n = \bigcup_{n\geq 0} D_n$. Using moreover that for all $n \geq 2$,

$$D_n = C_{n+1} - C_n = (X \setminus T^{-n-1}(A^\star)) \setminus (X \setminus T^{-n}(A^\star)) = T^{-n}(A^\star) \setminus T^{-n-1}(A^\star)$$

we get that

$$\begin{aligned} A \setminus A_{inf} &= A \cap \left((X \setminus T^{-1}(A^\star)) \cup \bigcup_{n\geq 1} \left(T^{-n}(A^\star) \setminus T^{-n-1}(A^\star)\right) \right) \\ &= \Big(A \cap \left(X \setminus A^+\right) \Big) \cup \left[A \cap \bigcup_{n\geq 1} T^{-n}\left(A^\star \setminus T^{-1}(A^\star)\right) \right] \\ &= \Big(A \cap \left(X \setminus A^+\right) \Big) \cup \left[A \cap \bigcup_{n\geq 1} T^{-n}\left(A^\star \setminus T^{-1}(A^\star)\right) \right] \\ &= (A \setminus A^+) \cup \left[A \cap \bigcup_{n\geq 1} T^{-n}(A^* \setminus A^+) \right] \\ &= (A \setminus A_{ret}) \cup \left[A \cap \bigcup_{n\geq 1} T^{-n}(A \setminus A_{ret}) \right] \\ &= A \cap \bigcup_{n\geq 0} T^{-n}(A \setminus A_{ret}). \end{aligned}$$

□

(S8.3) Let $(X, \mathcal{B}, \mu, T)$ be a MPS. If $A \in \mathcal{B}$ is such that $\mu(A) > 0$, then there exists $1 \leq N \leq \Phi$ such that

$$\mu(A \cap T^{-N}(A)) > 0,$$

where $\Phi = \left\lceil \dfrac{1}{\mu(A)} \right\rceil$.

Proof. Assume that $\mu(A \cap T^{-i}(A)) = 0$ for all $i = 1, \ldots, \Phi$. Then for all $m > n \in \{0, \ldots, \Phi\}$, if $1 \leq k := m - n \leq \Phi$, we have that

$$\begin{aligned}
\mu(T^{-n}(A) \cap T^{-m}(A)) &= \mu(T^{-n}(A \cap T^{-k}(A))) \\
&= \mu(A \cap T^{-k}(A)), \quad \text{as } T \text{ is measure preserving} \\
&= 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
1 = \mu(X) &\geq \mu\left(\bigcup_{i=0}^{\Phi} T^{-i}(A)\right) = \sum_{i=0}^{\Phi} \mu\left(T^{-i}(A)\right) \quad \text{by C.4.5.(iv)} \\
&= \sum_{i=0}^{\Phi} \mu(A), \quad \text{as } T \text{ is measure preserving} \\
&= \mu(A) \cdot (\Phi + 1) > 1, \quad \text{as } \Phi = \left\lceil \frac{1}{\mu(A)} \right\rceil > \frac{1}{\mu(A)}.
\end{aligned}$$

We have got thus a contradiction. □