SNSB
Summer Term 2013
Ergodic Theory and Additive Combinatorics
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Seminar 9

(S9.1) Let $f \in \mathcal{M}_{\mathbb{C}}(X, \mathcal{B})$ and $n \geq 1$.

(i) If f is T-invariant (a.e.), then $S_n f = f$ (a.e.).

(ii) $S_n f \in \mathcal{M}_{\mathbb{C}}(X, \mathcal{B})$.

(iii) $S_n f = \frac{1}{n} \sum_{k=0}^{n-1} U_{T^k} f$.

(iv) For any $p \geq 1$, $f \in L^p(X, \mathcal{B}, \mu)$ (resp. $L^p_{\mathbb{R}}(X, \mathcal{B}, \mu)$) implies $S_n f \in L^p(X, \mathcal{B}, \mu)$ (resp. $L^p_{\mathbb{R}}(X, \mathcal{B}, \mu)$).

(v) For all $x \in X$, $\frac{n+1}{n} S_{n+1}(x) - S_n f(Tx) = \frac{1}{n} f(x)$.

(vi) If $f \in \mathcal{M}_{\mathbb{R}}(X, \mathcal{B})$, then $\underline{f} \circ T = \underline{f}$ and $\overline{f} \circ T = \overline{f}$.

(vii) $\int_X S_n f \, d\mu = \int_X f \, d\mu$.

(viii) If $f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$ is nonnegative, then $S_n f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$ is nonnegative and $\|S_n f\|_1 = \|f\|_1$.

(S9.2) Let $A, B \in \mathcal{B}$ and $n \geq 1$.

(i) $S_n \chi_A = \frac{1}{n} \sum_{k=0}^{n-1} \chi_{T^{-k}(A)}$ and $\chi_B \cdot S_n \chi_A = \frac{1}{n} \sum_{k=0}^{n-1} \chi_{T^{-k}(A) \cap B}$.

(ii) $\int_X S_n \chi_A = \mu(A)$.

(iii) $\int_X \chi_B \cdot S_n \chi_A \, d\mu = \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}(A) \cap B)$.

(S9.3)

(i) Let X be a nonempty set, $(E_n)_{n\geq 1}$ be a sequence of subsets of X and $f : X \to \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \chi_{\cup_{i=1}^n E_i} f = \chi_{\cup_{i\geq 1} E_i} f. \tag{D.1}$$

(ii) Let $(X, \mathcal{B}, \mu)$ be a probability space, $f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu)$, $(E_n)_{n\geq 1}$ be an increasing sequence of measurable sets, and $E = \bigcup_{n\geq 1} E_n$. Prove that

$$\int_E f\, d\mu = \lim_{n\to\infty} \int_{E_n} f\, d\mu. \tag{D.2}$$

(S9.4)

Proposition . *Let $(X, \mathcal{B}, \mu, T)$ be a MPS. The following are equivalent*

(i) T is ergodic.

(ii) Whenever $f : X \to \mathbb{C}$ is measurable and $U_T f = f$, then f is constant a.e..

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(iv) Whenever $f : X \to \mathbb{R}$ is measurable and $U_T f = f$, then f is constant a.e..

(v) Whenever $f : X \to \mathbb{R}$ is measurable and $U_T f = f$ a.e., then f is constant a.e..