# Appendix A

# General notions

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ denote the sets of natural, integer, rational numbers, and real numbers, respectively. The subscript $+$ restricts the sets to the nonnegative numbers:

$$\mathbb{Z}_+ = \{x \in \mathbb{Z} \mid x \geq 0\} = \mathbb{N}, \quad \mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \geq 0\}, \quad \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}.$$

Furthermore, $\mathbb{N}^*$ denotes the set of positive natural numbers, that is $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
If $m, n \in \mathbb{Z}_+$, we use sometimes the notations $[m, n] := \{m, m+1, \ldots, n\}$, $[n] := \{1, \ldots, n\}$.
We also write $i = 1, \ldots, n$ instead of $i \in [n]$.

If $X$ is a set, we denote by $\mathcal{P}(X)$ the collection of its subsets and by $[X]^2$ the collection of 2-element subsets of $X$, i.e. $[X]^2 = \{\{x, y\} \mid x, y \in X\}$.
If $X$ is a finite set, the **size** of $X$ or the **cardinality** of $X$, denoted by $|X|$ is the number of elements of $X$.

Let $m, n \in \mathbb{N}^*$. We denote by $\mathbb{R}^{m\times n}$ the set of $m \times n$-matrices with entries from $\mathbb{R}$. Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ be a matrix. The transpose of $A$ is denoted by $A^T$. If $i = 1, \ldots, m$, we denote by $\mathbf{a}_i$ the $i$th row of $A$: $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n})$. If $I \subseteq \{1, \ldots, m\}$, we write $A_I$ for the submatrix of $A$ consisting of the rows in $I$ only. Thus, $\mathbf{a}_i = A_{\{i\}}$. We denote by $0_{m,n}$ the zero matrix in $\mathbb{R}^{m\times n}$, by $0_n$ the zero matrix in $\mathbb{R}^{n\times n}$ and by $I_n$ the identity matrix in $\mathbb{R}^{n\times n}$.

Let $n \in \mathbb{N}^*$. All vectors in $\mathbb{R}^n$ are column vectors. Let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n.$$

Then $x$ is a matrix in $\mathbb{R}^{n\times 1}$ and its transpose $x^T$ is a row vector, hence a matrix in $\mathbb{R}^{1\times n}$.

Furthermore, for $I \subseteq \{1, \ldots, m\}$, $x_I$ is the subvector of $x$ consisting of the components with indices in $I$. If $a \in \mathbb{R}$, we denote by $\mathbf{a}$ the vector in $\mathbb{R}^n$ whose components are all equal to $a$.

# Appendix B

# Euclidean space $\mathbb{R}^n$

The Euclidean space $\mathbb{R}^n$ is the $n$-dimensional real vector space with inner product

$$x^T y = \sum_{i=1}^{n} x_i y_i.$$

We let

$$\|x\| = (x^T x)^{1/2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

denote the Euclidean norm of a vector $x \in \mathbb{R}^n$.
For every $i = 1, \ldots, n$, we denote by $e_i$ the $i$th unit vector in $\mathbb{R}^n$. Thus, $e_1 = (1, 0, \ldots, 0, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$.
For vectors $x, y \in \mathbb{R}^n$ we write $x \leq y$ whenever $x_i \leq y_i$ for $i = 1, \ldots, n$. Similarly, $x < y$ whenever $x_i < y_i$ for $i = 1, \ldots, n$.
Let $x, y \in \mathbb{R}^n$. We say that $x, y$ are **parallel** if one of them is a scalar multiple of the other.

**Proposition B.0.1** (Cauchy-Schwarz inequality)**.** *For all $x, y \in \mathbb{R}^n$,*

$$|x^T y| \leq \|x\| \|y\|,$$

*with equality if and only if $x$ and $y$ are parallel.*

The (closed) **line segment** joining $x$ and $y$ is defined as

$$[x, y] = \{\lambda x + (1 - \lambda) y \mid \lambda \in [0, 1]\}.$$

The **open line segment** joining $x$ and $y$ is defined as

$$(x, y) = \{\lambda x + (1 - \lambda) y \mid \lambda \in (0, 1)\}.$$

**Definition B.0.2.** *A subset $L \subseteq \mathbb{R}^n$ is a* ***line*** *if there are $x, r \in \mathbb{R}^n$ with $r \neq \mathbf{0}$ such that*

$$L = \{x + \lambda r \mid \lambda \in \mathbb{R}\}.$$

We also say that $L$ is a line through point $x$ with direction vector $r \neq 0$ and denote it by $L_{x,r}$.

**Proposition B.0.3.** *A subset $L \subseteq \mathbb{R}^n$ is a line if and only if there are $x, y \in \mathbb{R}^n$ such that*

$$L = \{(1-\lambda)x + \lambda y \mid \lambda \in \mathbb{R}\}.$$

We also say that $L$ is the line through two points $x, y$ and denote it by $\overline{xy}$.

Given $r > 0$ and $x \in \mathbb{R}^n$, $B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$ is the **open ball** with center $x$ and radius $r$ and $\overline{B}_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$ is the **closed ball** with center $x$ and radius $r$.

**Definition B.0.4.** *A subset $X \subseteq \mathbb{R}^n$ is* bounded *if there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in X$.*

# Appendix C

# Linear algebra

**Definition C.0.1.** *A nonempty set $S \subseteq \mathbb{R}^n$ is a* ***(linear) subspace*** *if $\lambda_1 x_1 + \lambda_2 x_2 \in S$ whenever $x_1, x_2 \in S$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.*

Let $x_1, \ldots, x_m$ be points in $\mathbb{R}^n$. Any point $x \in \mathbb{R}^n$ of the form $x = \sum_{i=1}^{m} \lambda_i x_i$, with $\lambda_i \in \mathbb{R}$ for each $i = 1, \ldots, m$, is a **linear combination** of $x_1, \ldots, x_m$.

**Definition C.0.2.** *The* ***linear span*** *of a subset $X \subseteq \mathbb{R}^n$ (denoted by span$(X)$) is the intersection of all subspaces containing $X$.*

If $\text{span}(X) = \mathbb{R}^n$ we say that $X$ is a **spanning set** of $\mathbb{R}^n$ or that $X$ **spans** $\mathbb{R}^n$.

**Proposition C.0.3.** *(i) span$(\emptyset) = \{\mathbf{0}\}$.*

*(ii) For every $X \subseteq \mathbb{R}^n$, span$(X)$ consists of all linear combinations of points in $X$.*

*(iii) $S \subseteq \mathbb{R}^n$ is a subspace if and only if $S$ is closed under linear combinations if and only $S =$ span$(S)$.*

**Definition C.0.4.** *A set of vectors $X = \{x_1, \ldots, x_m\}$ is* ***linearly independent*** *if*

$$\sum_{i=1}^{m} \lambda_i x_i = 0 \quad \textit{implies} \quad \lambda_i = 0 \textit{ for each } i = 1, \ldots, m.$$

Is $X$ is not linearly independent, we say that $X$ is **linearly dependent**. We also say that $x_1, \ldots, x_m$ are linearly (in)dependent.

**Proposition C.0.5.** *Let $X = \{x_1, \ldots, x_m\}$ be a set of vectors in $\mathbb{R}^n$. Then $X$ is linearly dependent if and only if at least one of the vectors $x_i$ can be written as a linear combination of the other vectors in $X$.*

**Definition C.0.6.** *Let $S$ be a subspace of $\mathbb{R}^n$. A subset $B = \{x_1, \dots, x_m\} \subseteq S$ is a* ***basis*** *of $S$ if $B$ spans $S$ and $B$ is linearly independent.*

**Proposition C.0.7.** *Let $S$ be a subspace of $\mathbb{R}^n$ and $B$ be a basis of $S$ with $|B| = m$.*

*(i) Every vector in $S$ can be written in a unique way as a linear combination of vectors in $B$.*

*(ii) Every subset of $S$ containing more than $m$ vectors is linearly dependent.*

*(iii) Every other basis of $S$ has $m$ vectors.*

**Definition C.0.8.** *The* ***dimension*** *$dim(S)$ of a subspace $S$ of $\mathbb{R}^n$ is the number of vectors in a basis of $S$.*

**Proposition C.0.9.** *Let $S$ be a subspace of $\mathbb{R}^n$.*

*(i) If $S = \{0\}$, then $dim(S) = 0$, since its basis is empty.*

*(ii) $dim(S) \geq 1$ if and only if $S \neq \{0\}$.*

*(iii) If $X = \{x_1, \dots, x_m\} \subseteq S$ is a linearly independent set, then $m \leq dim(S)$.*

*(iv) If $X = \{x_1, \dots, x_m\} \subseteq S$ is a spanning set for $S$, then $m \geq dim(S)$.*

**Proposition C.0.10.** *Let $S$ be a subspace of dimension $m$ and $X = \{x_1, \dots, x_m\} \subseteq S$. Then $X$ is a basis of $S$ if and only if $X$ spans $S$ if and only if $X$ is linearly independent.*

**Proposition C.0.11.** *Suppose that $U$ and $V$ are subspaces of $\mathbb{R}^n$ such that $U \subseteq V$. Then*

*(i) $dim(U) \leq dim(V)$.*

*(ii) $dim(U) = dim(V)$ if and only if $U = V$.*

## C.1 Matrices

Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$.

**Definition C.1.1.** *The* ***column space*** *of $A$ is the linear span of the set of its columns. The* ***column rank*** *of $A$ is the dimension of the column space, the number of linearly independent columns.*

**Definition C.1.2.** *The* ***row space*** *of $A$ is the linear span of the set of its rows. The* ***row rank*** *of $A$ is the dimension of the row space, the number of linearly independent rows.*

**Proposition C.1.3.** *The row rank and column rank of $A$ are equal.*

*Proof.* See [3, Theorem 3.11, p. 131]. □

**Definition C.1.4.** *The **rank** of a matrix $A$, denoted by rank$(A)$, is its row rank or column rank.*

The $m \times n$ matrix $A$ has **full row rank** if its rank is $m$ and it has **full column rank** if its column rank is $n$.

**Theorem C.1.5.** *Let us consider the homogeneous system $Ax = \mathbf{0}$ (with $n$ unknowns and $m$ equations) and let $S := \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$ be its solution set. Then*

*(i) $S$ is a linear subspace of $\mathbb{R}^n$.*

*(ii) $\dim(S) = n - \text{rank}(A)$.*

*Proof.* See [3, Theorem 3.13, p. 131]. □

Thus, the homogeneous system $Ax = \mathbf{0}$ has a unique solution (namely $x = \mathbf{0}$) if and only if $\text{rank}(A) = n$.

Let $b \in \mathbb{R}^m$ and $A \mid b$ be the matrix $A$ augmented by $b$. Thus,

$$A \mid b = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & & & & \\ a_{i1} & a_{i2} & \dots & a_{in} & b_i \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

**Theorem C.1.6.** *Let us consider the linear system $Ax = b$ and let $S := \{x \in \mathbb{R}^n \mid Ax = b\}$ be its solution set.*

*(i) $S \neq \emptyset$ if and only if $\text{rank}(A) = \text{rank}(A \mid b)$.*

*(ii) If $S \neq \emptyset$ and $\overline{x}$ is a particular solution, then*

$$S = \overline{x} + \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}.$$

*(iii) The system has a unique solution if and only if $\text{rank}(A) = \text{rank}(A \mid b) = n$.*

*Proof.* See, for example, [3, Section III.3]. □

# Appendix D

# Affine sets

**Definition D.0.1.** *A set* $A \subseteq \mathbb{R}^n$ *is* ***affine*** *if* $\lambda_1 x_1 + \lambda_2 x_2 \in A$ *whenever* $x_1, x_2 \in A$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$ *satisfy* $\lambda_1 + \lambda_2 = 1$.

Geometrically, this means that $A$ contains the line through any pair of its points. Note that by this definition the empty set is affine.

**Example D.0.2.** (i) A point is an affine set.

(ii) Any linear subspace is an affine set.

(iii) Any line is an affine set.

(iv) Another example of an affine set is $P = \{x + \lambda_1 r_1 + \lambda_2 r_2 \mid \lambda_1, \lambda_2 \in \mathbb{R}\}$ which is a two-dimensional plane going through $x$ and spanned by the nonzero vectors $r_1$ and $r_2$.

**Definition D.0.3.** *We say that an affine set* $A$ *is* ***parallel*** *to another affine set* $B$ *if* $A = B + x_0$ *for some* $x_0 \in \mathbb{R}^n$, *i.e.* $A$ *is a translate of* $B$.

**Proposition D.0.4.** *Let* $A$ *be a nonempty subset of* $\mathbb{R}^n$. *Then* $A$ *is an affine set if and only if* $A$ *is parallel to a unique linear subspace* $S$, *i.e.,* $A = S + x_0$ *for some* $x_0 \in A$.

*Proof.* See [1, P.1.1, pag. 13]. □

**Remark D.0.5.** *An affine set is a linear subspace if and only if it contains the origin.*

*Proof.* To be done in the seminar. □

**Definition D.0.6.** *The* ***dimension*** *of a nonempty affine set* $A$, *denoted by* $dim(A)$, *is the dimension of the unique linear subspace parallel to* $A$. *By convention,* $dim(\emptyset) = -1$.

The maximal affine sets not equal to the whole space are of particular importance, these are the hyperplanes. More precisely,

**Definition D.0.7.** *A* ***hyperplane*** *in* $\mathbb{R}^n$ *is an affine set of dimension* $n-1$.

**Proposition D.0.8.** *Any hyperplane* $H \subseteq \mathbb{R}^n$ *may be represented by*

$$H = \{x \in \mathbb{R}^n \mid a^T x = \beta\} \quad \text{for some nonzero } a \in \mathbb{R}^n \text{ and } \beta \in \mathbb{R},$$

*i.e.* $H$ *is the solution set of a nontrivial linear equation. Furthermore, any set of this form is a hyperplane. Finally, the equation in this representation is unique up to a scalar multiple.*

*Proof.* See [1, P.1.2, pag. 13-14]. □

**Definition D.0.9.** *A* ***(closed) halfspace*** *in* $\mathbb{R}^n$ *is the set of all points* $x \in \mathbb{R}^n$ *that satisfy* $a^T x \leq \beta$ *for some* $a \in \mathbb{R}^n$ *and* $\beta \in \mathbb{R}$.

We shall use the following notations

$$\begin{aligned} H_{=}(a, \beta) &= \{x \in \mathbb{R}^n \mid a^T x = \beta\} \\ H_{\leq}(a, \beta) &= \{x \in \mathbb{R}^n \mid a^T x \leq \beta\} \\ H_{\geq}(a, \beta) &= \{x \in \mathbb{R}^n \mid a^T x \geq \beta\} \end{aligned}$$

Thus, each hyperplane $H_{=}(a, \beta)$ gives rise to a decomposition of the space in two halfspaces:

Affine sets are closely linked to systems of linear equations.

**Proposition D.0.10.** *Let* $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$. *Then the solution set* $\{x \in \mathbb{R}^n \mid Ax = b\}$ *of the system of linear equations* $Ax = b$ *is an affine set. Furthermore, any affine set may be represented in this way.*

*Proof.* See [1, P.1.3, pag. 13-14]. □

Let $x_1, \ldots, x_m$ be points in $\mathbb{R}^n$. An **affine combination** of $x_1, \ldots, x_m$ is a linear combination $\sum_{i=1}^{m} \lambda_i x_i$ with the property that $\sum_{i=1}^{m} \lambda_i = 1$.

**Definition D.0.11.** *The* ***affine hull*** *aff*$(X)$ *of a subset* $X \subseteq \mathbb{R}^n$ *is the intersection of all affine sets containing* $X$.

**Proposition D.0.12.** *(i) The affine hull aff*$(X)$ *of a subset* $X \subseteq \mathbb{R}^n$ *consists of all affine combinations of points in* $X$.

*(ii)* $A \subseteq \mathbb{R}^n$ *is affine if and only if* $A = \mathit{aff}(A)$.

*Proof.* See [1, P.1.4, pag. 16]. □

**Definition D.0.13.** *The* ***dimension*** *dim*$(X)$ *of a set* $X \subseteq \mathbb{R}^n$ *is the dimension of aff*$(X)$.

# Appendix E

# Convex sets

**Definition E.0.1.** *A set* $C \subseteq \mathbb{R}^n$ *is called* convex *if it contains line segments between each pair of its points, that is, if* $\lambda_1 x_1 + \lambda_2 x_2 \in C$ *whenever* $x_1, x_2 \in C$ *and* $\lambda_1, \lambda_2 \geq 0$ *satisfy* $\lambda_1 + \lambda_2 = 1$.

Equivalently, $C$ is convex if and only if $(1 - \lambda)C + \lambda C \subseteq C$ for every $\lambda \in [0, 1]$. Note that by this definition the empty set is convex.

**Example E.0.2.** (i) All affine sets are convex, but the converse does not hold.

(ii) More generally, the solution set of a family (finite or infinite) of linear inequalities $a_i^T x \leq b_i$, $i \in I$ is a convex set.

(iii) The open ball $B(a, r)$ and the closed ball $\overline{B}(a, r)$ are convex sets.

# Appendix F

# Graph Theory

Our presentation follows [2] and [9, Chapter 3].

## F.1 Graphs

**Definition F.1.1.** *A **graph** is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$.*

Thus, the elements of $E$ are 2-element subsets of $V$. To avoid notational ambiguities, we shall always assume tacitly that $V \cap E = \emptyset$. The elements of $V$ are the **vertices** (or **nodes** or **points**) of $G$, the elements of $E$ are its **edges**. The vertices of $G$ are denoted $x, y, z, u, v, v_1, v_2, \ldots$. The edge $\{x, y\}$ of $G$ is also denoted $[x, y]$ or $xy$.

**Definition F.1.2.** *The **order** of a graph $G$, written as $|G|$ is the number of vertices of $G$. The number of its edges is denoted by $\|G\|$.*

Graphs are **finite**, **infinite**, **countable** and so on according to their order. The empty graph $(\emptyset, \emptyset)$ is simply written $\emptyset$. A graph of order 0 or 1 is called **trivial**.

**Convention: Unless otherwise stated, our graphs will be finite.**
In the sequel, $G = (V, E)$ is a graph.

A graph with vertex set $V$ is said to be a graph **on** $V$. The vertex set of a graph $G$ is referred to as $V(G)$, its edge set as $E(G)$. We shall not always distinguish strictly between a graph and its vertex or edge set. For example, we may speak of a vertex $v \in G$ (rather than $v \in V(G)$), an edge $e \in G$, and so on.
A vertex $v$ is **incident** with an edge $e$ if $v \in e$; then $e$ is an edge at $v$. The set of all edges in $E$ at $v$ is denoted by $E(v)$. The **ends** of an edge $e$ are the two vertices incident with $e$. Two edges $e \neq f$ are **adjacent** if they have an end in common.

If $e = xy \in E$ is an edge, we say that $e$ **joins** its vertices $x$ and $y$, that $x$ and $y$ are **adjacent** (or **neighbours**), that $x$ and $y$ are the **ends** of the edge $e$.

If $F$ is a subset of $[V]^2$, we use the notations $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. Then $G - \{e\}$ and $G + \{e\}$ are abbreviated $G - e$ and $G + e$.

### F.1.1 The degree of a vertex

**Definition F.1.3.** *The **degree** (or valency) of a vertex $v$ is the number $|E(v)|$ of edges at $v$ and it is denoted by $d_G(v)$ or simply $d(v)$.*

A vertex of degree 0 is **isolated**, and a vertex of degree 1 is a **terminal** vertex. Obviously, the degree of a vertex is equal to the number of neighbours of $v$.

**Proposition F.1.4.** *The number of vertices of odd degree is always even.*

### F.1.2 Subgraphs

**Definition F.1.5.** *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs.*

*(i) $G'$ is a **subgraph** of $G$, written $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. If $G' \subseteq G$ we also say that $G$ is a **supergraph** of $G'$ or that $G'$ is **contained** in $G$.*

*(ii) If $G' \subseteq G$ and $G'$ contains all the edges $xy \in E$ with $x, y \in V'$, then $G'$ is an **induced subgraph** of $G$; we say that $V'$ **induces** or **spans** $G'$ in $G$ and write $G' = G[V']$.*

*(iii) If $G' \subseteq G$, we say that $G'$ is a **spanning** subgraph of $G$ if $V' = V$.*

### F.1.3 Paths, cycles

**Definition F.1.6.** *A **path** is a nonempty graph $P = (V(P), E(P))$ of the form*

$$V(P) = \{x_0, \dots, x_k\}, \quad E(P) = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\},$$

*where $k \geq 1$ and the $x_i$'s are all distinct.*

The vertices $x_0$ and $x_k$ are **linked** by $P$ and are called its **endvertices** or **ends**; the vertices $x_1, \dots, x_{k-1}$ are the **inner** vertices of $P$. The number of edges of a path is its **length**. The path of length $k$ is denoted $P^k$.
We often refer to a path by the natural sequence of its vertices, writing $P = x_0x_1 \dots x_k$ and saying that $P$ is a path **from** $x_0$ **to** $x_k$ (or **between** $x_0$ **and** $x_k$).
If a path $P$ is a subgraph of a graph $G = (V, E)$, we say that $P$ is a path **in** $G$.

**Definition F.1.7.** *Let $P = x_0 \dots x_k, k \geq 2$ be a path. The graph $P + x_k x_0$ is called a* ***cycle****.*

As in the case of paths, we usually denote a cycle by its (cyclic) sequence of vertices: $C = x_0 \dots x_k x_0$. The **length** of a cycle is the number of its edges (or vertices). The cycle of length $k$ is said to be a $k$**-cycle** and denoted $C^k$.

# F.2 Directed graphs

**Definition F.2.1.** *A* ***directed graph*** *(or* ***digraph****) is a pair $D = (V, A)$, where $V$ is a finite set and $A$ is a* ***multiset*** *of ordered pairs from $V$.*

Let us recall that a **multiset** (or **bag**) is a generalization of the notion of a set in which members are allowed to appear more than once.
The elements of $V$ are the **vertices** (or **nodes** or **points**) of $D$, the elements of $A$ are its **arcs** (or **directed edges**). The vertex set of a digraph $D$ is referred to as $V(D)$, its set of arcs as $A(D)$.
Since $A$ is a multiset, the same pair of vertices may occur several times in $A$. A pair occurring more than once in A is called a **multiple** arc, and the number of times it occurs is called its **multiplicity**. Two arcs are called **parallel** if they are represented by the same ordered pair of vertices. Also **loops** are allowed, that is, arcs of the form $(v, v)$.

**Definition F.2.2.** *Directed graphs without loops and multiple arcs are called* ***simple****, and directed graphs without loops are called* ***loopless****.*

Let $a = (u, v)$ be an arc. We say that $a$ **connects** $u$ and $v$, that $a$ **leaves** $u$ and **enters** $v$; $u$ and $v$ are called the **ends** of $a$, $u$ is called the **tail** of $a$ and $v$ is called the **head** of $a$. If there exists an arc connecting vertices $u$ and $v$, then $u$ and $v$ are called **adjacent** or **connected**. If there exists an arc $(u, v)$, then $v$ is called an **outneighbour** of $u$, and $u$ is called an **inneighbour** of $v$.

Each directed graph $D = (V, A)$ gives rise to an **underlying (undirected) graph**, which is the graph $G = (V, E)$ obtained by ignoring the orientation of the arcs:

$$E = \{\{u, v\} \mid (u, v) \in A\}.$$

If $G$ is the underlying (undirected) graph of a digraph $D$, we call $D$ an **orientation** of $G$. Terminology from undirected graphs is often transfered to directed graphs.

For any arc $a = (u, v) \in A$, we denote $a^{-1} := (v, u)$ and define $A^{-1} := \{a^{-1} \mid a \in A\}$. The **reverse** digraph $D^{-1}$ is defined by $D^{-1} = (V, A^{-1})$.

For any vertex $v$, we denote

| | | |
|---|---|---|
| $\delta_A^{in}(v) := \delta^{in}(v)$ | := | the set of arcs entering $v$, |
| $\delta_A^{out}(v) := \delta^{out}(v)$ | := | the set of arcs leaving $v$. |

**Definition F.2.3.** *The **indegree** $deg^{in}(v)$ of a vertex $v$ is the number of arcs entering $v$, i.e. $|\delta^{in}(v)|$. The **outdegree** $deg^{out}(v)$ of a vertex $v$ is the number of arcs leaving $v$, i.e. $|\delta^{out}(v)|$.*

For any $U \subseteq V$, we denote

| | | |
|---|---|---|
| $\delta_A^{in}(U) := \delta^{in}(U)$ | := | the set of arcs entering $U$, i.e. the set of arcs with head in $U$ and tail in $V \setminus U$, |
| $\delta_A^{out}(U) := \delta^{out}(U)$ | := | the set of arcs leaving $U$, i.e. the set of arcs with head in $V \setminus U$ and tail in $U$. |

## F.2.1 Subgraphs

One can define the concept of subgraph as for graphs.
Two subgraphs of $D$ are

(i) **vertex-disjoint** if they have no vertex in common;

(ii) **arc-disjoint** if they have no arc in common.

In general, we say that a family of $k$ subgraphs ($k \geq 3$) is (vertex, arc)-disjoint if the $k$ subgraphs are **pairwise** (vertex, arc)-disjoint, i.e. every two subgraphs from the family are (vertex, arc)-disjoint.

## F.2.2 Paths, circuits, walks

**Definition F.2.4.** *A **(directed) path** is a digraph $P = (V(P), A(P))$ of the form*

$$V = \{v_0, \dots, v_k\}, \quad E = \{(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)\},$$

*where $k \geq 1$ and the $v_i$'s are all distinct.*

The vertices $v_0$ and $v_k$ are called the **endvertices** or **ends** of $P$; the vertices $v_1, \dots, v_{k-1}$ are the **inner** vertices of $P$. The number of edges of a path is its **length**.
We often refer to a path by the natural sequence of its vertices, writing $P = v_0 v_1 \dots v_k$ and saying that $P$ is a path **from** $v_0$ **to** $v_k$ or that the path $P$ **runs from** $v_0$ **to** $v_k$.
If a path $P$ is a subgraph of a digraph $D = (V, A)$, we say that $P$ is a path **in** $G$.

**Notation F.2.5.** *We denote by $P^{-1} := (V(P), E(P)^{-1})$.*

**Definition F.2.6.** *Let $P = v_0 \dots v_k, k \geq 1$ be a path. The graph*

$$P + (v_k, v_0) = (\{v_0, \dots, v_k\}, \{(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k), (v_k, v_0)\}$$

*is called a **circuit**.*

As in the case of paths, we usually denote a circuit by its (cyclic) sequence of vertices: $C = v_0 \dots v_k v_0$. The **length** of a circuit is the number of its edges (or vertices). The circuit of length $k$ is said to be a **$k$-circuit** and denoted $C^k$.

**Definition F.2.7.** *A **walk** in $D$ is a nonempty alternating sequence $v_0 a_0 v_1 a_1 \dots a_{k-1} v_k$ of vertices and arcs of $D$ such that $a_i = (v_1, v_{i+1})$ for all $i = 0, \dots, k-1$. If $v_0 = v_k$, the walk is **closed**.*

Let $D = (V, A)$ be a digraph. For $s, t \in V$, a path in $D$ is said to be an **$s$-$t$ path** if it runs from $s$ to $t$, and for $S, T \subseteq V$, an **$S$-$T$ path** is a path in $D$ that runs from a vertex in $S$ to a vertex in $T$. A vertex $v \in V$ is called **reachable** from a vertex $s \in V$ (or from a set $S \subseteq V$) if there exists an $s$-$t$ path (or $S$-$t$ path).
Two $s$-$t$-paths are **internally vertex-disjoint** if they have no inner vertex in common.

**Definition F.2.8.** *A set $U$ of vertices is*

*(i) $S$-$T$ **disconnecting** if $U$ intersects each $S$-$T$-path.*

*(ii) an $s$-$t$ **vertex-cut** if $s, t \notin U$ and each $s$-$t$-path intersects $U$.*

We say that $v_0 a_0 v_1 a_1 \dots a_{k-1} v_k$ is a walk of length $k$ from $v_0$ to $v_k$ or between $v_0$ and $v_k$. If all vertices in a walk are distinct, then the walk defines obviously a path in $D$.