Techniques of combinatorial optimization

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January 17, 2017

Contents

Abstract

The material in these notes is taken from several existing sources, among which the main ones are

- lecture notes from Chandra Chekuri' s course "Topics in Combinatorial Optimization" at the University of Illinois at Urbana-Champaign:

  https://courses.engr.illinois.edu/cs598csc/sp2010/

- lecture notes from Michel Goemans's course "Combinatorial Optimization" at MIT:

  http://www-math.mit.edu/~goemans/18433S13/18433.html

- A. Schrijver, A course in Combinatorial Optimization, University of Amsterdam, 2013:

  http://homepages.cwi.nl/~lex/files/dict.pdf

- Geir Dahl, An introduction to convexity, polyhedral theory and combinatorial optimization, University of Oslo, 1997:

  http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.78.5286

- A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, 3 Volumes, Springer, 2003
- D. Jungnickel, Graphs, Networks and Algorithms, 4th edition, Springer, 2013.
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Chapter 1

Polyhedra and Linear Programming

1.1 Optimization problems

An **optimization problem** (or **mathematical programming problem**) is a maximization problem

$$(P): \quad \text{maximize } \{f(x) \mid x \in A\} \tag{1.1}$$

or a minimization problem

$$(P): \quad \text{minimize } \{f(x) \mid x \in A\} \tag{1.2}$$

where $f : A \to \mathbb{R}$ is a given function. Each point in A is called a **feasible point**, or a **feasible solution** and A is the **feasible region** or **feasible set**. An optimization problem is called **feasible** if it has some feasible solution; otherwise, it is called **unfeasible**. The function f is called the **objective function** or the **cost function**.
Two maximization problems

$$(P): \text{ maximize } \{f(x) \mid x \in A\} \quad \text{and} \quad (Q): \text{ maximize } \{g(y) \mid y \in B\}$$

are **equivalent** if for each feasible solution $x \in A$ of (P) there is a corresponding feasible solution $y \in B$ of (Q) such that $f(x) = g(y)$ and vice versa. Similarly for minimization problems.
A point $x^* \in A$ is an **optimal solution** of the

(i) problem (1.1) if $f(x^*) \geq f(x)$ for all $x \in A$.

(ii) problem (1.2) if $f(x^*) \leq f(x)$ for all $x \in A$.

The **optimal value** $v(P)$ of (1.1) is defined as $v(P) = \sup\{f(x) \mid x \in A\}$. Similarly, the **optimal value** $v(P)$ of (1.2) is defined as $v(P) = \inf\{f(x) \mid x \in A\}$. Thus, if x^* is an optimal solution, then $f(x^*) = v(P)$. Note that there may be several optimal solutions.

An optimization problem (P) is **bounded** if $v(P)$ is finite. For many bounded problems of interest in optimization, this supremum (infimum) is attained, and then we may replace sup (inf) by max (min).
We say that the maximization problem (1.1) is **unbounded** if for any $M \in \mathbb{R}$ there is a feasible solution x^M with $f(x^M) \geq M$, and we then write $v(P) = \infty$. Similarly, the minimization problem (1.2) is **unbounded** if for any $m \in \mathbb{R}$ there is a feasible solution x^m with $f(x^m) \leq m$; we then write $v(P) = -\infty$.
If (1.1) is unfeasible, we define $v(P) = -\infty$, as we are maximizing over the empty set. If (1.2) is unfeasible, we define $v(P) = \infty$, as we are minimizing over the empty set.
Thus, for an optimization problem (P) there are three possibilities:

(i) (P) is unfeasible

(ii) (P) is unbounded

(iii) (P) is bounded.

1.2 Polyhedra

A **linear inequality** is an inequality of the form $a^T x \leq \beta$, where $a, x \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Note that a **linear equality (equation)** $a^T x = \beta$ may be written as the two linear inequalities $a^T x \leq \beta$, $-a^T x \leq -\beta$.
A **system of linear inequalities**, or **linear system** for short, is a finite set of linear inequalities, so it may be written in matrix form as

$$(S1) \qquad Ax \leq b,$$

where $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. For every $i = 1, \ldots, m$, the ith inequality of the system $Ax \leq b$ is the linear inequality $\mathbf{a}_i x \leq b_i$, where $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ is the ith row of A. Hence, (S1) can be written as

$$(S1') \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, 2, \ldots, m.$$

We say that two linear systems are **equivalent** if they have the same solution set. A linear system $Ax \leq b$ is called **real** (resp. **rational**) if all the elements in A and b are real (resp. rational). Note that a rational linear system is equivalent to a linear system with all coefficients being integers; we just multiply each inequality by a suitably large integer.
A linear system is **consistent** (or **solvable**, or **feasible**) if it has at least one solution, i.e., there is an x_0 satisfying $Ax_0 \leq b$.

Definition 1.2.1. *A **polyhedron** in $\mathbb{R}^n$ is the intersection of finitely many halfspaces.*

One can easily see that a subset $P \subseteq \mathbb{R}^n$ is a polyhedron if and only if $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and some vector $b \in \mathbb{R}^m$. A polyhedron is **real** (resp. **rational**) if it is the solution set of a real (resp. rational) linear system.

Definition 1.2.2. *The **dimension** $dim(P)$ of a polyhedron $P \subseteq \mathbb{R}^n$ is the dimension of the affine hull of P. If $dim(P) = n$, we say that P is **full-dimensional**.*

Proposition 1.2.3. *Any polyhedron is a convex set.*

Proof. Exercise. □

Example 1.2.4. (i) Affine sets are polyhedra.

(ii) Singletons are polyhedra of dimension 0.

(iii) Lines are polyhedra of dimension 1.

(iv) The unit cube $C_3 = \{x \in \mathbb{R}^3 \mid 0 \leq x_i \leq 1 \text{ for all } i = 1, 2, 3\}$ in $\mathbb{R}^3$ is a full-dimensional polyhedron.

Proof. Exercise. □

1.3 Solvability of systems of linear inequalities

Theorem 1.3.1 (Theorem of the Alternatives).
Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. For the system $Ax \leq b$, exactly one of the following two alternatives hold:

(i) The system is solvable.

(ii) There exists $y \in \mathbb{R}^m$ such that $y \geq \mathbf{0}$, $y^T A = \mathbf{0}^T$ and $y^T b < 0$.

Proof. Supplementary exercise. □

From the Theorem of the Alternatives one can derive the Farkas lemma.

Lemma 1.3.2 (Farkas Lemma).
The system $Ax = b, x \geq \mathbf{0}$ has no solution if and only if there exists $y \in \mathbb{R}^m$ such that $y^T A \geq \mathbf{0}^T, y^T b < 0$.

Proof. Let us denote (S1): $Ax = b, x \geq \mathbf{0}$ and (S2): $y^T A \geq \mathbf{0}^T, y^T b < 0$. We can rewrite (S1) as $Ax \leq b, -Ax \leq -b, -x \leq \mathbf{0}$, hence as $\begin{pmatrix} A \\ -A \\ -I_n \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ \mathbf{0} \end{pmatrix}$. Apply then Theorem of the Alternatives to conclude that (S1) has no solution if and only if the system

$$(S3): \quad z \geq \mathbf{0},\, z^T \begin{pmatrix} A \\ -A \\ -I_n \end{pmatrix} = \mathbf{0}^T,\, z^T \begin{pmatrix} b \\ -b \\ \mathbf{0} \end{pmatrix} < 0$$

has a solution. Let us prove now that (S3) is solvable if and only if (S2) is solvable.
"$\Rightarrow$" Let $z \in \mathbb{R}^{2m+n}$ be a solution of (S3). Then $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ with $u, v \in \mathbb{R}^m$ and $w \in \mathbb{R}^n$ satisfying $u, v, w \geq \mathbf{0}$, $u^T A - v^T A - w^T = \mathbf{0}^T$ and $u^T b - v^T b < 0$. Take $y := u - v$. Then $y \in \mathbb{R}^m$, $y^T A = w^T \geq \mathbf{0}^T$ and $y^T b < 0$, that is y is a solution of (S2).
"$\Leftarrow$" Let $y \in \mathbb{R}^m$ be a solution of (S2). Take $w := A^T y \in \mathbb{R}^n$ (so, $w^T = y^T A$) and $u, v \in \mathbb{R}^m$ such that $u, v \geq \mathbf{0}$ and $y = u - v$ (for example, $u_i = \max\{y_i, 0\}, v_i = \max\{-y_i, 0\}$). Then $z := \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ is a solution of (S3). □

In the sequel we give some variants of Farkas lemma.

Lemma 1.3.3 (Farkas lemma - variant)**.** *The system $Ax = b$ has a solution $x \geq \mathbf{0}$ if and only if $y^T b \geq 0$ for each $y \in \mathbb{R}^m$ with $y^T A \geq \mathbf{0}^T$.*

Proof. Exercise. □

Lemma 1.3.4 (Farkas lemma - variant)**.** *The system $Ax \leq b$ has a solution if and only if $y^T b \geq 0$ for each $y \geq \mathbf{0}$ with $y^T A = \mathbf{0}^T$.*

Proof. Exercise. □

1.4 Linear programming

Linear programming, abbreviated to LP, concerns the problem of maximizing or minimizing a linear functional over a polyhedron:

$$\max\{c^T x \mid Ax \leq b\} \quad \text{or} \quad \min\{c^T x \mid Ax \leq b\}, \tag{1.3}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

An LP problem will be also called a **linear program**.
We shall refer to the maximization problem

$$(P) \qquad \max\{c^T x \mid Ax \le b\}$$

as the **primal LP**.
The primal LP has its associated **dual LP**:

$$(D) \qquad \min\{b^T y \mid y \ge \mathbf{0}, y^T A = c^T\} = \min\{b^T y \mid y \ge \mathbf{0}, A^T y = c\}.$$

Thus, we have n primal variables and m dual variables.
The following result follows from an immediate application of the Theorem of Alternatives and Farkas Lemma 1.3.2.

Lemma 1.4.1. *(i) (P) is unfeasible if and only if there exists $u \in \mathbb{R}^m$ such that $u \ge \mathbf{0}$, $u^T A = \mathbf{0}^T$ and $u^T b < 0$.*

(ii) (D) is unfeasible if and only if there exists $u \in \mathbb{R}^n$ such that $Au \ge \mathbf{0}, c^T u < 0$.

Proposition 1.4.2 (Weak Duality)**.** *Let x be a feasible solution of the primal LP and y be a feasible solution of the dual LP. Then*

(i) $c^T x \le b^T y$.

(ii) If $c^T x = b^T y$, then x and y are optimal.

Proof. We have that $c^T x = (y^T A)x = y^T(Ax) \le y^T b = b^T y$, since $y \ge \mathbf{0}$. □

The main result in the theory of linear programming is the Strong Duality Theorem:

Theorem 1.4.3 (Strong Duality)**.** *Assume that the primal and dual LPs are feasible. Then they are bounded and*

$$\max\{c^T x \mid Ax \le b\} = \min\{b^T y \mid y \ge \mathbf{0}, y^T A = c^T\}.$$

Proof. Supplementary exercise. □

As an immediate consequence, we have that

Corollary 1.4.4. *Let x be a feasible solution of the primal LP and y be a feasible solution of the dual LP. Then they are optimal solutions to (P) and (D) if and only if $b^T y = c^T x$.*

Proposition 1.4.5. *Let (P) and (D) be the primal and dual LPs.*

(i) If both (P) and (D) are feasible, then they are bounded.

(ii) If either (P) or (D) is unfeasible, then the other is either unfeasible or unbounded.

(iii) If either (P) or (D) is unbounded, then the other is unfeasible.

(iv) If either (P) or (D) is bounded, then the other is bounded too.

Proof. Exercise. □

1.5 Polytopes

Let $x^1, \dots, x^m$ be points in $\mathbb{R}^n$. A **convex combination** of $x^1, \dots, x^m$ is a linear combination $\sum_{i=1}^{m} \lambda_i x^i$ with the property that $\lambda_i \geq 0$ for all $i = 1, \dots, m$ and $\sum_{i=1}^{m} \lambda_i = 1$.

Definition 1.5.1. *The **convex hull** of a subset $X \subseteq \mathbb{R}^n$, denoted by conv(X), is the intersection of all convex sets containing X.*

If $X = \{x^1, \dots, x^k\}$, we write $\text{conv}(x^1, \dots, x^k)$ for $\text{conv}(X)$.

Proposition 1.5.2. *(i) The convex hull conv(X) of a subset $X \subseteq \mathbb{R}^n$ consists of all convex combinations of points in X.*

(ii) $C \subseteq \mathbb{R}^n$ is convex if and only if C is closed under convex combinations if and only if $C = conv(C)$.

Proof. See [1, P.1.6, pag. 19 and P.1.7, pag. 20]. □

Definition 1.5.3. *A **polytope** is a set $P \subseteq \mathbb{R}^n$ which is the convex hull of a finite number of points.*

Thus, P is a polytope iff there are $x^1, \dots, x^k \in \mathbb{R}^n$ such that

$$P = \text{conv}(x^1, \dots, x^k) = \left\{ \sum_{i=1}^{k} \lambda_i x^i \mid \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

We recall that

$$\|x\| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

is the Euclidean norm of a vector $x \in \mathbb{R}^n$.
A subset $X \subseteq \mathbb{R}^n$ is **bounded** if there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in X$.

The following fundamental result is also known as the Finite Basis Theorem for Polytopes:

Theorem 1.5.4 (Minkowski (1896), Steinitz (1916), Weyl (1935))**.**
A nonempty set P is a polytope if and only if it is a bounded polyhedron.

1.6 Integer linear programming

A vector $x \in \mathbb{R}^n$ is called **integer** if each component is an integer, i.e., if x belongs to $\mathbb{Z}^n$. Many combinatorial optimization problems can be described as maximizing a linear function $c^T x$ over the **integer** vectors in some polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Thus, this type of problems can be described as:

$$(ILP) \quad \max\{c^T x \mid Ax \leq b; x \in \mathbb{Z}^n\}.$$

Such problems are called **integer linear programming** problems, for short, **ILP** problems. They consist of maximizing a linear function over the intersection $P \cap \mathbb{Z}^n$ of a polyhedron P with the set $\mathbb{Z}^n$ of integer vectors. It is obvious that one has always the following inequalities:

$$\begin{aligned} \max\{c^T x \mid Ax \leq b; x \in \mathbb{Z}^n\} &\leq \max\{c^T x \mid Ax \leq b\}, \\ \min\{b^T y \mid y \geq \mathbf{0}, y^T A = c^T; y \in \mathbb{Z}^m\} &\geq \min\{b^T y \mid y \geq \mathbf{0}, y^T A = c^T\}. \end{aligned}$$

It is easy to make an example where strict inequalities holds.
This implies that generally one will have strict inequality in the following duality relation:

$$\max\{c^T x \mid Ax \leq b; x \in \mathbb{Z}^n\} \leq \min\{b^T y \mid y \geq \mathbf{0}, y^T A = c^T; y \in \mathbb{Z}^m\}.$$

1.7 Integer polyhedra

Let $P \subseteq \mathbb{R}^n$ be a nonempty polyhedron. We define its **integer hull** P_I by

$$P_I = \text{conv}(P \cap \mathbb{Z}^n),$$

so this is the convex hull of the intersection between P and the lattice $\mathbb{Z}^n$ of integer points. Note that P_I may be empty although P is not.

Proposition 1.7.1. *If P is bounded, then P_I is a polyhedron.*

Proof. Assume that P is bounded and let $M \in \mathbb{N}$ be such that $\|x\| \leq M$ for all $x \in P$, so $|x_i| \leq M$ for all $i = 1, \ldots, n$. It follows that $P \cap Z^n \subseteq \{-M, -M+1, \ldots, M-1, M\}^n$, hence P contains a finite number of integer points, and therefore P_I is a polytope. By the finite basis theorem for polytopes (Theorem 1.5.4), we get that P_I is a polyhedron. □

Definition 1.7.2. *A polyhedron is called **integer** if $P = P_I$.*

An equivalent description of integer polyhedra is given by the following result (see e.g., [1, Proposition 5.4, p. 113]).

Theorem 1.7.3. *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a nonempty polyhedron. The following are equivalent:*

(i) P is integer.

(ii) For each $c \in \mathbb{R}^n$, the LP problem $\max\{c^T x \mid x \in P\}$ has an integer optimal solution if it is bounded.

As an immediate consequence, it follows that if a polyhedron $P = \{x \mid Ax \leq b\}$ is integer and the LP $\max\{c^T x \mid Ax \leq b\}$ is bounded, we have that

$$\max\{c^T x \mid Ax \leq b; x \in \mathbb{Z}^n\} = \max\{c^T x \mid Ax \leq b\}.$$

1.8 Totally unimodular lattices

Total unimodularity of matrices is an important tool in integer linear programming.

Definition 1.8.1. *A matrix A is called* ***totally unimodular*** *(TU) if each square submatrix of A has determinant equal to $0, +1$, or -1.*

In particular, each entry of a totally unimodular matrix is $0, +1$, or -1. Obviously, every submatrix of a TU matrix is also TU.
The property of total unimodularity is preserved under a number of matrix operations, for instance:

(i) transpose;

(ii) augmenting with the identity matrix;

(iii) multiplying a row or column by -1;

(iv) interchanging two rows or columns;

(v) duplication of rows or columns.

In order to determine if a matrix is *TU*, the following criterion due to Ghouila and Houri (1962) is useful.

Proposition 1.8.2. *Let $A \in \mathbb{R}^{m \times n}$. The following are equivalent:*

(i) A is TU.

(ii) Each collection R of rows of A can be partitioned into classes R_1 and R_2 such that the sum of rows in R_1 minus the sum of rows in R_2 is a vector with entries $0, -1, 1$ only.

(iii) *Each collection C of columns of A can be partitioned into classes C_1 and C_2 such that the sum of columns in C_1 minus the sum of columns in C_2 is a vector with entries* $0, -1, 1$ *only.*

Proof. See e.g. [8, Theorem 19.3]. □

Let us detail (ii) from the above proposition. It says that each collection R of rows of $A = (a_{ij})$ can be partitioned into classes R_1 and R_2 such that for all $j = 1, \ldots, n$, if we define

$$x_j := \sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij},$$

then $x_j \in \{0, -1, 1\}$.
A link between total unimodularity and integer linear programming is given by the following fundamental result.

Theorem 1.8.3. *Let $A \in \mathbb{R}^{m \times n}$ be a TU matrix and let $b \in \mathbb{Z}^m$. Then the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is integer.*

Proof. See [1, Theorem 5.7]. □

An important converse result is due to Hoffman and Kruskal (1956):

Theorem 1.8.4. *Let $A \in \mathbb{R}^{m \times n}$. Then A is TU if and only if the polyhedron $P = \{x \in \mathbb{R}^n \mid x \geq \mathbf{0}, Ax \leq b\}$ is integer for every $b \in \mathbb{Z}^m$.*

Proof. See [10, Corollary 8.2a, p. 137]. □

It follows that each linear programming problem with integer data and totally unimodular constraint matrix has integer optimal primal and dual solutions:

Proposition 1.8.5. *Let $A \in \mathbb{R}^{m \times n}$ be a TU matrix, let $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. Assume that the primal LP $\max\{c^T x \mid Ax \leq b\}$ and dual LP $\min\{b^T y \mid y \geq \mathbf{0}, y^T A = c^T\}$ are bounded. Then they have integer optimal solutions.*

Proof. Exercise. □

Proposition 1.8.6. *Let $A \in \mathbb{R}^{m \times n}$ be a TU matrix, let b, b', d, d' be vectors in $(\mathbb{Z} \cup \{-\infty, +\infty\})^m$ with $b \leq b'$ and $d \leq d'$. Then*

$$P = \{x \in \mathbb{R}^n \mid b \leq Ax \leq b', d \leq x \leq d'\}$$

is an integer polyhedron.

Proof. Exercise. □

1.9 Polyhedral combinatorics

A $\{0,1\}$-**valued vector** is a vector with all entries in $\{0,1\}$. An **integer vector** is a vector with all entries integer. If E is a nonempty finite set, we identify the concept of a function $x : E \to \mathbb{R}$ with that of a vector x in $\mathbb{R}^E$. Its components are denoted equivalently by $x(e)$ or x_e. An **integer function** is an integer-valued function.

A **set system** is a pair $(E, \mathcal{F})$, where E is a nonempty **finite** set and $\mathcal{F}$ is a family of subsets of E, called the **feasible sets**. Let $w : E \to \mathbb{R}_+$ be a **weight function**. Define

$$w(X) := \sum_{e \in X} w(e) \quad \text{for each } X \in \mathcal{F}.$$

Thus, $w(X)$ is the total weight of the elements in X. Then

$$\text{maximize}\{w(X) \mid X \in \mathcal{F}\} \quad \text{or} \quad \text{minimize}\{w(X) \mid X \in \mathcal{F}\} \tag{1.4}$$

are **combinatorial optimization problems**.

For a subset $X \subseteq E$, the **incidence vector** of X (with respect to E) is the vector $\chi^X \in \{0,1\}^E$ defined as

$$\chi^X(e) = \begin{cases} 1 & \text{if } e \in X \\ 0 & \text{if } e \notin X. \end{cases}$$

Thus, the incidence vector χ^X is a vector in the space $\mathbb{R}^E$. Considering the weight function w also as a vector in $\mathbb{R}^E$, it follows that for every $x \in \mathbb{R}^E$,

$$w^T \chi^X = \sum_{e \in E} w(e) \chi^X(e) = \sum_{e \in X} w(e) = w(X).$$

Proposition 1.9.1. *Let $P := conv\{\chi^X \mid X \in \mathcal{F}\}$ be the convex hull (in $\mathbb{R}^E$) of the incidence vectors of the elements of $\mathcal{F}$. Then*

$$\max\{w^T x \mid x \in P\} = \max\{w(X) \mid X \in \mathcal{F}\}.$$

Proof. " $\geq$ " is trivial, since $w(X) = w^T \chi^X$ and $\chi^X \in P$.
" $\leq$ " P is the convex hull of finitely many vectors, hence it is a polytope. By Theorem 1.5.4, we get that P is a bounded polyhedron. Then the mapping

$$f : P \to \mathbb{R}, \quad f(x) = w^T x$$

is a continuous function on a bounded subset of $\mathbb{R}^n$. As a consequence, f is bounded and attains its maximum and minimum. Thus, the LP problem

$$\max\{w^T x \mid x \in P\}$$

is bounded and has an optimal solution x^*. As $x^* \in P$, there are $X_1, \ldots, X_k \in \mathcal{F}$ such that $x^* = \sum_{i=1}^k \lambda_i \chi^{X_i}$ for some $\lambda_1, \ldots, \lambda_k \geq 0, \sum_{i=1}^k \lambda_i = 1$. Since

$$w^T x^* = \sum_{i=1}^k \lambda_i w^T \chi^{X_i} = \sum_{i=1}^k \lambda_i w(X_i),$$

there exists at least one $j = 1, \ldots, k$ such that $w(X_j) \geq w^T x^*$. Thus, $\max\{w(X) \mid X \in \mathcal{F}\} \geq w^T x^*$. □

The previous result and Theorem 1.5.4 are the starting point of polyhedral combinatorics.

Chapter 2

Matchings in bipartite graphs

Let $G = (V, E)$ be a graph and $w : E \to \mathbb{R}_+$ be a weight function.

Definition 2.0.1. *A* ***matching*** $M \subseteq E$ *is a set of disjoint edges, i.e. such that every vertex of* V *is incident to at most one edge of* M.

We are interested in the following problem:

Maximum weight matching problem(MWMP): Find a matching M of maximum weight.

By letting $w(e) := 1$ for all $e \in E$, we obtain as a particular case the problem

Maximum matching problem: Find a matching M of maximum cardinality.

Thus, we want to solve

$$(MWMP) \quad \max\{w(M) \mid M \text{ matching in } G\}.$$

If we take $\mathcal{F}$ to be the set of matchings in G, we can apply Proposition 1.9.1 to conclude that (MWMP) is equivalent to the problem

$$\max\{w^T x \mid x \in \text{conv}\{\chi^M \mid M \text{ matching in } G\}.$$

The set

$$\text{conv}\{\chi^M \mid M \text{ matching in } G\}$$

is a polytope in $\mathbb{R}^E$, called the **matching polytope** of G and denoted by $P_{matching}(G)$. By Theorem 1.5.4, it is a bounded polyhedron:

$$P_{matching}(G) = \{x \in \mathbb{R}^E \mid Cx \leq d\}$$

for some matrix C and some vector d. Then (MWMP) is equivalent to

$$\max\{w^T x \mid Cx \leq d\}. \tag{2.1}$$

In this way we have formulated the original combinatorial problem as a linear programming problem. This enables us to apply linear programming methods to study the original problem.
The question at this point is, however, how to find the matrix C and the vector d. We know that C and d do exist, but we must know them in order to apply linear programming methods.

Let us give a solution for bipartite graphs.

2.1 (MWMP) for bipartite graphs

Definition 2.1.1. *A graph $G = (V, E)$ is* ***bipartite*** *if V admits a partition into two sets V_1 and V_2 such that every edge $e \in E$ has one end in V_1 and the other one in V_2.*
We say that $\{V_1, V_2\}$ is a ***bipartition*** *of G.*

Let us recall that the $V \times E$-**incidence matrix** of G is the $V \times E$-matrix $A = (a_{ve})_{v\in V, e\in E}$ defined as follows:

$$a_{ve} = \begin{cases} 1 & \text{if } e \in E(v), \\ 0 & \text{otherwise.} \end{cases}$$

In the above definition, $E(v)$ is the set of all edges in E at v. It follows that for all $v \in V$, $\sum_{e\in E} a_{ve} = \sum_{e\in E(v)} a_{ve} = d(v)$, where $d(v)$ is the degree of v.
The following characterization of bipartite graphs is very useful.

Proposition 2.1.2. *G is bipartite if and only if G contains no odd cycle (i.e. cycle of odd length).*

Proof. Exercise. □

Theorem 2.1.3. *A graph $G = (V, E)$ is bipartite if and only if its incidence matrix A is totally unimodular.*

Proof. " ⇒ " Assume that G is bipartite and let $\{V_1, V_2\}$ be a bipartition of G. We apply Proposition 1.8.2 to prove that A is TU. Let $R \subseteq V$ be the index set of an arbitrary collection of rows of A and define $R_1 := R \cap V_1$ and $R_2 := R \cap V_2$. Then R_1, R_2 form a partition of R. We have to prove that for every $e \in E$, if we define

$$a_e := \sum_{w\in R_1} a_{we} - \sum_{w\in R_2} a_{we},$$

then $a_e \in \{0, 1, -1\}$. Let $e = uv \in E$. We have the following cases:

(i) $u, v \notin R$. Then $a_{we} = 0$ for all $w \in R_1, R_2$. Hence $a_e = 0$.

(ii) $u \in R$ and $v \notin R$. If $u \in R_1$, then $\sum_{w \in R_1} a_{we} = a_{ue} = 1$ and $\sum_{w \in R_2} a_{we} = 0$. Thus, $a_e = 1$. We get similarly that, if $u \in R_2$, then $a_e = -1$.

(iii) $v \in R$ and $u \notin R$. Similarly.

(iv) $u, v \in R$. Then we can have either $u \in R_1, v \in R_2$ or $u \in R_2, v \in R_1$. Suppose that $u \in R_1$ and $v \in R_2$, the other case being similar. Then $\sum_{w \in R_1} a_{we} = a_{ue} = 1$ and $\sum_{w \in R_2} a_{we} = a_{ve} = 1$, so $a_e = 0$.

" $\Leftarrow$ " Assume that G is not bipartite. By Proposition 2.1.2, G has a cycle $C_k = v_0 v_1 \dots v_{k-1} v_0$, with k odd, $k \geq 3$. Let B the submatrix of A obtained by taking the rows $v_0, \dots v_{k-1}$ and the columns $v_0 v_1, \dots, v_{k-1} v_0$. Then B is the incidence matrix of C_k and one can easily see that $|det(B)| = 2$. It follows that A is not TU. □

Theorem 2.1.4. *The matching polytope $P_{matching}(G)$ of a bipartite graph G is equal to the set of all vectors $x \in \mathbb{R}^E$ satisfying:*

$$\begin{aligned} P_{matching}(G) &= \left\{ x \in \mathbb{R}^E \mid \ x_e \geq 0 \textit{ for each } e \in E \textit{ and } \textstyle\sum_{e \in E(v)} x_e \leq 1 \textit{ for each } v \in V \ \right\} \\ &= \{x \in \mathbb{R}^E \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\}, \end{aligned}$$

where A is the $V \times E$-incidence matrix of G, $\mathbf{0}$ is the constant 0-vector in $\mathbb{R}^V$ and $\mathbf{1}$ is the constant 1-vector in $\mathbb{R}^V$.

Proof. Denote $P := \{x \in \mathbb{R}^E \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\}$. We have to prove that $P_{matching}(G) = P$.
" $\subseteq$ " Since P is convex, it is enough to show that $\chi^M \in P$ for each matching M of G. This can be easily verified. Obviously, $\chi^M_e \geq 0$ for all $e \in E$. Furthermore, for every $v \in V$, we have that there is at most one edge $e \in E(v) \cap M$, hence $\sum_{e \in E(v)} \chi^M_e \leq 1$.
" $\supseteq$ " Since G is bipartite, we can apply Theorem 2.1.3 to conclude that its incidence matrix A is totally unimodular. The total unimodularity of A implies, by Theorem 1.8.4, that the polyhedron P is integer, hence $P = \text{conv}(P \cap \mathbb{Z}^E)$.

Claim: If $x \in P \cap \mathbb{Z}^E$, then $x = \chi^M$ for some matching M of G.
Proof of Claim: We have that $x_e \geq 0$ for all $e \in E$ and, from the second condition, $x_e \leq 1$ for all e. Since x is integer, it follows that x is a $\{0, 1\}$-valued vector. If we define $M := \{e \in E \mid x_e = 1\}$, we have that $x = \chi^M$. Let us prove that M is a matching of G. If $e_1, e_2 \in M$ are not disjoint, then there is some $v \in V$ such that $e_1, e_2 \in E(v)$. It follows that $\sum_{e \in E(v)} x_e \geq x_{e_1} + x_{e_2} = 2$, a contradiction. ■
It follows that $P = \text{conv}(P \cap \mathbb{Z}^E) \subseteq \text{conv}\{\chi^M \mid M \text{matching in } G\} = P_{matching}(G)$. □

Thus,

$$P_{matching}(G) = \{x \in \mathbb{R}^E \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} = \{x \in \mathbb{R}^E \mid Cx \leq d\},$$

where $C = \begin{pmatrix} -I_E \\ A \end{pmatrix}$ (with I_E the $E \times E$-identity matrix) and $d = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$.
We therefore can apply linear programming techniques to handle (MWMP). Thus we can find a maximum-weight matching in a bipartite graph in polynomial time, with any polynomial-time linear programming algorithm.

2.2 Min-max relations and König's theorem

We prove first a variant of the Strong Duality theorem 1.4.3.

Proposition 2.2.1 (Strong Duality - variant)**.** *Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Then*

$$\max\{c^T x \mid x \geq \mathbf{0}, Ax \leq b\} = \min\{y^T b \mid y \geq \mathbf{0}, y^T A \geq c^T\}.$$

(assuming both sets are nonempty).

Proof. Exercise. □

In the sequel, G is a bipartite graph and A is the $V \times E$ incidence matrix of G.
Applying Proposition 2.2.1, we get the following min-max relation:

Proposition 2.2.2.

$$\max\{w^T x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} = \min\{y^T \mathbf{1} \mid y \geq \mathbf{0}, y^T A \geq w^T\}$$

We have thus that

$$\max\{w(M) \mid M \text{ matching in } G\} \;=\; \min\{y^T \mathbf{1} \mid y \geq \mathbf{0}, y^T A \geq w^T\},$$

If we take $w(e) := 1$ for all e (i.e. $w = \mathbf{1}$ in $\mathbb{R}^E$), we get that

$$\max\{|M| \mid M \text{ matching in } G\} = \min\{y^T \mathbf{1} \mid y \geq \mathbf{0}, y^T A \geq \mathbf{1}\} \tag{2.2}$$

In the sequel, we show that we can derive from this König's matching theorem.

Definition 2.2.3. *A **vertex cover** of G is a set of vertices intersecting each edge.*

Theorem 2.2.4 (König (1931))**.** *The maximum cardinality of a matching in a bipartite graph is equal to the minimum cardinality of a vertex cover.*

Proof. We can apply Proposition 1.8.5 to conclude that $\min\{y^T\mathbf{1} \mid y \geq \mathbf{0}, y^TA \geq \mathbf{1}\}$ is attained by an integer optimal solution y^* and that $(y^*)^T\mathbf{1}$ is the maximum cardinality of a matching in G.
Remark that for every $y = (y_v)_{v\in V}$ and every edge $e = uw \in E$, we have that $(y^TA)_e = \sum_{v\in V} y_v a_{ve} = y_u + y_w$.

Claim: y^* is a $\{0,1\}$-valued vector.

Proof of Claim: Assume that there exists $v_0 \in V$ such that $y^*_{v_0} \geq 2$. Define then y' as follows: $y'_v = y^*_v$ for $v \neq v_0$ and $y'_{v_0} = 1$. Obviously $y' \geq \mathbf{0}$ and one can easily see that for every $e = uw \in V$, $(y'^TA)_e = y'_u + y'_w \geq \mathbf{1}$. On the other hand, $y'^T\mathbf{1} < (y^*)^T\mathbf{1}$, a contradiction. ■
Let $W \subseteq V$ be an arbitrary vertex cover of G and let $\chi^W \subseteq \mathbb{R}^V$ be its incidence vector. Then $(\chi^W)^T\mathbf{1} = |W|$ and $\chi^W \geq \mathbf{0}$. Furthermore, $((\chi^W)^TA)_e \geq 1$ for every edge e of G, since e has at least one end $v \in W$, so $\chi^W_v = 1$. It follows that we must have that $|W| = (\chi^W)^T\mathbf{1} \geq (y^*)^T\mathbf{1}$ for every vertex cover W of G.
Let us define $W_0 := \{v \in V \mid y^*_v = 1\}$. Then $y^* = \chi^{W_0}$ and $(y^*)^T\mathbf{1} = |W_0|$. It remains to prove that W_0 is a vertex cover of G. If $e \in G$ is arbitrary, then, since $((y^*)^TA)_e \geq 1$, there is $v \in V$ such that $y^*_v = 1$, i.e. $v \in W_0$. □

König's matching theorem is an example of a min-max formula that can be derived from a polyhedral characterization. The polyhedral description together with linear programming duality also gives a certificate of optimality of a matching M: to convince that a certain matching M has maximum size, it is possible and sufficient to display a vertex cover of size $|M|$. In other words, it yields a good characterization for the maximum-size matching problem in bipartite graphs.

One can also derive the weighted version of König's matching theorem:

Theorem 2.2.5 (Egerváry (1931))**.** *Let $G = (V,E)$ be a bipartite graph and $w : E \to \mathbb{N}$ be a weight function. The maximum weight of a matching in G is equal to the minimum value of $\sum_{v\in V} y_v$, where y ranges over all functions $y : V \to \mathbb{N}$ such that $y_u + y_v \geq w(e)$ for each edge $e = uv$ of G.*

Proof. Exercise. □

Chapter 3

Flows and cuts

This material is mostly from [9, Chapters 10,13] and [6, Chapter 8].

We assume that all directed graphs are loopless.
Convention: If E is a finite set and $g : E \to \mathbb{R}$ is a mapping, for any $F \subseteq E$, we define $g(F) = \sum_{x \in F} g(x)$.

Definition 3.0.1. *A **flow network** is a quadruple $N = (D, c, s, t)$, where $D = (V, A)$ is a directed graph, $s, t \in V$ are two distinguished points and $c : A \to \mathbb{R}_+$ is a **capacity** function.*

We say that s is the **source**, t is the **sink** and $c(a)$ is the **capacity** of the arc $a \in A$.

In the sequel, $N = (D, c, s, t)$ is a flow network.

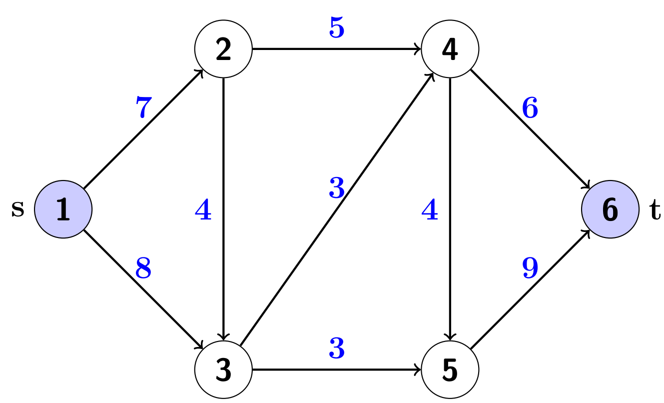

Figure 3.1: A flow network

Our main motivation is to transport as many units as possible simultaneously from s to t.

A solution to this problem will be called a **maximum flow**. We give in the sequel formal definitions.

Definition 3.0.2. *Let $f : A \to \mathbb{R}_+$ be a function. We say that*

*(i) f is a **flow** if $f(a) \leq c(a)$ for each $a \in A$.*

*(ii) f satisfies the **flow conservation law** at vertex $v \in V$ if*

$$\sum_{a \in \delta^{in}(v)} f(a) = \sum_{a \in \delta^{out}(v)} f(a) \tag{3.1}$$

*(iii) f is an s-t-**flow** if f is a flow satisfying the flow conservation law at all vertices except s and t.*

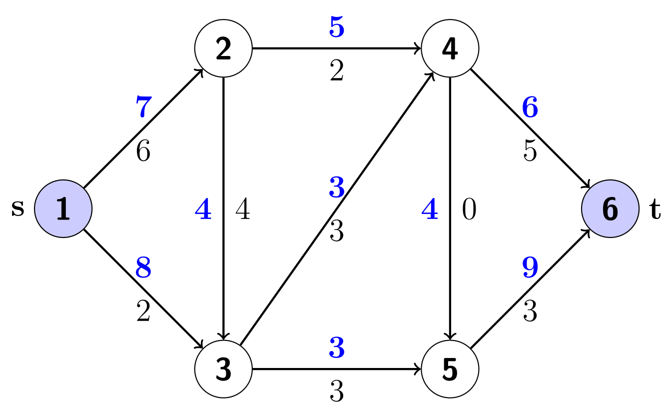

Figure 3.2: A flow network and a flow

Notation 3.0.3. *If $f : A \to \mathbb{R}_+$ is a flow and $v \in V$, we use the following notation:*

$$in_f(v) = \sum_{a \in \delta^{in}(v)} f(a) = f(\delta^{in}(v)), \quad out_f(v) := \sum_{a \in \delta^{out}(v)} f(a) = f(\delta^{out}(v)).$$

Thus, $in_f(v)$ is the amount of flow entering v and $out_f(v)$ is the amount of flow leaving v. The flow conservation law at v says that these should be equal.

Definition 3.0.4. *The **value** of an s-t flow f is defined as :*

$$value(f) := out_f(s) - in_f(s) = \sum_{a \in \delta^{out}(s)} f(a) - \sum_{a \in \delta^{in}(s)} f(a).$$

Hence, the value is the net amount of flow leaving s. One can prove that this is equal to the net amount of flow entering t (exercise!).
The **Maximum Flow Problem** is then

(Max-Flow): Find an s-t flow of maximum value.

An s-t flow of maximum value is also called simply **maximum flow**.

To formulate a min-max relation, we need the notion of a cut. A subset B of A is called a **cut** if $B = \delta^{out}(U)$ for some $U \subseteq V$. In particular, $\emptyset$ is a cut.

Definition 3.0.5. *An s-t **cut** is a cut $\delta^{out}(U)$ such that $s \in U$ and $t \notin U$. The **capacity** of an s-t cut $\delta^{out}(U)$ is*

$$c(\delta^{out}(U)) = \sum_{a \in \delta^{out}(U)} c(a).$$

The **Minimum Cut Problem** is then

(Min-Cut): Find an s-t cut of minimum capacity.

An s-t cut of minimum capacity is also called simply **minimum cut**.

One of the central results of flow network theory is the Max-Flow Min-Cut theorem, proved by Ford and Fulkerson [1954,1956b] for undirected graphs and by Dantzig and Fulkerson [1955,1956] for directed graphs.

Theorem 3.0.6 (Max-Flow Min-Cut theorem)**.** *Let $N = (D, c, s, t)$ be a network flow. Then the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut.*

We shall give two proofs to this theorem, one using polyhedra and linear programming, the other one using the Ford-Fulkerson algorithm.

Let us introduce first a useful notion. For any $f : A \to \mathbb{R}$, we define the **excess function** as the mapping

$$\text{excess}_f : \mathcal{P}(V) \to \mathbb{R}, \quad \text{excess}_f(U) = f(\delta^{in}(U)) - f(\delta^{out}(U)) \quad \text{for every } U \subseteq V. \tag{3.2}$$

Set $\text{excess}_f(v) := \text{excess}_f(\{v\})$ for every $v \in V$. Hence, if f is an s-t flow, the flow conservation law says that $\text{excess}_f(v) = 0$ for every $v \in V \setminus \{s, t\}$. Furthermore, the value of f is equal to $-\text{excess}_f(s)$.

Lemma 3.0.7. *(i) $\text{excess}_f(V) = 0$.*

(ii) For every $U \subseteq V$, $\text{excess}_f(U) = \sum_{v \in U} \text{excess}_f(v)$.

Proof. (i) Obviously, since $\delta^{in}(V) = \delta^{out}(V) = \emptyset$.

(ii) Let us denote the left-hand term of the equality with (L) and the right-hand term of the equality with (R). The equality follows by counting, for each $a \in A$, the multiplicity of $f(a)$ in (L) and (R).

Given an arbitrary arc $a = (x, y) \in A$, we have the following cases:

(a) $x, y \notin U$. Then $a \notin \delta^{in}(U) \cup \delta^{out}(U)$ and $a \notin \delta^{in}(v) \cup \delta^{out}(v)$ for any $v \in U$. Thus $f(a)$ does not appear in (L) or (R).

(b) $x, y \in U$. Then $a \notin \delta^{in}(U) \cup \delta^{out}(U)$, hence $f(a)$ does not appear in (L). Furthermore, we have that $a \in \delta^{in}(y) \cap \delta^{out}(x)$, so, $f(a) \in f(\delta^{in}(y))$ and $f(a) \in f(\delta^{out}(x))$, hence in (R) we have $-f(a) + f(a) = 0$.

(c) $x \in U, y \notin U$. Then $a \in \delta^{out}(U)$ and $a \notin \delta^{in}(U)$, hence in (L) we have $-f(a)$. Furthermore, $a \in \delta^{out}(x)$, so in (R) we have $-f(a)$ too.

(d) $x \notin U, y \in U$. Then $a \in \delta^{in}(U)$ and $a \notin \delta^{out}(U)$, hence in (L) we have $f(a)$. Furthermore, $a \in \delta^{in}(y)$, so in (R) we have $f(a)$ too.

□

A first result towards obtaining the max-min relation is the following "weak duality":

Proposition 3.0.8. *Assume that f is an s-t flow and that $\delta^{out}(U)$ is an s-t cut. Then*

$$value(f) \leq c(\delta^{out}(U)). \tag{3.3}$$

Equality holds if and only if $f(a) = 0$ for all $a \in \delta^{in}(U)$ and $f(a) = c(a)$ for all $a \in \delta^{out}(U)$.

Proof. Remark that, since $s \in U$ and $t \notin U$, we have by Lemma 3.0.7.(ii) that

$$\text{excess}_f(U) = \sum_{v \in U} \text{excess}_f(v) = \sum_{v \in U \setminus \{s\}} \text{excess}_f(v) + \text{excess}_f(s) = \text{excess}_f(s),$$

by the flow conservation law (3.1). It follows that

$$\begin{aligned} \text{value}(f) &= -\text{excess}_f(s) = -\text{excess}_f(U) = f(\delta^{out}(U)) - f(\delta^{in}(U)) \\ &\leq f(\delta^{out}(U)) \\ &\leq c(\delta^{out}(U)). \end{aligned}$$

with equality if and only if $f(\delta^{in}(U)) = 0$ and $f(\delta^{out}(U)) = c(\delta^{out}(U))$. Since $f(a) \geq 0$ for all $a \in A$, we have that $f(\delta^{in}(U)) = 0$ iff $f(a) = 0$ for all $a \in \delta^{in}(U)$. Since $f(a) \leq c(a)$ for all $a \in A$, we have that $f(\delta^{out}(U)) = c(\delta^{out}(U))$ iff $f(a) = c(a)$ for all $a \in \delta^{out}(U)$. □

As an immediate consequence, we get

Corollary 3.0.9. *If f is some s-t flow whose value equals the capacity of some s-t cut $\delta^{out}(U)$, then f is a maximum flow and $\delta^{out}(U)$ is a minimum cut.*

3.1 An LP formulation of the Maximum Flow Problem

Let us show that the Maximum Flow Problem has an LP formulation. We want to solve the problem

$$(\mathbf{Max-Flow}): \quad \max\{\text{value}(f) \mid f \text{ is an } s-t \text{ flow}\}.$$

As $f, c : A \to \mathbb{R}$, they can be seen as vectors in $\mathbb{R}^A$, hence we shall use the notation f_a, c_a for $f(a), c(a)$.
Let us recall that the **incidence matrix** (or $V \times A$ **incidence matrix**) of $D = (V, A)$ is the $V \times A$-matrix $M = (m_{va})_{v\in V, a\in A}$ defined as follows:

$$m_{va} = \begin{cases} 1 & \text{if } v \text{ is a head of } a \text{ (i.e. } a = (u, v) \text{ for some } u \in V) \\ -1 & \text{if } v \text{ is a tail of } a \text{ (i.e. } a = (v, u) \text{ for some } u \in V) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for every $v \in V$, we have that $m_{va} = 1$ if $a \in \delta^{in}(v)$, $m_{va} = -1$ if $a \in \delta^{out}(v)$ and $m_{va} = 0$ otherwise.

Proposition 3.1.1. *The incidence matrix M of a directed graph $D = (V, A)$ is totally unimodular.*

Proof. Exercise. □

For every $v \in V$ let us denote with $\mathbf{m}_v$ the v-th line of M. Then

$$\mathbf{m}_v f = \sum_{a\in A} m_{va} f_a = \sum_{a\in\delta^{in}(v)} f_a - \sum_{a\in\delta^{out}(v)} f_a = in_f(v) - out_f(v).$$

In particular,

$$\mathbf{m}_t f = in_f(t) - out_f(t) = out_f(s) - in_f(s) = \text{value}(f).$$

Let M_0 be the matrix obtained from M by deleting the rows $\mathbf{m}_s, \mathbf{m}_t$, corresponding to s and t. The fact that f satisfies the flow conservation law for all vertices $v \neq s, t$ can be written as $M_0 f = \mathbf{0}$. Then **(Max-Flow)** is equivalent with the following linear programming problem

$$(\mathbf{Max-Flow})_{LP}: \quad \max\{\mathbf{m}_t f \mid M_0 f = \mathbf{0}, \mathbf{0} \leq f \leq c\}.$$

It is obvious that $f \equiv \mathbf{0}$ is a feasible solution. Furthermore, $(\mathbf{Max-Flow})_{LP}$ is bounded, since $value(f) \leq \sum_{a\in\delta^{out}(s)} f_a \leq c(\delta^{out}(s))$. It follows from linear programming that

Proposition 3.1.2. *The Maximum Flow Problem always has an optimal solution.*

Another important consequence is the Integrity Theorem, due to Dantzig and Fulkerson [1955,1956]:

Theorem 3.1.3 (The Integrity theorem)**.** *If all capacities are integers, then there exists an integer flow of maximum value.*

Proof. We have that

$$\max\{\mathbf{m}_t f \mid M_0 f = \mathbf{0}, \mathbf{0} \le f \le c\} = \max\{\mathbf{m}_t f \mid \mathbf{0} \le M_0 f \le \mathbf{0}, \mathbf{0} \le f \le c\}.$$

Since M is totally unimodular, M_0 is also totally unimodular, as a submatrix of M. As c is an integer vector by hypothesis, we can apply Proposition 1.8.6 with $b = b' = \mathbf{0}$ and $d = \mathbf{0}, d' = c$ to conclude that the polyhedron

$$P = \{f \in \mathbb{R}^A \mid M_0 f = \mathbf{0}, \mathbf{0} \le f \le c\}$$

is integer. Apply now Proposition 1.7.3.(ii) to conclude that $\max\{\mathbf{m}_t f \mid x \in P\}$ has an integer optimal solution. □

3.1.1 Proof of the Max-Flow Min-Cut Theorem 3.0.6

First, let us remark that, by LP-duality, we have that

$$\begin{aligned} \max\{\mathbf{m}_t f \mid M_0 f = \mathbf{0}, \mathbf{0} \le f \le c\} &= \max\{(\mathbf{m}_t^T)^T f \mid C' f \le c'\} \\ &= \min\{c'^T w \mid w \ge \mathbf{0}, w^T C' = \mathbf{m}_t\} \\ &= \min\{c'^T w \mid w \ge \mathbf{0}, C'^T w = \mathbf{m}_t^T\}, \end{aligned}$$

where $C' = \begin{pmatrix} M_0 \\ -M_0 \\ I \\ -I \end{pmatrix}$ and $c' = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ c \\ \mathbf{0} \end{pmatrix}$.

Claim: There are integer vectors r, z such that $r \ge \mathbf{0}$, $z_s = 0$, $z_t = -1$, $z^T M + r^T \ge \mathbf{0}$ and $r^T c$ is the maximum value of an s-t flow.

Proof of Claim: (**Supplementary**)

Since C'^T is totally unimodular and $\mathbf{m}_t^T$ is an integer vector, we can apply Proposition 1.8.6 with $b = b' = \mathbf{m}_t^T$, $d = \mathbf{0}, d' = +\infty$ and Proposition 1.7.3.(ii) to conclude that $\min\{c'^T w \mid w \ge \mathbf{0}, C'^T w = \mathbf{m}_t^T\}$ has an integer optimal solution w^*.

Let $w^* = \begin{pmatrix} w^1 \\ w^2 \\ w^3 \\ w^4 \end{pmatrix}$. Then $w^{*T} c' = w^{3T} c$, $\quad w^1, w^2, w^3, w^4 \ge \mathbf{0}$ and $w^{1T} M_0 - w^{2T} M_0 + (w^{3T} - w^{4T}) = \mathbf{m}_t$. Denote, for simplicity, $w := w^1 - w^2$. Then $w \in \mathbb{Z}^{V \setminus \{s,t\}}$, $w^T M_0 + w^{3T} \ge$

$\mathbf{m}_t + w^{4^T} \geq \mathbf{m}_t$. Extend w to $z \in \mathbb{Z}^V$ by defining $z_t := -1, z_s := 0$ and $z_v := w_v$ for all $v \neq s, t$. Let us take $r := w^3$. Then $r \in \mathbb{Z}^A, r \geq \mathbf{0}$, $w^T M_0 + r^T \geq \mathbf{m}_t$ and

$$r^T c = w^{*T} c' = \min\{c'^T w \mid w \geq \mathbf{0}, C'^T w = \mathbf{m}_t^T\} = \max\{\mathbf{m}_t f \mid M_0 f = \mathbf{0}, \mathbf{0} \leq f \leq c\}.$$

It remains to prove that $z^T M + r^T \geq \mathbf{0}$, i.e. that for every $a = (u, v) \in A$, we have that $(z_v - z_u) + r_a \geq 0$. We have the following cases:

(i) $u = s, v = t$. Then $(w^T M_0 + r^T)_a = r_a + 0 \geq m_{ta} = 1$. Thus, $(z_t - z_s) + r_a = -1 + r_a \geq 0$.

(ii) $u = s, v \notin \{s, t\}$. Then $z_v = w_v$, $(w^T M_0 + r^T)_a = r_a + w_v \geq m_{ta} = 0$. It follows that $(z_v - z_s) + r_a = w_v + r_a \geq 0$.

(iii) $u = t, v = s$. Then $(w^T M_0 + r^T)_a = r_a + 0 \geq m_{ta} = -1$. Thus, $(z_s - z_t) + r_a = 1 + r_a \geq 0$.

(iv) $u = t, v \notin \{s, t\}$. Then $z_v = w_v$, $(w^T M_0 + r^T)_a = r_a + w_v \geq m_{ta} = -1$. Thus, $(z_v - z_t) + r_a = w_v + 1 + r_a \geq 0$.

(v) $u, v \notin \{s, t\}$. Then $z_u = w_u, z_v = w_v$, and $(z_v - z_u) + r_a = (w_v - w_u) + r_a = (w^T M_0 + r^T)_a \geq m_{ta} = 0$.

(vi) $u \notin \{s, t\}, v = s$. Then $z_u = w_u$, $(w^T M_0 + r^T)_a = -w_u + r_a \geq m_{ta} = 0$. Thus, $(z_s - z_u) + r_a = -w_u + r_a \geq 0$.

(vii) $u \notin \{s, t\}, v = t$. Then $z_u = w_u$, $(w^T M_0 + r^T)_a = -w_u + r_a \geq m_{ta} = 1$. Thus, $(z_t - z_u) + r_a = -1 - w_u + r_a \geq 0$.

■

Define now

$$U := \{v \in V \mid z_v \geq 0\}.$$

Then U is a subset of V containing s and not containing t, so $\delta^{out}(U)$ is an s-t cut.
Claim: $c(\delta^{out}(U)) \leq r^T c$.
Proof of Claim: We have that $c(\delta^{out}(U)) = \sum_{a \in \delta^{out}(U)} c(a)$.
Let $a = (u, v) \in \delta^{out}(U)$. Then $u \in U$ and $v \notin U$, hence $z_u \geq 0$ and $z_v \leq -1$ (since z is integer). Since $0 \leq (z^T M + r^T)_a = (z_v - z_u) + r_a$, we must have $r_a \geq z_u - z_v \geq -z_v \geq 1$. Thus,

$$\begin{aligned} r^T c &= \sum_{a \in A} r_a c(a) \geq \sum_{a \in \delta^{out}(U)} r_a c(a) \quad \text{since } r, c \geq \mathbf{0} \\ &\geq \sum_{a \in \delta^{out}(U)} c(a) = c(\delta^{out}(U)). \end{aligned}$$

■

Thus, we have found an s-t cut with capacity less or equal than the maximum value of an s-t flow. Apply now Proposition 3.0.8 to conclude that the Max-Flow Min-Cut Theorem 3.0.6 holds.

3.2 Ford-Fulkerson algorithm

In the following, $D = (V, A)$ is a digraph, (D, c, s, t) is a flow network.
We define first the concepts of **residual graph** and **augmenting path**, which are very important in studying flows.
For each arc $a = (u, v) \in A$, we define a^{-1} to be a new arc from v to u. We call a^{-1} the **reverse** arc of a and vice versa. For any $B \subseteq A$, let $B^{-1} = \{a^{-1} \mid a \in B\}$.
We consider in the sequel the digraph $\overline{D} = (V, A \cup A^{-1})$. Note that if $a = (u, v) \in A$ and $a' = (v, u) \in A$, then a^{-1} and a' are two distinct parallel arcs in $\overline{D}$. We shall usually denote the arcs of $\overline{D}$ with $e, e_0, e_1, \ldots$.

Definition 3.2.1. *Let $f : A \to \mathbb{R}_+$ be an s-t flow.*

*(i) The **residual capacity** c_f associated to f is defined by*

$$c_f : A(\overline{D}) \to \mathbb{R}_+, \quad c_f(e) = \begin{cases} c(a) - f(a) & \text{if } e = a \in A \\ f(a) & \text{if } e = a^{-1}, a \in A. \end{cases}$$

*(ii) The **residual graph** is the graph $D_f = (V, A(D_f))$, where*

$$A(D_f) = \{e \in A(\overline{D}) \mid c_f(e) > 0\} = \{a \in A \mid c(a) > f(a)\} \cup \{a^{-1} \mid a \in A, f(a) > 0\}.$$

*(iii) An f-**augmenting path** is an s-t path in the residual graph D_f.*

Let P be an s-t path in D_f. The following notation will be useful in the sequel:

$$A^{-1}(P) := \{a \in A \mid a^{-1} \in A(P)\}.$$

We define $\chi^P : A \to \mathbb{R}$ as follows: for every $a \in A$,

$$\chi^P(a) = \begin{cases} 1 & \text{if } a \in A(P) \\ -1 & \text{if } a \in A^{-1}(P) \text{ (i.e. } a^{-1} \in A(P)) \\ 0 & \text{otherwise.} \end{cases}$$

For $\gamma \geq 0$, let us denote

$$f_P^\gamma : A \to \mathbb{R}, \quad f_P^\gamma = f + \gamma\chi^P.$$

Then for every $a \in A$, we have that

$$f_P^\gamma(a) = \begin{cases} f(a) + \gamma & \text{if } a \in A(P) \\ f(a) - \gamma & \text{if } a \in A^{-1}(P) \\ f(a) & \text{otherwise.} \end{cases}$$

Lemma 3.2.2. *If $\gamma = \min_{e\in A(P)} c_f(e)$, then f_P^γ is an s-t flow with value$(f_P^\gamma) =$ value$(f)+\gamma$.*

Proof. We denote for simplicity $g := f_P^\gamma$. First, let us remark that $\gamma > 0$, since $c_f(e) > 0$ for every arc e of the residual graph. Furthermore, $\gamma = \min\{\min\{c(a) - f(a) \mid a \in A(P)\}, \min\{f(a) \mid a \in A^{-1}(P)\}\}$. It follows that $f(a)+\gamma \leq c(a)$ if $a \in A(P)$ and $0 \leq f(a)-\gamma$ if $a \in A^{-1}(P)$. As a consequence, $0 \leq g(a) \leq c(a)$ for all $a \in A$.
Assume that $P = v_0v_1 \ldots v_kv_{k+1}, k \geq 0, v_0 := s, v_{k+1} := t$.
Since $\chi^P(a) = 0$ for all $a \notin A(P) \cup A^{-1}(P)$, it follows that for every $v \in V$, we have that

$$\begin{aligned}
in_g(v) &= \sum_{a\in\delta^{in}(v)} g(a) = \sum_{a\in\delta^{in}(v)} f(a) + \gamma \sum_{a\in\delta^{in}(v)} \chi^P(a) = in_f(v) + \gamma \sum_{a\in\delta^{in}(v)} \chi^P(a) \\
&= in_f(v) + \sum_{a\in L(v)} \chi^P(a), \\
out_g(v) &= \sum_{a\in\delta^{out}(v)} g(a) = \sum_{a\in\delta^{out}(v)} f(a) + \gamma \sum_{a\in\delta^{out}(v)} \chi^P(a) = out_f(v) + \gamma \sum_{a\in\delta^{out}(v)} \chi^P(a) \\
&= out_f(v) + \sum_{a\in R(v)} \chi^P(a),
\end{aligned}$$

where $L(v) := \delta^{in}(v) \cap (A(P) \cup A^{-1}(P)), R(v) := \delta^{out}(v) \cap (A(P) \cup A^{-1}(P))$. Thus,

$$out_g(v) - in_g(v) = out_f(v) - in_f(v) + \gamma \left(\sum_{a\in R(v)} \chi^P(a) - \sum_{a\in L(v)} \chi^P(a) \right).$$

Claim 1: value$(g) =$ value$(f) + \gamma$.
Proof of Claim:

$$\text{value}(g) = \text{value}(f) + \gamma \left(\sum_{a\in R(s)} \chi^P(a) - \sum_{a\in L(s)} \chi^P(a) \right)$$

Let $e := (s, v_1) \in A(P)$. We have two cases:

(i) $e \in A$. Then $L(s) = \emptyset, R(s) = \{e\}, \chi^P(e) = 1$.

(ii) $e \in A^{-1}(P)$, so $e = a^{-1}$ with $a = (v_1, s) \in A$. Then $L(s) = \{a\}, \chi^P(a) = -1, R(s) = \emptyset$.

In both cases, one gets value$(g) =$ value$(f) + \gamma$. ■
Claim 2: g satisfies the flow conservation law at every $v \in V \setminus \{s, t\}$.
Proof of Claim: Let $v \in V, v \neq s, t$. Then

$$out_g(v) - in_g(v) = \gamma \left(\sum_{a\in R(v)} \chi^P(a) - \sum_{a\in L(v)} \chi^P(a) \right),$$

since f satisfies the flow conservation law at v. Thus, we have to prove that

$$\sum_{a\in R(v)} \chi^P(a) - \sum_{a\in L(v)} \chi^P(a) = 0. \tag{3.4}$$

If $v \notin P$, then this is obvious, since $\chi^P(a) = 0$ for every arc $a \in A$ incident with v. If $P = st$, then we do not have what to prove. Assume now that $v = v_i$ for some $i = 1, \ldots, k$, where $k \geq 1$. Let $e_1 = (v_{i-1}, v_i), e_2 = (v_i, v_{i+1})$ be the arcs incident with v in P. We have the following cases:

(i) $e_1, e_2 \in A$. Then $L(v) = \{e_1\}, \chi^P(e_1) = 1, R(v) = \{e_2\}, \chi^P(e_2) = 1$.

(ii) $e_1 \in A$, $e_2 = a_2^{-1}$, with $a_2 = (v_{i+1}, v_i) \in A$. Then $L(v) = \{e_1, a_2\}, \chi^P(e_1) = 1, \chi^P(a_2) = -1, R(v) = \emptyset$.

(iii) $e_2 \in A$, $e_1 = a_1^{-1}$, with $a_1 = (v_i, v_{i-1}) \in A$. Then $L(v) = \emptyset, R(v) = \{e_2, a_1\}, \chi^P(e_2) = 1, \chi^P(a_1) = -1$.

(iv) $e_1 = a_1^{-1}$ and $e_2 = a_2^{-1}$, with $a_1 = (v_i, v_{i-1}) \in A, a_2 = (v_{i+1}, v_i) \in A$. Then $L(v) = \{a_2\}, \chi^P(a_2) = -1, R(v) = \{a_1\}, \chi^P(a_1) = -1$.

In all cases, one gets (3.4). ■

Thus, the proof is concluded. □

To **augment** f along P by γ means to replace the flow f with the flow f_P^γ. Using these concepts, the following algorithm for the Maximum Flow Problem, due to Ford and Fulkerson [1957], is natural.

Ford-Fulkerson Algorithm

Input: A flow network (D, c, s, t)

Output: An s-t flow of maximum value.

Step 1 Set $f(a) := 0$ for all $a \in A(D)$.

Step 2 Find an f-augmenting path P. **If** none exists **then stop**.

Step 3 Compute $\gamma := \min_{e\in A(P)} c_f(e)$. Augment f along P by γ and **go to** Step 2.

As we proved in Lemma 3.2.2, the choice of γ guarantees that f continues to be a flow. To find an f-augmenting path, we just have to find any s-t-path in the residual graph D_f.
We will see that when the algorithm stops, then f is indeed an s-t flow of maximum value. First, we prove the following important result.

Proposition 3.2.3. *Suppose that f is an s-t flow such that the residual graph D_f has no s-t paths. If we let S be the set of vertices reachable in D_f from s, then $\delta^{out}(S)$ is an s-t cut in D such that*

$$value(f) = c(\delta^{out}(S)).$$

In particular, f is an s-t flow of maximum value and $\delta^{out}(S)$ is an s-t cut in D of minimum capacity.

Proof. Since D_f has no s-t paths, it follows that $t \notin S$. Since $s \in S$, we get that $\delta^{out}(S)$ is an s-t cut in D. We apply Proposition 3.0.8 to get the result. Remark that if $a \in \delta^{out}_A(S)$, then $a = (u, v)$ with $u \in S$ and $v \notin S$, so v is not reachable in D_f from s. As a consequence, $a \notin A(D_f)$, hence $f(a) = c(a)$. If $a \in \delta^{in}(S)$, then $a = (u, v)$ with $u \notin S$ and $v \in S$, so u is not reachable in D_f from s. As a consequence, $a^{-1} = (v, u) \notin A(D_f)$, hence $f(a) = 0$.
It follows by Proposition 3.0.8 that value$(f) = c(\delta^{out}(S))$. As a consequence, f is an s-t flow of maximum value and $\delta^{out}(S)$ is an s-t cut in D of minimum capacity. □

Theorem 3.2.4. *An s-t flow f has maximum value if and only if there is no f-augmenting path.*

Proof. "$\Rightarrow$" If there is an f-augmenting path p, then Step 3 of the Ford-Fulkerson algorithm computes an s-t flow of greater value than f, hence f is not of maximal value.
"$\Leftarrow$" By Proposition 3.2.3. □

By linear programming (Proposition 3.1.2), we know that there exists a maximal s-t flow. Then, as an immediate consequence of the previous two results, we get the Max-Flow Min-Cut Theorem 3.0.6.

Another important consequence is:

Theorem 3.2.5. *If all capacities are integer (i.e. $c : A \to \mathbb{Z}_+$), then the Ford-Fulkerson algorithm terminates and the s-t flow of maximum value is integer.*

Proof. Let
$$N := c(\delta^{out}(s)) \in \mathbb{Z}_+.$$
Let f_i be the s-t flow at iteration i. One can easily see by induction on i that f_i is integer and that value$(f_{i+1}) \geq$ value$(f_i) + 1$. Since for any s-t flow f we have that value$(f) \leq N$, it follows that the Ford-Fulkerson algorithm terminates after at most N iterations. Since the flow at every iteration is integer, it follows that the maximal flow is also integer. □

One can easily see that the Ford-Fulkerson algorithm terminates also when all capacities are rational. However, if we allow irrational capacities, the algorithm might not terminate at all (see [9, Section 10.4a]).

3.3 Circulations

Let $D = (V, A)$ be a digraph.

Definition 3.3.1. *A mapping $f : A \to \mathbb{R}$ is a **circulation** if for each $v \in V$, one has*

$$\sum_{a\in\delta^{in}(v)} f(a) = \sum_{a\in\delta^{out}(v)} f(a). \tag{3.5}$$

Thus, f satisfies the flow conservation law (3.1) at every vertex $v \in V$. Hence, f is a circulation if and only if $in_f(v) = out_f(v)$ for all $v \in V$ if and only if $\text{excess}_f(v) = 0$ for all $v \in V$.

We point out the following useful result, whose proof is immediate.

Lemma 3.3.2. *Assume that $v \in V$ and $f_1, \ldots, f_n : A \to \mathbb{R}$ are mappings satisfying the flow conservation law (3.1) at v. Then any linear combination of $f_1, \ldots, f_n$ satisfies (3.1) at v.*

Proof. Exercise. □

Let us recall that for any subgraph D' of D, $\chi^{D'}$ denotes its characteristic function, defined by

$$\chi^{D'} : A \to \{0, 1\}, \quad \chi^{D'}(a) = \begin{cases} 1 & \text{if } a \in D' \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.3.3. *(i) Any linear combination of circulations is a circulation.*

(ii) If C is a circuit in D, then χ^C is a nonnegative circulation.

Proof. (i) By Lemma 3.3.2.

(ii) Let $C := v_0 v_1 \ldots v_{k-1} v_k v_0, k \geq 1$ be a circuit in D. Then $\chi^C((v_0, v_1)) = \chi^C((v_1, v_2)) = \ldots = \chi^C((v_{k-1}, v_k)) = \chi^C((v_k, v_0)) = 1$ and $\chi^C(a) = 0$ for all the other arcs $a \in A$. For an arbitrary $v \in V$ we have the following cases:

(a) $v \notin C$. Then $in_{\chi^C}(v) = out_{\chi^C}(v) = 0$.

(b) $v \in C$, so $v = v_i$ for some $i = 0, \ldots, k$. Then

$$\begin{aligned} in_{\chi^C}(v_i) &= \sum_{a\in\delta^{in}(v_i)} \chi^C(a) = \chi^C(a_i) + 0 = 1, \\ out_{\chi^C}(v_i) &= \sum_{a\in\delta^{out}(v_i)} \chi^C(a) = \chi^C(b_i) + 0 = 1, \end{aligned}$$

where $a_i = \begin{cases} (v_k, v_0) & \text{if } i = 0 \\ (v_{i-1}, v_i) & \text{otherwise} \end{cases}$ and $b_i = \begin{cases} (v_k, v_0) & \text{if } i = k \\ (v_i, v_{i+1}) & \text{otherwise.} \end{cases}$

□

Definition 3.3.4. *The **support** of a mapping $f : A \to \mathbb{R}$ is the set*

$$supp(f) := \{a \in A \mid f(a) \neq 0\}.$$

If $supp(f) \neq \emptyset$, then $(V, supp(f))$ is a nontrivial subgraph of D.

Proposition 3.3.5. *Assume that there exists a nonnegative circulation f in D with nonempty support. Then $(V, supp(f))$ contains a circuit.*

Proof. By hypothesis, there exists $a = (u, v) \in A$ with $a \in supp(f)$, so $f(a) > 0$, since f is nonnegative. Take $v_0 := v$. Since $a \in \delta^{in}(v)$, we have that $in_f(v) \geq f(a) > 0$. It follows that $out_f(v) > 0$, so we must have $a_1 = (v, v_1) \in \delta^{out}(v)$ such that $f(a_1) > 0$. As D is loopless, we have that $v_1 \neq v$.
Since $a_1 \in \delta^{in}(v_1)$, we must have $a_2 = (v_1, v_2) \in \delta^{out}(v_1)$ with $f(a_2) > 0$. If $v_2 = v_0$, then we have found a circuit $C = v_0 v_1 v_0$ and we stop. If $v_2 \neq v_0$, then we reason similarly to get a sequence of different vertices $v_0, v_1, v_2, v_3, \ldots$ with $(v_i, v_{i+1}) \in supp(f), i = 0, 1, 2, \ldots,$. Since D is finite, we must stop after a finite number of steps. Thus, there exists N such that $v_N = v_i$ for some $i = 0, \ldots, N-2$. It follows that $C := v_i v_{i+1} \ldots v_{N-1} v_i$ is a circuit in $(V, supp(f))$. □

Proposition 3.3.6. *A function $f : A \to \mathbb{R}_+$ is a circulation if and only if there exist $N \in \mathbb{Z}_+$, positive real numbers $\mu_1, \ldots, \mu_N$ and circuits $C_1, \ldots, C_N$ in D such that*

$$f = \sum_{i=1}^{N} \mu_i \chi^{C_i}. \tag{3.6}$$

Furthermore, if f is integer, then the μ_i's can be chosen to be integer.

Proof. "$\Leftarrow$" By Lemma 3.3.3.
"$\Rightarrow$" We use induction on $|supp(f)|$. If $|supp(f)| = 0$, the result is trivial. So assume that $|supp(f)| > 0$. Then, by Proposition 3.3.5, the subgraph $(V, supp(f))$ of D contains a circuit C. Let $\mu := \min_{a \in A(C)} f(a) > 0$ and define

$$f' := f - \mu\chi^C, \quad \text{so } f'(a) = \begin{cases} f(a) - \mu & \text{if } a \in A(C) \\ f(a) & \text{otherwise} \end{cases}.$$

Then f' is a nonnegative circulation.
Claim: $|supp(f')| < |supp(f)|$.
Proof of Claim: Obviously, $supp(f') \subseteq supp(f)$. We show that the inclusion is strict. Take $a_0 \in A(C)$ with $f(a_0) = \mu$. Then $a_0 \in supp(f)$, but $f'(a_0) = 0$, hence $a_0 \notin supp(f')$. ■

Then by the induction hypothesis, there exist numbers $L \in \mathbb{Z}_+$, $\mu_1, \ldots, \mu_L > 0$ and circuits $C_1, \ldots, C_L$ in D such that

$$f' = \sum_{i=1}^{L} \mu_i \chi^{C_i}. \tag{3.7}$$

Take $N := L+1$, $\mu_N := \mu$ and $C_N := C$. Then the result follows. □

3.4 Flow Decomposition Theorem

In this section we give a proof of the Flow Decomposition theorem, due to Gallai [1958], Ford and Fulkerson [1962].

Theorem 3.4.1. *[Flow Decomposition Theorem]*
Let $D = (V, A)$ be a digraph, $N = (D, c, s, t)$ a flow network and f be an s-t-flow in N with value$(f) \geq 0$. Then there exist $K, L \in \mathbb{Z}_+$, positive numbers $w_1, \ldots, w_K, \mu_1, \ldots, \mu_L$, s-t paths $P_1, \ldots, P_K$ and circuits $C_1, \ldots, C_L$ in N such that

$$f = \sum_{i=1}^{K} w_i \chi^{P_i} + \sum_{j=1}^{L} \mu_j \chi^{C_j} \quad \textit{and} \quad \textit{value}(f) = \sum_{i=1}^{K} w_i.$$

Moreover, if f is integer then the w_i's, μ_j's can be chosen to be integer.

Proof. We have two cases:
Case 1: value$(f) = 0$. Then $in_f(v) = out_f(v)$ for all $v \in V$, hence f is a circulation. The result follows (with $K = 0$) by Proposition 3.3.6.

Case 2: value$(f) > 0$. We show that we can reduce the problem to Case 1. Consider a new vertex x and add arcs (x, s), (t, x), both carrying flow value(f). Formally, we define the graph $D' := (V', A')$, where $V' := V \cup \{x\}$, $A' = A \cup \{(x, s), (t, x)\}$ and a function

$$f' : A' \to \mathbb{R}, \quad f'(a) = \begin{cases} f(a) & \text{if } a \in A \\ \text{value}(f) & \text{otherwise.} \end{cases}$$

Claim: f' is a nonnegative circulation in D'.
Proof of Claim: It is obvious that f' satisfies the flow circulation law (3.1) at every vertex $v \in V' \setminus \{s, t, x\}$. Since

$$\begin{aligned} in_{f'}(x) &= \sum_{a \in \delta^{in}(x)} f'(a) = f'((t, x)) = \text{value}(f), \\ out_{f'}(x) &= \sum_{a \in \delta^{out}(x)} f'(a) = f'((x, s)) = \text{value}(f), \end{aligned}$$

f' satisfies (3.1) at x. Furthermore,

$$\begin{aligned}
in_{f'}(s) &= \sum_{a\in\delta^{in}(s)} f'(a) = f'((x,s)) + \sum_{a\in\delta_A^{in}(s)} f'(a) = \text{value}(f) + \sum_{a\in\delta_A^{in}(s)} f(a)\\
&= \text{value}(f) + in_f(s) = (out_f(s) - in_f(s)) + in_f(s) = out_f(s),\\
out_{f'}(s) &= \sum_{a\in\delta^{out}(s)} f'(a) = \sum_{a\in\delta_A^{out}(s)} f'(a) = \sum_{a\in\delta_A^{out}(s)} f(a) = out_f(s),
\end{aligned}$$

hence f' satisfies (3.1) at s. Finally,

$$\begin{aligned}
in_{f'}(t) &= \sum_{a\in\delta^{in}(t)} f'(a) = \sum_{a\in\delta_A^{in}(t)} f'(a) = \sum_{a\in\delta_A^{in}(t)} f(a) = in_f(t),\\
out_{f'}(t) &= \sum_{a\in\delta^{out}(t)} f'(t) = f'((t,x)) + \sum_{a\in\delta_A^{out}(t)} f'(a) = \text{value}(f) + \sum_{a\in\delta_A^{out}(t)} f(a)\\
&= \text{value}(f) + out_f(t) = (in_f(t) - out_f(t)) + out_f(t) = in_f(t),
\end{aligned}$$

hence f' satisfies (3.1) at t. ■

We can apply Proposition 3.3.6 to f' to get $K, L \in \mathbb{Z}_+$, positive numbers $w_1, \dots, w_K, \mu_1, \dots, \mu_L$, $F_1, \dots, F_K$ circuits in D' containing x and $C_1, \dots, C_L$ circuits in D such that

$$f' = \sum_{i=1}^{K} w_i \chi^{F_i} + \sum_{j=1}^{L} \mu_j \chi^{C_j}.$$

If F_i is a circuit in D' containing x, then we must have $F_i = P_i + (t,x) + (x,s)$ for some s-t path P_i. Furthermore, $\chi^{F_i}(a) = \chi^{P_i}(a)$ for all $a \in A$. It follows that

$$f = \sum_{i=1}^{K} w_i \chi^{F_i} + \sum_{j=1}^{L} \mu_j \chi^{C_j} = \sum_{i=1}^{K} w_i \chi^{P_i} + \sum_{j=1}^{L} \mu_j \chi^{C_j}.$$

Finally, let us remark that for all $j = 1, \dots, L$,

$$\text{value}(\chi^{C_j}) = out_{\chi^{C_j}}(s) - in_{\chi^{C_j}}(s) = 0,$$

since χ^{C_j} is a circulation, by Lemma 3.3.3.(ii). Furthermore, for all $i = 1, \dots, K$, $\text{value}(\chi^{P_i}) = 1$, since P_i is an s-t path. Hence, $\text{value}(f) = \sum_{i=1}^{K} w_i$.

□

Let us recall that two subgraphs of D are

(i) **vertex-disjoint** if they have no vertex in common;

(ii) **arc-disjoint** if they have no arc in common.

In general, we say that a family of k subgraphs ($k \geq 3$) is (vertex, arc)-disjoint if the k subgraphs are **pairwise** (vertex, arc)-disjoint, i.e. every two subgraphs from the family are (vertex, arc)-disjoint.

By taking $c : A \to \mathbb{R}_+, c(a) = 1$ for all $a \in A$, we obtain a network $N = (D, c, s, t)$ that has all capacities equal to 1. We say that N is a **unit capacity** network. Then, the capacity of any subset $B \subseteq A$ is its size, i.e. $c(B) = |B|$. Furthermore, any integer s-t-flow f in N is a $\{0, 1\}$-flow, i.e. $f : A \to \{0, 1\}$.
The Flow Decomposition Theorem 3.4.1 gives us in this case

Proposition 3.4.2. *Let $D = (V, A)$ be a digraph, $N = (D, s, t)$ be a unit capacity network and f be an s-t $\{0, 1\}$-flow in N with value$(f) \geq 0$. Then there exist $K, L \in \mathbb{Z}_+$, s-t paths $P_1, \ldots, P_K$ and circuits $C_1, \ldots, C_L$ in N such that*

$$f = \sum_{i=1}^{K} \chi^{P_i} + \sum_{j=1}^{L} \chi^{C_j} \quad \textit{and} \quad \textit{value}(f) = K.$$

Furthermore, the family $\{P_1, \ldots, P_K, C_1, \ldots, C_L\}$ is arc-disjoint.

Proof. Exercise. □

3.5 Minimum-cost flows

Let $D = (V, A)$ be a digraph and let $k : A \to \mathbb{R}$, called the **cost** function. For any function $f : A \to \mathbb{R}$, the **cost** of f is, by definition

$$\text{cost}(f) := \sum_{a \in A} k(a) f(a). \tag{3.8}$$

The following is the **minimum-cost flow problem** (or **min-cost flow problem**):

given: a flow network $N = (D, c, s, t)$, a cost function $k : A \to \mathbb{R}$ and a value $\varphi \in \mathbb{R}_+$
find: a minimum-cost s-t flow f in N of value φ.

This problem includes the problem of finding an s-t flow of maximum value that has minimum cost among all s-t flows of maximum value.

Assume that $d, c : A \to \mathbb{R}$ are mappings satisfying $d(a) \leq c(a)$ for each arc $a \in A$. We call d the **demand** mapping and c the **capacity** mapping.

Definition 3.5.1. *A circulation f is said to be* ***feasible*** *(with respect to the constraints d and c) if*

$$d(a) \leq f(a) \leq c(a) \quad \text{for each arc } a \in A.$$

We point out that it is quite possible that no feasible circulations exist.
The **minimum-cost circulation** problem is the following:

given: a digraph $D = (V, A)$, $d, c : A \to \mathbb{R}$ and a cost function $k : A \to \mathbb{R}$
find: a feasible circulation f of minimum cost.

One can easily reduce the minimum-cost flow problem to the minimum-cost circulation problem.
Let $a_0 := (t, s)$ be a new arc and define the extended digraph $D' := (V, A')$, where $A' = A \cup \{a_0\}$. For every $f : A \to \mathbb{R}$ and $\varphi \in \mathbb{R}$, let us denote

$$f_\varphi : A' \to \mathbb{R}, \quad f_\varphi(a_0) = \varphi, \; f_\varphi(a) = f(a) \text{ for all } a \in A.$$

Define $d(a_0) := c(a_0) := \varphi$, $k(a_0) := 0$, and $d(a) := 0$ for each arc $a \in A$.

Proposition 3.5.2. *The following are equivalent*

(i) $f' : A' \to \mathbb{R}$ is a minimum-cost feasible circulation in D'

(ii) $f' = f_\varphi$ for some minimum-cost s-t flow f in N of value φ.

Proof. It is obvious that a mapping $f' : A' \to \mathbb{R}$ is feasible w.r.t. d, c if and only if $f' = f_\varphi$ for some $f : A \to \mathbb{R}$ satisfying $0 \leq f \leq c$.
Claim: f_φ is a circulation in D' if and only if f satisfies the flow conservation law at all $v \neq s, t$ and value$(f) = \varphi$.
Proof of Claim: Remark that

(i) for all $v \neq s, t$, we have that $in_f(v) = in_{f_\varphi}(v)$ and $out_f(v) = out_{f_\varphi}(v)$,

(ii) $in_{f_\varphi}(s) = in_f(s) + \varphi$, $out_{f_\varphi}(s) = out_f(s)$

(iii) $out_{f_\varphi}(t) = out_f(t) + \varphi$, $in_{f_\varphi}(t) = in_f(t)$.

■

Thus, $f' : A' \to \mathbb{R}$ is a feasible circulation in D' if and only if $f' = f_\varphi$ for some s-t flow f in N of value φ.
Remark, finally, that

$$\text{cost}(f_\varphi) = \sum_{a \in A'} k(a) f_\varphi(a) = k(a_0) f_\varphi(a_0) + \sum_{a \in A} k(a) f_\varphi(a) = 0 + \sum_{a \in A} k(a) f(a) = \text{cost}(f).$$

□

Thus, a minimum-cost feasible circulation in D' gives a minimum-cost flow of value φ in the original flow network N.

3.5.1 Minimum-cost circulations and the residual graph

Let $D = (V, A)$ be a digraph, $d, c : A \to \mathbb{R}$, and f be a feasible circulation in D. Let $k : A \to \mathbb{R}$ be a cost function.

Recall the notation $\overline{D} = (V, A \cup A^{-1})$.

Definition 3.5.3. *(i) The **residual capacity** c_f associated to f is defined by*

$$c_f : A(\overline{D}) \to \mathbb{R}_+, \quad c_f(e) = \begin{cases} c(a) - f(a) & \text{if } e = a \in A \\ f(a) - d(a) & \text{if } e = a^{-1}, a \in A. \end{cases}$$

*(ii) The **residual graph** is the graph $D_f = (V, A(D_f))$, where*

$$A(D_f) = \{e \in A(\overline{D}) \mid c_f(e) > 0\} = \{a \in A \mid c(a) > f(a)\} \cup \{a^{-1} \mid a \in A, f(a) > d(a)\}.$$

We extend the cost function k to A^{-1} by defining

$$k(a^{-1}) := -k(a) \quad \text{for each } a \in A.$$

Lemma 3.5.4. *Let f', f be feasible circulations in D and define $g : A \cup A^{-1} \to \mathbb{R}$ as follows: for all $a \in A$,*

$$g(a) = \max\{0, f'(a) - f(a)\}, \quad g(a^{-1}) = \max\{0, f(a) - f'(a)\}.$$

Then

(i) g is a circulation in $\overline{D}$;

(ii) $cost(g) = cost(f') - cost(f)$;

(iii) $g(e) = 0$ for all $e \notin A(D_f)$.

Proof. One can easily see that $g(a) - g(a^{-1}) = f'(a) - f(a)$ for all $a \in A$.

(i) We get that

$$\begin{aligned} in_g(v) &= \sum_{e \in \delta^{in}_{\overline{D}}(v)} g(e) = \sum_{a \in \delta^{in}_D(v)} g(a) + \sum_{a \in \delta^{out}_D(v)} g(a^{-1}) \\ out_g(v) &= \sum_{e \in \delta^{out}_{\overline{D}}(v)} g(e) = \sum_{a \in \delta^{out}_D(v)} g(a) + \sum_{a \in \delta^{in}_D(v)} g(a^{-1}). \end{aligned}$$

Thus,

$$\begin{aligned} out_g(v) - in_g(v) &= \sum_{a\in\delta_D^{out}(v)} (g(a) - g(a^{-1})) - \sum_{a\in\delta_D^{in}(v)} (g(a) - g(a^{-1})) \\ &= \sum_{a\in\delta_D^{out}(v)} (f'(a) - f(a)) - \sum_{a\in\delta_D^{in}(v)} (f'(a) - f(a)) \\ &= out_{f'}(v) - out_f(v) - in_{f'}(v) + in_f(v) = 0. \end{aligned}$$

since f and f' are circulations.

(ii) We have that

$$\begin{aligned} \mathrm{cost}(g) &= \sum_{e\in A\cup A^{-1}} k(e)g(e) = \sum_{a\in A} k(a)g(a) + \sum_{a\in A} k(a^{-1})g(a^{-1}) \\ &= \sum_{a\in A} k(a)(g(a) - g(a^{-1})) = \sum_{a\in A} k(a)(f'(a) - f(a)) = \mathrm{cost}(f') - \mathrm{cost}(f). \end{aligned}$$

(iii) Let $e \notin A(D_f)$. We have two cases:

(a) $e = a \in A$. Then $c(a) = f(a)$, so $f'(a) \le f(a)$. It follows that $g(e) = g(a) = 0$.

(b) $e = a^{-1}, a \in A$. Then $d(a) = f(a)$, so $f'(a) \ge f(a)$. It follows that $g(e) = g(a^{-1}) = 0$.

□

Let C be a circuit in D_f. We define $\psi^C : A \to \mathbb{R}$ as follows: for every $a \in A$,

$$\psi^C(a) = \begin{cases} 1 & \text{if } a \text{ is an arc of } C \\ -1 & \text{if } a^{-1} \text{ is an arc of } C \\ 0 & \text{otherwise.} \end{cases}$$

For $\gamma \ge 0$, let us denote

$$f_C^\gamma : A \to \mathbb{R}, \quad f_C^\gamma = f + \gamma\psi^C.$$

Lemma 3.5.5. *Let* $\gamma := \min_{e\in A(C)} c_f(e)$. *Then* f_C^γ *is a feasible circulation with* $cost(f_C^\gamma) = cost(f) + \gamma cost(\psi^C)$.

Proof. Exercise. □

The following result is fundamental.

Theorem 3.5.6. *f is a minimum-cost feasible circulation if and only if each circuit of D_f has nonnegative cost.*

Proof. "$\Rightarrow$" Assume by contradiction that there exists a circuit C in D_f with negative cost. Applying Lemma 3.5.5, there exists $\gamma > 0$ such that f_C^γ is a feasible circulation with $\text{cost}(f_C^\gamma) < \text{cost}(f)$. It follows that the cost of f is not minimum, a contradiction.
"$\Leftarrow$" Suppose that each circuit in D_f has nonnegative cost. Let f' be any feasible circulation and define g as in Lemma 3.5.4. Then g is a circulation in $\overline{D}$, $g(e) = 0$ for all $e \notin A(D_f)$ and $\text{cost}(g) = \text{cost}(f') - \text{cost}(f)$.
We can apply Proposition 3.3.6 to get $L \in \mathbb{Z}_+$, $\mu_1, \ldots, \mu_L > 0$ and circuits $C_1, \ldots, C_L$ in $\overline{D}$ such that

$$g = \sum_{i=1}^{L} \mu_i \chi^{C_i}. \tag{3.9}$$

Claim: For each $i = 1, \ldots, L$, C_i is a circuit in D_f.
Proof of Claim: If $e \in C_i$, then $\chi^{C_i}(e) = 1$, so $g(e) \geq \mu_i > 0$. Thus, we must have $e \in A(D_f)$. ■

It follows that $\text{cost}(g) = \sum_{i=1}^{L} \mu_i \text{cost}(\chi^{C_i}) \geq 0$, so $\text{cost}(f') \geq \text{cost}(f)$. □

Theorem 3.5.6 gives us a method to improve a given circulation f:

> Choose a negative-cost circuit C in the residual graph D_f, and reset $f := f_C^\gamma$, where γ is as in Lemma 3.5.5.
> If no such circuit exists, f is a minimum-cost circulation.

It is not difficult to see that for rational data this leads to a finite algorithm.

3.6 Hofmann's circulation theorem

Let $D = (V, A)$ be a digraph. We consider mappings $d, c : A \to \mathbb{R}$ satisfying $d(a) \leq c(a)$ for each arc $a \in A$.
In the sequel, we shall prove Hoffman's circulation theorem, which gives a characterization of the existence of feasible circulations. We get this result as an application of the Max-Flow Min-Cut Theorem. We refer to [9, Theorem 11.2] for a direct proof.
We assume for simplicity that the constraints d, c are nonnegative. However, the below proof can be adapted to the general case.

Add to D two new vertices s and t and all arcs $(s, v), (v, t)$ for $v \in V$. We denote the new digraph by H. Thus, $V(H) = V \cup \{s, t\}$ and $A(H) = A \cup \{(s, v), (v, t) \mid v \in V\}$. We define a capacity function on H as follows:

$$\begin{aligned} c'(a) &= c(a) - d(a) \text{ for all } a \in A \\ c'((s,v)) &= d(\delta_A^{in}(v)) = \sum_{a \in \delta_A^{in}(v)} d(a) \text{ for all } v \in V \\ c'((v,t)) &= d(\delta_A^{out}(v)) = \sum_{a \in \delta_A^{out}(v)} d(a) \text{ for all } v \in V. \end{aligned}$$

Since $0 \leq d(a) \leq c(a)$ for all a, it follows that we have got a flow network $N = (H, c', s, t)$.

Lemma 3.6.1. *(i)* $c'(\delta^{out}(s)) = c'(\delta^{in}(t)) = d(A)$.

(ii) For any s-t flow g in N, value$(g) \leq d(A)$ and equality holds if and only if $g((s,v)) = c'((s,v))$ for all $v \in V$ if and only if $g((v,t)) = c'((v,t))$ for all $v \in V$.

Proof. (Supplementary)

(i)

$$\begin{aligned} c'(\delta^{out}(s)) &= \sum_{v \in V} c'((s,v)) = \sum_{v \in V} d(\delta_A^{in}(v)) = d(A) \\ c'(\delta^{in}(t)) &= \sum_{v \in V} c'((v,t)) = \sum_{v \in V} d(\delta_A^{out}(v)) = d(A). \end{aligned}$$

(ii) If we take $U_1 := \{s\}$ and $U_2 := V \cup \{s\}$, we have that $\delta^{out}(U_1) = \delta^{out}(s)$ and $\delta^{out}(U_2) = \delta^{in}(t)$, hence, by (i), $c'(\delta^{out}(U_1)) = c'(\delta^{out}(U_2)) = d(A)$. Apply now Proposition 3.0.8.

□

Theorem 3.6.2. *There exists a feasible circulation in D if and only if the maximum value of an s-t flow on N is $d(A)$.*

Proof. (Supplementary) "⇐" Let g be an s-t flow in N of maximum value $d(A)$. We define $f : A \to \mathbb{R}$ by

$$f(a) = g(a) + d(a) \quad \text{for all } a \in A.$$

We shall prove that f is a feasible circulation in D. Since $0 \leq g(a) \leq c'(a) = c(a) - d(a)$ for all $a \in A$, we get that f is feasible w.r.t. d, c. It remains to check the flow conservation law at every vertex $v \in V$. We have that

$$\begin{aligned} in_g(v) &= \sum_{a\in\delta_A^{in}(v)} g(a) + g((s,v)) = \sum_{a\in\delta_A^{in}(v)} g(a) + c'((s,v) \\ &= \sum_{a\in\delta_A^{in}(v)} f(a) - \sum_{a\in\delta_A^{in}(v)} d(a) + \sum_{a\in\delta_A^{in}(v)} d(a) = in_f(v) \\ out_g(v) &= \sum_{a\in\delta_A^{out}(v)} g(a) + g((v,t)) = \sum_{a\in\delta_A^{out}(v)} g(a) + c'((v,t) \\ &= \sum_{a\in\delta_A^{out}(v)} f(a) - \sum_{a\in\delta_A^{out}(v)} d(a) + \sum_{a\in\delta_A^{out}(v)} d(a) = out_f(v). \end{aligned}$$

Since g is an s-t flow, we have that $in_g(v) = out_g(v)$, so $in_f(v) = out_f(v)$.
"⇒" Let f be a feasible circulation in D. Define $g : A(H) \to \mathbb{R}$ as follows:

$$g(a) = f(a) - d(a) \text{ for all } a \in A, \quad g((s,v)) = c'((s,v)), \quad g((v,t)) = c'((v,t)).$$

As f is feasible, we have that $0 \le g \le c'$. As above, we get that g satisfies the flow conservation law at every vertex $v \in V \setminus \{s,t\}$. Finally,

$$\text{value}(g) = g(\delta^{out}(s)) = \sum_{v\in V} g((s,v)) = \sum_{v\in V} c'((s,v)) = d(A).$$

□

Theorem 3.6.3 (Hoffman's Circulation Theorem)**.** *There exists a feasible circulation in D if and only if for each subset U of V,*

$$\sum_{a\in\delta^{in}(U)} d(a) \le \sum_{a\in\delta^{out}(U)} c(a). \tag{3.10}$$

Proof. (Supplementary) "⇒" If there exists a feasible circulation f, then $\text{excess}_f(v) = 0$ for all $v \in V$. Thus, by Lemma 3.0.7.(ii), we get that for all $U \subseteq V$, $\text{excess}_f(U) = 0$, that is, $f(\delta^{in}(U)) = f(\delta^{out}(U))$. It follows that

$$\sum_{a\in\delta^{in}(U)} d(a) \le \sum_{a\in\delta^{in}(U)} f(a) = f(\delta^{in}(U)) = f(\delta^{out}(U)) = \sum_{a\in\delta^{out}(U)} f(a) \le \sum_{a\in\delta^{out}(U)} c(a).$$

"⇐" By Theorem 3.6.2 and the Max-Flow Min-Cut Theorem, there exists a feasible circulation in$D$ if and only if the maximum value of an s-t flow on N is $d(A)$ if and only if the minimum capacity of an s-t cut in N is $d(A)$.

We shall prove that if (3.10) holds for all $U \subseteq V$, then the minimum capacity of an s-t cut in N is $d(A)$.
Every s-t cut in N is of the form $\delta^{out}(U \cup \{s\})$, where $U \subseteq V$. Let us denote for simplicity

$$L_U := \sum_{a \in \delta_A^{out}(U)} c(a) - \sum_{a \in \delta_A^{in}(U)} d(a). \tag{3.11}$$

Claim: For every $U \subseteq V$, we have that $c'(\delta^{out}(U \cup \{s\})) = L_U + d(A)$.
Proof of Claim: Let $U \subseteq V$. Then

$$\begin{aligned} c'(\delta^{out}(U \cup \{s\})) &= \sum_{a \in \delta^{out}(U \cup \{s\})} c'(a) = \sum_{v \notin U} c'((s,v)) + \sum_{v \in U} c'((v,t)) + \sum_{a \in \delta_A^{out}(U)} c'(a) \\ &= \sum_{v \notin U} d(\delta_A^{in}(v)) + \sum_{v \in U} d(\delta_A^{out}(v)) + \sum_{a \in \delta_A^{out}(U)} c(a) - \sum_{a \in \delta_A^{out}(U)} d(a) \\ &= L_U + \left(\sum_{v \notin U} d(\delta_A^{in}(v)) + \sum_{v \in U} d(\delta_A^{out}(v)) + \sum_{a \in \delta_A^{in}(U)} d(a) - \sum_{a \in \delta_A^{out}(U)} d(a) \right). \end{aligned}$$

Let us denote

$$S_1 := \sum_{v \notin U} d(\delta_A^{in}(v)), \;\; S_2 := \sum_{v \in U} d(\delta_A^{out}(v)), \;\; S_3 := \sum_{a \in \delta_A^{in}(U)} d(a) \text{ and } S_4 := \sum_{a \in \delta_A^{out}(U)} d(a).$$

We have to prove that $S_1 + S_2 + S_3 - S_4 = d(A) = \sum_{a \in A} d(a)$. Let $a = (u_1, u_2) \in A$. We have four cases:

(i) $u_1, u_2 \in U$. Then $d(a)$ appears only in S_2.

(ii) $u_1, u_2 \notin U$. Then $d(a)$ appears in S_1.

(iii) $u_1 \in U$, $u_2 \notin U$. Then $d(a)$ appears in S_1, S_2, S_4

(iv) $u_1 \notin U$, $u_2 \in U$. Then $d(a)$ appears in S_3.

■

Since, by (3.10), $L_U \geq 0$ for all $U \subseteq V$, we have got that the capacity of any s-t cut in N is at least $d(A)$. Furthermore, $c'(\delta^{out}(s)) = d(A)$, hence there exists an s-t cut in N with capacity $d(A)$. The proof is concluded. □

As a consequence of the proofs above, one has moreover

Corollary 3.6.4. *If c and d are integer and there exists a feasible circulation f in D, then there exists an integer-valued feasible circulation f'.*

Chapter 4

Combinatorial applications

4.1 Menger's Theorems

We assume that $D = (V, A)$ is a digraph and $s, t \in V$. In this section we study the maximum number of pairwise disjoint s-t paths in D. One of the central results is a min-max theorem due to Menger [1927].
We give a proof of this result using networks flows and the Max-Flow Min-Cut Theorem. In the sequel, $N = (D, s, t)$ is a unit capacity network.

Proposition 4.1.1. *Let $k \in \mathbb{Z}_+$.*

(i) If N has an s-t $\{0,1\}$-flow f with value$(f) = k$, then D has k arc-disjoint s-t paths.

(ii) If D has k arc-disjoint s-t paths, then N has an s-t $\{0,1\}$-flow f with value$(f) = k$.

Proof. (i) Apply Proposition 3.4.2.

(ii) Let $P_1, \ldots, P_k$ be k arc-disjoint s-t paths in D and take $f := \chi^{P_1} + \ldots + \chi^{P_k}$. One can easily see that f is an s-t $\{0,1\}$-flow with value$(f) = k$ (exercise!).

□

Corollary 4.1.2. *The maximum number of arc-disjoint s-t paths in D coincides with the value of the maximum flow in N.*

Proof. Let M be the maximum number of arc-disjoint s-t paths and L be the value of the maximum flow. We have that $M \leq L$, by Proposition 4.1.1.(ii). By the Integrity Theorem 3.1.3, there exists an integer s-t flow f of maximum value L. Then f must be a $\{0,1\}$-flow. Apply now Proposition 4.1.1.(i) to conclude that $L \leq M$. □

An immediate consequence of the previous corollary and of the Max-Flow Min-Cut Theorem is Menger's Theorem:

Theorem 4.1.3 (Menger's theorem (directed arc-disjoint version))**.** *The maximum number of arc-disjoint s-t-paths is equal to the minimum size of an s-t-cut.*

In the sequel, we show how can we get other versions of Menger's theorem.

Definition 4.1.4. *A subset $B \subseteq A$ is said to be an s-t* disconnecting arc set *if B intersects each s-t path.*

If $B \subseteq A$ is an s-t disconnecting arc set, we also say simply that B is *s-t disconnecting* or that B is *s-t separating* or that B *disconnects* or *separates* s and t or that B is an *arc separator* for s and t (see [5, Section 7.1]).

Lemma 4.1.5. *(i) Each s-t cut is an s-t disconnecting arc set.*

(ii) Each s-t disconnecting arc set of minimum size is an s-t cut.

(iii) The minimum size of an s-t disconnecting arc set coincides with the minimum size of an s-t cut.

Proof. Exercise. □

As an immediate consequence of the previous proposition and Menger's Theorem 4.1.3 we get the following version:

Theorem 4.1.6. *The maximum number of arc-disjoint s-t-paths is equal to the minimum size of an s-t disconnecting arc set.*

Another version of Menger's Theorem is the variant on internally vertex-disjoint s-t-paths.

Definition 4.1.7. *Two s-t-paths are* internally vertex-disjoint *if they have no inner vertex in common.*

Definition 4.1.8. *A set U of vertices is called an s-t* vertex-cut *(or a* vertex separator *for s and t) if $s, t \notin U$ and each s-t-path intersects U.*

Theorem 4.1.9 (Menger's theorem (directed internally vertex-disjoint version))**.**
Let s and t be two nonadjacent vertices of D. Then the maximum number of internally vertex-disjoint s-t-paths is equal to the minimum size of an s-t vertex-cut.

Proof. Make a digraph D' as follows from D: replace any vertex $v \in V$ by two vertices v^{in}, v^{out} and make an arc (v^{in}, v^{out}); moreover, replace each arc (u, v) by (u^{out}, v^{in}). Thus,

$$V(D') = \{v^{in}, v^{out} \mid v \in A\} \text{ and } A(D') = \{(v^{in}, v^{out}) \mid v \in V\} \cup \{(u^{out}, v^{in}) \mid (u, v) \in A\}.$$

Since s and t are nonadjacent, $(s^{out}, t^{in}) \notin A(D')$.

If $P = sv_1 \dots v_k t$ is an s-t path in D, then

$$P' := s^{out} v_1^{in} v_1^{out} v_2^{in} v_2^{out} \dots v_{k-1}^{out} v_k^{in} v_k^{out} t^{in}$$

is an s^{out}-t^{in} path in D'. Furthermore, any s^{out}-t^{in} path in D' is of the form P' for some s-t path P in D.

Claim 1: Two s-t paths P, Q in D are internally vertex-disjoint if and only the s^{out}-t^{in} paths P', Q' in D' are arc-disjoint.

Proof of Claim: " $\Rightarrow$ " Assume that P', Q' have an arc a' in common. Then $a' = (v^{in}, v^{out})$ or $a' = (v^{out}, w^{in})$. In both cases, we must have that v is a common vertex of P, Q.
" $\Leftarrow$ " If P, Q have a common vertex v, then (v^{in}, v^{out}) is a common arc of P', Q'. ■

For any $U \subseteq V$, let $U' \subseteq A(D')$ be defined by

$$U' := \{(v^{in}, v^{out}) \mid v \in U\}.$$

Then U and U' have the same size.

Claim 2: Let $U \subseteq V \setminus \{s, t\}$. Then U is an s-t vertex-cut in D if and only if U' is an s^{out}-t^{in} disconnecting arc set in D'.

Proof of Claim: Remark that for any s-t path P in D and any $v \in V, v \neq s, t$, we have that $v \in P$ if and only if $(v^{in}, v^{out}) \in P'$. ■

Claim 3: There exists $U \subseteq V \setminus \{s, t\}$ such that U' is an s^{out}-t^{in} disconnecting arc set of minimum size.

Proof of Claim: If $B \subseteq A(D')$ is an s^{out}-t^{in} disconnecting arc set, remark that

(i) $B' = B \setminus \{(s^{in}, s^{out}), (t^{in}, t^{out})\}$ continues to be an s^{out}-t^{in} disconnecting arc set.

(ii) if B contains the arc (u^{out}, v^{in}) and one of the arcs $(u^{in}, u^{out}), (v^{in}, v^{out})$ for some $u, v \in V \setminus \{s, t\}$, then $B' = B \setminus \{(u^{out}, v^{in})\}$ continues to be an s^{out}-t^{in} disconnecting arc set.

(iii) if B contains both arcs $(s^{out}, v^{in}), (v^{in}, v^{out})$ for some $v \in V \setminus \{s, t\}$, then $B' = B \setminus \{(s^{out}, v^{in})\}$ continues to be an s^{out}-t^{in} disconnecting arc set.

(iv) if B contains both arcs $(u^{out}, t^{in}), (u^{in}, u^{out})$ for some $u \in V \setminus \{s, t\}$, then $B' = B \setminus \{(u^{out}, t^{in})\}$ continues to be an s^{out}-t^{in} disconnecting arc set.

Let $B \subseteq A(D')$ be an s-t disconnecting arc set of minimum size. Since B is minimal, we have that $(s^{in}, s^{out}), (t^{in}, t^{out}) \notin B$, by (i) above. If B contains an arc of the form (u^{out}, v^{in}), then we replace it with

$$B' := \begin{cases} B \setminus \{(u^{out}, v^{in})\} \cup \{(u^{in}, u^{out})\} & \text{if } u \neq s. \\ B \setminus \{(u^{out}, v^{in})\} \cup \{(v^{in}, v^{out})\} & \text{if } u = s, \end{cases}$$

which, by (ii)-(iv) above, is again an s^{out}-t^{in} disconnecting arc set and has the same size as B. By applying repeatedly this procedure we get the claim. ■

Claim 4: The minimum size of an s-t vertex-cut in D coincides with the minimum size of an s^{out}-t^{in} disconnecting arc set in D'.

Proof of Claim: Let m_1 be first minimum and m_2 be the second minimum.
If $U \subseteq V \setminus \{s,t\}$ is an s-t vertex-cut in D with $|U| = m_1$, then, by Claim 2, we have that U' is an s^{out}-t^{in} disconnecting arc set with $|U'| = |U| = m_1$. Thus, $m_1 \geq m_2$.
By Claim 3, there exists $W \subseteq V \setminus \{s,t\}$ such that W' is an s^{out}-t^{in} disconnecting arc set with $|W| = |W'| = m_2$. Since, by Claim 2, W is an s-t vertex-cut in D, we get that $m_2 \geq m_1$. ■
Apply now Theorem 4.1.6 for D', s^{out}, t^{in} to get the result. □

4.2 Maximum matching in bipartite graphs (Supplementary)

Let $G = (V,E)$ be a graph. Let us recall that a *matching* in G is a set $M \subseteq E$ of pairwise disjoint edges and a *vertex cover* of G is a set of vertices intersecting each edge of G. A *maximum matching* is a matching of maximum size and a *minimum vertex cover* is a vertex cover of minimum size. Let us define

$$\begin{aligned} \nu(G) &:= \text{the maximum size of a matching in } G, \\ \tau(G) &:= \text{the minimum size of a vertex cover in } G. \end{aligned}$$

These numbers are called the *matching number* and the *vertex cover number* of G, respectively. One can easily see that, for any graph G,

Lemma 4.2.1. $\nu(G) \leq \tau(G)$.

Proof. Exercise. □

However, if G is bipartite, equality holds, which is the content of König's Matching Theorem 2.2.4. In Section 2, we gave a proof of this theorem using linear programming methods. In the sequel, we give another proof using the directed internally vertex-disjoint version of Menger's Theorem.
For the rest of the section, we assume that G is a bipartite graph with classes X and Y. Thus, $X \cap Y = \emptyset, X \cup Y = V$ and $E \subseteq \{uv \mid u \in X, v \in Y\}$. We write also $G = (X \cup Y, E)$. We associate to the bipartite graph $G = (X \cup Y, E)$ a unit capacity network $N = (D, s, t)$ as follows. Let s, t be new vertices and consider the digraph $D = (V \cup \{s,t\}, A)$, where

$$A = \{(u,v) \mid uv \in E, u \in X, v \in Y\} \cup \{(s,v) \mid v \in X\} \cup \{(v,t) \mid v \in Y\}.$$

Proposition 4.2.2. *Let $k \in \mathbb{Z}_+$. The following are equivalent* *(i) N has an s-t $\{0,1\}$-flow with value k*

(ii) G has a matching of size k

(iii) D has k internally-vertex disjoint s-t paths.

Proof. "$(i) \Rightarrow (ii)$" Let f be an s-t $\{0,1\}$-flow with value$(f) = k$. Then there are exactly k vertices $u_1, \ldots, u_k \in X$ such that $f((s, u_i)) = 1$ for all $i = 1, \ldots, k$. Furthermore, by the flow conservation law and the fact that f is a $\{0,1\}$-flow, we get that for every i there exists a unique $v_i \in Y$ such that (u_i, v_i) is an arc in D with $f((u_i, v_i)) = 1$.
Claim: $v_i \neq v_j$ for $i \neq j$.
Proof of Claim: Assume that there are $i \neq j$ such that $v_i = v_j = v$. Then $(u_i, v), (u_j, v) \in A$, so $in_f(v) \geq f((u_i, v)) + f((u_j, v)) \geq 2$, while $out_f(v) = f(v, t) \leq 1$. We have got a contradiction. ■
Take $M := \{u_i v_i \mid i = 1, \ldots, k\}$. Then M is a matching of size k.
"(ii)⇒(ii)" Let $M = \{u_i v_i \mid i = 1, \ldots, k\}$ be a matching of size k. Let $P_i := s u_i v_i t$ for every $i = 1, \ldots, k$. Then $P_1, \ldots, P_k$ are k internally-vertex disjoint s-t paths.
"(iii)⇒(i)" Let $P_1, \ldots, P_k$ be internally-vertex disjoint s-t paths. Take $f := \chi^{P_1} + \ldots + \chi^{P_k}$. Then f is an s-t $\{0,1\}$-flow with value$(f) = k$. □

Proposition 4.2.3. *$\nu(G)$ coincides with the maximum value of a flow in N.*

Proof. Let F be the value of the maximum flow. We have that $\nu(G) \leq F$, by Proposition 4.2.2. By the Integrity Theorem 3.1.3, there exists an integer s-t flow f of maximum value F. Then f must be a $\{0,1\}$-flow. Apply now Proposition 4.2.2 to conclude that $F \leq \nu(G)$. □

Thus, we can apply the Ford-Fulkerson algorithm for the network N to find a maximum matching in G.

Theorem 4.2.4 (König (1931))**.** *If G is a bipartite graph, $\nu(G) = \tau(G)$.*

Proof. By Lemma 4.2.2, we have that $\nu(G)$ is equal to the maximum number of internally-vertex disjoint s-t paths in D.
One can easily see that $U \subseteq X \cup Y$ is a vertex cover in G iff U intersects every $uv \in E$ iff U intersects every path $P = suvt$ in D iff U is an s-t vertex cut in D.
Finally, apply Menger's Theorem 4.1.9 to get the result. □

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Appendix A

General notions

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ denote the sets of natural, integer, rational numbers, and real numbers, respectively. The subscript + restricts the sets to the nonnegative numbers:

$$\mathbb{Z}_+ = \{x \in \mathbb{Z} \mid x \geq 0\} = \mathbb{N}, \quad \mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \geq 0\}, \quad \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}.$$

Furthermore, $\mathbb{N}^*$ denotes the set of positive natural numbers, that is $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
If $m, n \in \mathbb{Z}_+$, we use sometimes the notations $[m, n] := \{m, m+1, \ldots, n\}$, $[n] := \{1, \ldots, n\}$. We also write $i = 1, \ldots, n$ instead of $i \in [n]$.

If X is a set, we denote by $\mathcal{P}(X)$ the collection of its subsets and by $[X]^2$ the collection of 2-element subsets of X, i.e. $[X]^2 = \{\{x, y\} \mid x, y \in X\}$.
If X is a finite set, the **size** of X or the **cardinality** of X, denoted by $|X|$ is the number of elements of X.

Let $m, n \in \mathbb{N}^*$. We denote by $\mathbb{R}^{m \times n}$ the set of $m \times n$-matrices with entries from $\mathbb{R}$. Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ be a matrix. The transpose of A is denoted by A^T. If $i = 1, \ldots, m$, we denote by $\mathbf{a}_i$ the ith row of A: $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n})$. If $I \subseteq \{1, \ldots, m\}$, we write A_I for the submatrix of A consisting of the rows in I only. Thus, $\mathbf{a}_i = A_{\{i\}}$. We denote by $0_{m,n}$ the zero matrix in $\mathbb{R}^{m \times n}$, by 0_n the zero matrix in $\mathbb{R}^{n \times n}$ and by I_n the identity matrix in $\mathbb{R}^{n \times n}$.

Let $n \in \mathbb{N}^*$. All vectors in $\mathbb{R}^n$ are column vectors. Let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n.$$

Then x is a matrix in $\mathbb{R}^{n \times 1}$ and its transpose x^T is a row vector, hence a matrix in $\mathbb{R}^{1 \times n}$.

Furthermore, for $I \subseteq \{1, \ldots, m\}$, x_I is the subvector of x consisting of the components with indices in I. If $a \in \mathbb{R}$, we denote by $\mathbf{a}$ the vector in $\mathbb{R}^n$ whose components are all equal to a.

Appendix B

Euclidean space $\mathbb{R}^n$

The Euclidean space $\mathbb{R}^n$ is the n-dimensional real vector space with inner product

$$x^T y = \sum_{i=1}^{n} x_i y_i.$$

We let

$$\|x\| = (x^T x)^{1/2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

denote the Euclidean norm of a vector $x \in \mathbb{R}^n$.
For every $i = 1, \ldots, n$, we denote by e_i the ith unit vector in $\mathbb{R}^n$. Thus, $e_1 = (1, 0, \ldots, 0, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$.
For vectors $x, y \in \mathbb{R}^n$ we write $x \leq y$ whenever $x_i \leq y_i$ for $i = 1, \ldots, n$. Similarly, $x < y$ whenever $x_i < y_i$ for $i = 1, \ldots, n$.
Let $x, y \in \mathbb{R}^n$. We say that x, y are **parallel** if one of them is a scalar multiple of the other.

Proposition B.0.1 (Cauchy-Schwarz inequality)**.** *For all $x, y \in \mathbb{R}^n$,*

$$|x^T y| \leq \|x\| \|y\|,$$

with equality if and only if x and y are parallel.

The (closed) **line segment** joining x and y is defined as

$$[x, y] = \{\lambda x + (1 - \lambda) y \mid \lambda \in [0, 1]\}.$$

The **open line segment** joining x and y is defined as

$$(x, y) = \{\lambda x + (1 - \lambda) y \mid \lambda \in (0, 1)\}.$$

Definition B.0.2. *A subset $L \subseteq \mathbb{R}^n$ is a* ***line*** *if there are $x, r \in \mathbb{R}^n$ with $r \neq \mathbf{0}$ such that*

$$L = \{x + \lambda r \mid \lambda \in \mathbb{R}\}.$$

We also say that L is a line through point x with direction vector $r \neq 0$ and denote it by $L_{x,r}$.

Proposition B.0.3. *A subset $L \subseteq \mathbb{R}^n$ is a line if and only if there are $x, y \in \mathbb{R}^n$ such that*

$$L = \{(1 - \lambda)x + \lambda y \mid \lambda \in \mathbb{R}\}.$$

We also say that L is the line through two points x, y and denote it by $\overline{xy}$.

Given $r > 0$ and $x \in \mathbb{R}^n$, $B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$ is the **open ball** with center x and radius r and $\overline{B}_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$ is the **closed ball** with center x and radius r.

Definition B.0.4. *A subset $X \subseteq \mathbb{R}^n$ is* bounded *if there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in X$.*

Appendix C

Linear algebra

Definition C.0.1. *A nonempty set $S \subseteq \mathbb{R}^n$ is a* ***(linear) subspace*** *if $\lambda_1 x_1 + \lambda_2 x_2 \in S$ whenever $x_1, x_2 \in S$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.*

Let $x_1, \ldots, x_m$ be points in $\mathbb{R}^n$. Any point $x \in \mathbb{R}^n$ of the form $x = \sum_{i=1}^{m} \lambda_i x_i$, with $\lambda_i \in \mathbb{R}$ for each $i = 1, \ldots, m$, is a **linear combination** of $x_1, \ldots, x_m$.

Definition C.0.2. *The* ***linear span*** *of a subset $X \subseteq \mathbb{R}^n$ (denoted by span(X)) is the intersection of all subspaces containing X.*

If $\text{span}(X) = \mathbb{R}^n$ we say that X is a **spanning set** of $\mathbb{R}^n$ or that X **spans** $\mathbb{R}^n$.

Proposition C.0.3. *(i) span$(\emptyset) = \{\mathbf{0}\}$.*

(ii) For every $X \subseteq \mathbb{R}^n$, span(X) consists of all linear combinations of points in X.

(iii) $S \subseteq \mathbb{R}^n$ is a subspace if and only if S is closed under linear combinations if and only $S =$ span(S).

Definition C.0.4. *A set of vectors $X = \{x_1, \ldots, x_m\}$ is* ***linearly independent*** *if*

$$\sum_{i=1}^{m} \lambda_i x_i = 0 \quad \textit{implies} \quad \lambda_i = 0 \textit{ for each } i = 1, \ldots, m.$$

Is X is not linearly independent, we say that X is **linearly dependent**. We also say that $x_1, \ldots, x_m$ are linearly (in)dependent.

Proposition C.0.5. *Let $X = \{x_1, \ldots, x_m\}$ be a set of vectors in $\mathbb{R}^n$. Then X is linearly dependent if and only if at least one of the vectors x_i can be written as a linear combination of the other vectors in X.*

Definition C.0.6. *Let S be a subspace of $\mathbb{R}^n$. A subset $B = \{x_1, \ldots, x_m\} \subseteq S$ is a* ***basis*** *of S if B spans S and B is linearly independent.*

Proposition C.0.7. *Let S be a subspace of $\mathbb{R}^n$ and B be a basis of S with $|B| = m$.*

(i) Every vector in S can be written in a unique way as a linear combination of vectors in B.

(ii) Every subset of S containing more than m vectors is linearly dependent.

(iii) Every other basis of S has m vectors.

Definition C.0.8. *The* ***dimension*** *$dim(S)$ of a subspace S of $\mathbb{R}^n$ is the number of vectors in a basis of S.*

Proposition C.0.9. *Let S be a subspace of $\mathbb{R}^n$.*

(i) If $S = \{0\}$, then $dim(S) = 0$, since its basis is empty.

(ii) $dim(S) \geq 1$ if and only if $S \neq \{0\}$.

(iii) If $X = \{x_1, \ldots, x_m\} \subseteq S$ is a linearly independent set, then $m \leq dim(S)$.

(iv) If $X = \{x_1, \ldots, x_m\} \subseteq S$ is a spanning set for S, then $m \geq dim(S)$.

Proposition C.0.10. *Let S be a subspace of dimension m and $X = \{x_1, \ldots, x_m\} \subseteq S$. Then X is a basis of S if and only if X spans S if and only if X is linearly independent.*

Proposition C.0.11. *Suppose that U and V are subspaces of $\mathbb{R}^n$ such that $U \subseteq V$. Then*

(i) $dim(U) \leq dim(V)$.

(ii) $dim(U) = dim(V)$ if and only if $U = V$.

C.1 Matrices

Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$.

Definition C.1.1. *The* ***column space*** *of A is the linear span of the set of its columns. The* ***column rank*** *of A is the dimension of the column space, the number of linearly independent columns.*

Definition C.1.2. *The* ***row space*** *of A is the linear span of the set of its rows. The* ***row rank*** *of A is the dimension of the row space, the number of linearly independent rows.*

Proposition C.1.3. *The row rank and column rank of A are equal.*

Proof. See [3, Theorem 3.11, p. 131]. □

Definition C.1.4. *The **rank** of a matrix A, denoted by $\text{rank}(A)$, is its row rank or column rank.*

The $m \times n$ matrix A has **full row rank** if its rank is m and it has **full column rank** if its column rank is n.

Theorem C.1.5. *Let us consider the homogeneous system $Ax = \mathbf{0}$ (with n unknowns and m equations) and let $S := \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$ be its solution set. Then*

(i) S is a linear subspace of $\mathbb{R}^n$.

(ii) $\dim(S) = n - \text{rank}(A)$.

Proof. See [3, Theorem 3.13, p. 131]. □

Thus, the homogeneous system $Ax = \mathbf{0}$ has a unique solution (namely $x = \mathbf{0}$) if and only if $\text{rank}(A) = n$.

Let $b \in \mathbb{R}^m$ and $A \mid b$ be the matrix A augmented by b. Thus,

$$A \mid b = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & & & & \\ a_{i1} & a_{i2} & \dots & a_{in} & b_i \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

Theorem C.1.6. *Let us consider the linear system $Ax = b$ and let $S := \{x \in \mathbb{R}^n \mid Ax = b\}$ be its solution set.*

(i) $S \neq \emptyset$ if and only if $\text{rank}(A) = \text{rank}(A \mid b)$.

(ii) If $S \neq \emptyset$ and $\overline{x}$ is a particular solution, then

$$S = \overline{x} + \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}.$$

(iii) The system has a unique solution if and only if $\text{rank}(A) = \text{rank}(A \mid b) = n$.

Proof. See, for example, [3, Section III.3]. □

Appendix D

Affine sets

Definition D.0.1. *A set $A \subseteq \mathbb{R}^n$ is* ***affine*** *if $\lambda_1 x_1 + \lambda_2 x_2 \in A$ whenever $x_1, x_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfy $\lambda_1 + \lambda_2 = 1$.*

Geometrically, this means that A contains the line through any pair of its points. Note that by this definition the empty set is affine.

Example D.0.2. (i) A point is an affine set.

(ii) Any linear subspace is an affine set.

(iii) Any line is an affine set.

(iv) Another example of an affine set is $P = \{x + \lambda_1 r_1 + \lambda_2 r_2 \mid \lambda_1, \lambda_2 \in \mathbb{R}\}$ which is a two-dimensional plane going through x and spanned by the nonzero vectors r_1 and r_2.

Definition D.0.3. *We say that an affine set A is* ***parallel*** *to another affine set B if $A = B + x_0$ for some $x_0 \in \mathbb{R}^n$, i.e. A is a translate of B.*

Proposition D.0.4. *Let A be a nonempty subset of $\mathbb{R}^n$. Then A is an affine set if and only if A is parallel to a unique linear subspace S, i.e., $A = S + x_0$ for some $x_0 \in A$.*

Proof. See [1, P.1.1, pag. 13]. □

Remark D.0.5. *An affine set is a linear subspace if and only if it contains the origin.*

Proof. To be done in the seminar. □

Definition D.0.6. *The* ***dimension*** *of a nonempty affine set A, denoted by $dim(A)$, is the dimension of the unique linear subspace parallel to A. By convention, $dim(\emptyset) = -1$.*

The maximal affine sets not equal to the whole space are of particular importance, these are the hyperplanes. More precisely,

Definition D.0.7. *A* ***hyperplane*** *in* $\mathbb{R}^n$ *is an affine set of dimension* $n-1$*.*

Proposition D.0.8. *Any hyperplane* $H \subseteq \mathbb{R}^n$ *may be represented by*

$$H = \{x \in \mathbb{R}^n \mid a^T x = \beta\} \quad \text{for some nonzero } a \in \mathbb{R}^n \text{ and } \beta \in \mathbb{R},$$

i.e. H is the solution set of a nontrivial linear equation. Furthermore, any set of this form is a hyperplane. Finally, the equation in this representation is unique up to a scalar multiple.

Proof. See [1, P.1.2, pag. 13-14]. □

Definition D.0.9. *A* ***(closed) halfspace*** *in* $\mathbb{R}^n$ *is the set of all points* $x \in \mathbb{R}^n$ *that satisfy* $a^T x \leq \beta$ *for some* $a \in \mathbb{R}^n$ *and* $\beta \in \mathbb{R}$*.*

We shall use the following notations

$$\begin{aligned} H_{=}(a,\beta) &= \{x \in \mathbb{R}^n \mid a^T x = \beta\} \\ H_{\leq}(a,\beta) &= \{x \in \mathbb{R}^n \mid a^T x \leq \beta\} \\ H_{\geq}(a,\beta) &= \{x \in \mathbb{R}^n \mid a^T x \geq \beta\} \end{aligned}$$

Thus, each hyperplane $H_{=}(a,\beta)$ gives rise to a decomposition of the space in two halfspaces:

Affine sets are closely linked to systems of linear equations.

Proposition D.0.10. *Let* $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$*. Then the solution set* $\{x \in \mathbb{R}^n \mid Ax = b\}$ *of the system of linear equations* $Ax = b$ *is an affine set. Furthermore, any affine set may be represented in this way.*

Proof. See [1, P.1.3, pag. 13-14]. □

Let $x_1, \ldots, x_m$ be points in $\mathbb{R}^n$. An **affine combination** of $x_1, \ldots, x_m$ is a linear combination $\sum_{i=1}^{m} \lambda_i x_i$ with the property that $\sum_{i=1}^{m} \lambda_i = 1$.

Definition D.0.11. *The* ***affine hull*** *aff*(X) *of a subset* $X \subseteq \mathbb{R}^n$ *is the intersection of all affine sets containing* X*.*

Proposition D.0.12. *(i) The affine hull aff*(X) *of a subset* $X \subseteq \mathbb{R}^n$ *consists of all affine combinations of points in* X*.*

(ii) $A \subseteq \mathbb{R}^n$ *is affine if and only if* $A = \mathit{aff}(A)$*.*

Proof. See [1, P.1.4, pag. 16]. □

Definition D.0.13. *The* ***dimension*** *dim*(X) *of a set* $X \subseteq \mathbb{R}^n$ *is the dimension of aff*(X)*.*

Appendix E

Convex sets

Definition E.0.1. *A set* $C \subseteq \mathbb{R}^n$ *is called* convex *if it contains line segments between each pair of its points, that is, if* $\lambda_1 x_1 + \lambda_2 x_2 \in C$ *whenever* $x_1, x_2 \in C$ *and* $\lambda_1, \lambda_2 \geq 0$ *satisfy* $\lambda_1 + \lambda_2 = 1$.

Equivalently, C is convex if and only if $(1 - \lambda)C + \lambda C \subseteq C$ for every $\lambda \in [0, 1]$. Note that by this definition the empty set is convex.

Example E.0.2. (i) All affine sets are convex, but the converse does not hold.

(ii) More generally, the solution set of a family (finite or infinite) of linear inequalities $a_i^T x \leq b_i$, $i \in I$ is a convex set.

(iii) The open ball $B(a, r)$ and the closed ball $\overline{B}(a, r)$ are convex sets.

Appendix F

Graph Theory

Our presentation follows [2] and [9, Chapter 3].

F.1 Graphs

Definition F.1.1. *A* ***graph*** *is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$.*

Thus, the elements of E are 2-element subsets of V. To avoid notational ambiguities, we shall always assume tacitly that $V \cap E = \emptyset$. The elements of V are the **vertices** (or **nodes** or **points**) of G, the elements of E are its **edges**. The vertices of G are denoted $x, y, z, u, v, v_1, v_2, \ldots$. The edge $\{x, y\}$ of G is also denoted $[x, y]$ or xy.

Definition F.1.2. *The* ***order*** *of a graph G, written as $|G|$ is the number of vertices of G. The number of its edges is denoted by $\|G\|$.*

Graphs are **finite**, **infinite**, **countable** and so on according to their order. The empty graph $(\emptyset, \emptyset)$ is simply written $\emptyset$. A graph of order 0 or 1 is called **trivial**.

Convention: Unless otherwise stated, our graphs will be finite.
In the sequel, $G = (V, E)$ is a graph.

A graph with vertex set V is said to be a graph **on** V. The vertex set of a graph G is referred to as $V(G)$, its edge set as $E(G)$. We shall not always distinguish strictly between a graph and its vertex or edge set. For example, we may speak of a vertex $v \in G$ (rather than $v \in V(G)$), an edge $e \in G$, and so on.
A vertex v is **incident** with an edge e if $v \in e$; then e is an edge at v. The set of all edges in E at v is denoted by $E(v)$. The **ends** of an edge e are the two vertices incident with e. Two edges $e \neq f$ are **adjacent** if they have an end in common.

If $e = xy \in E$ is an edge, we say that e **joins** its vertices x and y, that x and y are **adjacent** (or **neighbours**), that x and y are the **ends** of the edge e.

If F is a subset of $[V]^2$, we use the notations $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. Then $G - \{e\}$ and $G + \{e\}$ are abbreviated $G - e$ and $G + e$.

F.1.1 The degree of a vertex

Definition F.1.3. *The* ***degree*** *(or valency) of a vertex v is the number $|E(v)|$ of edges at v and it is denoted by $d_G(v)$ or simply $d(v)$.*

A vertex of degree 0 is **isolated**, and a vertex of degree 1 is a **terminal** vertex. Obviously, the degree of a vertex is equal to the number of neighbours of v.

Proposition F.1.4. *The number of vertices of odd degree is always even.*

F.1.2 Subgraphs

Definition F.1.5. *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs.*

(i) G' is a ***subgraph*** *of G, written $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. If $G' \subseteq G$ we also say that G is a* ***supergraph*** *of G' or that G' is* ***contained*** *in G.*

(ii) If $G' \subseteq G$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an ***induced subgraph*** *of G; we say that V'* ***induces*** *or* ***spans*** *G' in G and write $G' = G[V']$.*

(iii) If $G' \subseteq G$, we say that G' is a ***spanning*** *subgraph of G if $V' = V$.*

F.1.3 Paths, cycles

Definition F.1.6. *A* ***path*** *is a nonempty graph $P = (V(P), E(P))$ of the form*

$$V(P) = \{x_0, \dots, x_k\}, \quad E(P) = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\},$$

where $k \geq 1$ and the x_i's are all distinct.

The vertices x_0 and x_k are **linked** by P and are called its **endvertices** or **ends**; the vertices $x_1, \dots, x_{k-1}$ are the **inner** vertices of P. The number of edges of a path is its **length**. The path of length k is denoted P^k.
We often refer to a path by the natural sequence of its vertices, writing $P = x_0x_1 \dots x_k$ and saying that P is a path **from** x_0 **to** x_k (or **between** x_0 **and** x_k).
If a path P is a subgraph of a graph $G = (V, E)$, we say that P is a path **in** G.

Definition F.1.7. *Let $P = x_0 \dots x_k, k \geq 2$ be a path. The graph $P + x_k x_0$ is called a* ***cycle****.*

As in the case of paths, we usually denote a cycle by its (cyclic) sequence of vertices: $C = x_0 \dots x_k x_0$. The **length** of a cycle is the number of its edges (or vertices). The cycle of length k is said to be a k**-cycle** and denoted C^k.

F.2 Directed graphs

Definition F.2.1. *A* ***directed graph*** *(or* ***digraph****) is a pair $D = (V, A)$, where V is a finite set and A is a* ***multiset*** *of ordered pairs from V.*

Let us recall that a **multiset** (or **bag**) is a generalization of the notion of a set in which members are allowed to appear more than once.
The elements of V are the **vertices** (or **nodes** or **points**) of D, the elements of A are its **arcs** (or **directed edges**). The vertex set of a digraph D is referred to as $V(D)$, its set of arcs as $A(D)$.
Since A is a multiset, the same pair of vertices may occur several times in A. A pair occurring more than once in A is called a **multiple** arc, and the number of times it occurs is called its **multiplicity**. Two arcs are called **parallel** if they are represented by the same ordered pair of vertices. Also **loops** are allowed, that is, arcs of the form (v, v).

Definition F.2.2. *Directed graphs without loops and multiple arcs are called* ***simple****, and directed graphs without loops are called* ***loopless****.*

Let $a = (u, v)$ be an arc. We say that a **connects** u and v, that a **leaves** u and **enters** v; u and v are called the **ends** of a, u is called the **tail** of a and v is called the **head** of a. If there exists an arc connecting vertices u and v, then u and v are called **adjacent** or **connected**. If there exists an arc (u, v), then v is called an **outneighbour** of u, and u is called an **inneighbour** of v.

Each directed graph $D = (V, A)$ gives rise to an **underlying (undirected) graph**, which is the graph $G = (V, E)$ obtained by ignoring the orientation of the arcs:

$$E = \{\{u, v\} \mid (u, v) \in A\}.$$

If G is the underlying (undirected) graph of a digraph D, we call D an **orientation** of G. Terminology from undirected graphs is often transfered to directed graphs.

For any arc $a = (u, v) \in A$, we denote $a^{-1} := (v, u)$ and define $A^{-1} := \{a^{-1} \mid a \in A\}$. The **reverse** digraph D^{-1} is defined by $D^{-1} = (V, A^{-1})$.

For any vertex v, we denote

$\delta_A^{in}(v) := \delta^{in}(v)$:= the set of arcs entering v,

$\delta_A^{out}(v) := \delta^{out}(v)$:= the set of arcs leaving v.

Definition F.2.3. *The* ***indegree*** $deg^{in}(v)$ *of a vertex* v *is the number of arcs entering* v*, i.e.* $|\delta^{in}(v)|$*. The* ***outdegree*** $deg^{out}(v)$ *of a vertex* v *is the number of arcs leaving* v*, i.e.* $|\delta^{out}(v)|$*.*

For any $U \subseteq V$, we denote

$\delta_A^{in}(U) := \delta^{in}(U)$:= the set of arcs entering U, i.e. the set of arcs with head in U and tail in $V \setminus U$,

$\delta_A^{out}(U) := \delta^{out}(U)$:= the set of arcs leaving U, i.e. the set of arcs with head in $V \setminus U$ and tail in U.

F.2.1 Subgraphs

One can define the concept of subgraph as for graphs.
Two subgraphs of D are

(i) **vertex-disjoint** if they have no vertex in common;

(ii) **arc-disjoint** if they have no arc in common.

In general, we say that a family of k subgraphs ($k \geq 3$) is (vertex, arc)-disjoint if the k subgraphs are **pairwise** (vertex, arc)-disjoint, i.e. every two subgraphs from the family are (vertex, arc)-disjoint.

F.2.2 Paths, circuits, walks

Definition F.2.4. *A* ***(directed) path*** *is a digraph* $P = (V(P), A(P))$ *of the form*

$$V = \{v_0, \dots, v_k\}, \quad E = \{(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)\},$$

where $k \geq 1$ *and the* v_i*'s are all distinct.*

The vertices v_0 and v_k are called the **endvertices** or **ends** of P; the vertices $v_1, \dots, v_{k-1}$ are the **inner** vertices of P. The number of edges of a path is its **length**.
We often refer to a path by the natural sequence of its vertices, writing $P = v_0 v_1 \dots v_k$ and saying that P is a path **from** v_0 **to** v_k or that the path P **runs from** v_0 **to** v_k.
If a path P is a subgraph of a digraph $D = (V, A)$, we say that P is a path **in** G.

Notation F.2.5. *We denote by* $P^{-1} := (V(P), E(P)^{-1})$.

Definition F.2.6. *Let $P = v_0 \dots v_k, k \geq 1$ be a path. The graph*

$$P + (v_k, v_0) = (\{v_0, \dots, v_k\}, \{(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k), (v_k, v_0)\}$$

*is called a **circuit**.*

As in the case of paths, we usually denote a circuit by its (cyclic) sequence of vertices: $C = v_0 \dots v_k v_0$. The **length** of a circuit is the number of its edges (or vertices). The circuit of length k is said to be a k**-circuit** and denoted C^k.

Definition F.2.7. *A **walk** in D is a nonempty alternating sequence $v_0 a_0 v_1 a_1 \dots a_{k-1} v_k$ of vertices and arcs of D such that $a_i = (v_1, v_{i+1})$ for all $i = 0, \dots, k-1$. If $v_0 = v_k$, the walk is **closed**.*

Let $D = (V, A)$ be a digraph. For $s, t \in V$, a path in D is said to be an s**-**t **path** if it runs from s to t, and for $S, T \subseteq V$, an S**-**T **path** is a path in D that runs from a vertex in S to a vertex in T. A vertex $v \in V$ is called **reachable** from a vertex $s \in V$ (or from a set $S \subseteq V$) if there exists an s-t path (or S-t path).
Two s-t-paths are **internally vertex-disjoint** if they have no inner vertex in common.

Definition F.2.8. *A set U of vertices is*

*(i) S-T **disconnecting** if U intersects each S-T-path.*

*(ii) an s-t **vertex-cut** if $s, t \notin U$ and each s-t-path intersects U.*

We say that $v_0 a_0 v_1 a_1 \dots a_{k-1} v_k$ is a walk of length k from v_0 to v_k or between v_0 and v_k. If all vertices in a walk are distinct, then the walk defines obviously a path in D.