FMI, CS, Master I
Techniques of Combinatorial Optimization
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Seminar 1

(S1.1) Any polyhedron is a convex set.

Proof. Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Assume that $x, y \in P$ and $\lambda_1, \lambda_2 \geq 0$ are such that $\lambda_1 + \lambda_2 = 1$. We get that

$$A(\lambda_1 x + \lambda_2 y) = \lambda_1 Ax + \lambda_2 Ay \leq \lambda_1 b + \lambda_2 b = b,$$

hence $\lambda_1 x_1 + \lambda_2 x_2 \in P$. It follows that P is convex. □

(S1.2) Prove that

(i) Affine sets are polyhedra.

(ii) Singletons are polyhedra of dimension 0.

(iii) Lines are polyhedra of dimension 1.

(iv) The unit cube $C_3 = \{x \in \mathbb{R}^3 \mid 0 \leq x_i \leq 1 \text{ for all } i = 1, 2, 3\}$ in $\mathbb{R}^3$ is a full-dimensional polyhedron.

Proof. (i) Let D be an affine set. By Proposition D.0.10, we have that $D = \{x \in \mathbb{R}^n \mid Ax = b\}$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then

$$D = \{x \in \mathbb{R}^n \mid Ax \leq b \text{ and } -Ax \leq -b\} = \left\{x \in \mathbb{R}^n \mid \begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix}\right\}.$$

Thus, D is a polyhedron.

(ii) Let $b \in \mathbb{R}^n$. Then $\{b\} = b + \{0\}$, hence $\{b\}$ is an affine set of dimension 0.

(iii) Let $x_0, r \in \mathbb{R}^n, r \neq 0$ and $L_{x_0,r}$ be the line through x_0 with direction vector r. Since $L_{x_0,r}$ is affine, it is a polyhedron too. Furthermore, $L_{x_0,r} = x_0 + \text{span}(r)$, hence $L_{x_0,r}$ is an affine set of dimension 1.

(iv) We have that $x \in C_3$ if and only if x is a solution of the system $x \leq \mathbf{1}, -x \leq \mathbf{0}$. Thus, C_3 is a polyhedron. Since $0 \in C_3 \subseteq \text{aff}(C_3)$, it follows that $\text{aff}(C_3)$ is a linear space. Since $e_1, e_2, e_3 \in C_3 \subseteq \text{aff}(C_3)$, we get that $\dim(\text{aff}(C_3)) = 3$.

□

(S1.3) [Farkas lemma - variant] The system $Ax = b$ has a solution $x \geq \mathbf{0}$ if and only if $y^T b \geq 0$ for each $y \in \mathbb{R}^m$ with $y^T A \geq \mathbf{0}^T$.

Proof. Farkas Lemma has the logical form $\neg P \leftrightarrow \exists y(Q(y) \wedge R(y))$, where

$$P \equiv \exists x(Ax = b \wedge x \geq \mathbf{0}),\ Q(y) \equiv y^T A \geq \mathbf{0}^T,\ R(y) \equiv y^T b < 0.$$

It follows that

$$\begin{aligned} P &\leftrightarrow \neg\exists y(Q(y) \wedge R(y)) \leftrightarrow \forall y \neg(Q(y) \wedge R(y))) \leftrightarrow \forall y(\neg Q(y) \vee \neg R(y)) \\ &\leftrightarrow \forall y(Q(y) \rightarrow \neg R(y)). \end{aligned}$$

□

(S1.4) [Farkas lemma - variant] The system $Ax \leq b$ has a solution if and only if $y^T b \geq 0$ for each $y \geq \mathbf{0}$ with $y^T A = \mathbf{0}^T$.

Proof. Theorem of the Alternatives has the logical form $\neg P \leftrightarrow \exists y(Q(y) \wedge R(y))$, where

$$P \equiv \exists x(Ax \leq b),\ Q(y) \equiv y \geq \mathbf{0} \wedge y^T A = \mathbf{0}^T,\ R(y) \equiv y^T b < 0.$$

It follows that

$$\begin{aligned} P &\leftrightarrow \neg\neg P \leftrightarrow \neg\exists y(Q(y) \wedge R(y)) \leftrightarrow \forall y \neg(Q(y) \wedge R(y))) \leftrightarrow \forall y(\neg Q(y) \vee \neg R(y)) \\ &\leftrightarrow \forall y(Q(y) \rightarrow \neg R(y)). \end{aligned}$$

□