FMI, CS, Master I
Techniques of Combinatorial Optimization
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Seminar 2

(S2.1) Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Then

$$\max\{c^T x \mid x \geq \mathbf{0}, Ax \leq b\} = \min\{b^T y \mid y \geq \mathbf{0}, y^T A \geq c^T\}.$$

(assuming both sets are nonempty).

Proof. Remark that

$$\{x \in \mathbb{R}^n \mid x \geq \mathbf{0}, Ax \leq b\} = \{x \in \mathbb{R}^n \mid Cx \leq d\},$$

where $C = \begin{pmatrix} -I_n \\ A \end{pmatrix} \in \mathbb{R}^{(n+m)\times n}$ and $d = \begin{pmatrix} \mathbf{0} \\ b \end{pmatrix} \in \mathbb{R}^{n+m}$.

Hence, $\max\{c^T x \mid x \geq \mathbf{0}, Ax \leq b\}$ is the primal problem

$$(P) \qquad \max\{c^T x \mid Cx \leq d\}.$$

The dual problem associated to (P) is

$$(D) \qquad \min\{d^T z \mid z \geq \mathbf{0}, z^T C = c^T\}.$$

It suffices to prove that

$$\min\{d^T z \mid z \geq \mathbf{0}, z^T C = c^T\} = \min\{b^T y \mid y \geq \mathbf{0}, y^T A \geq c^T\}.$$

Let $P_1 := \{z \in \mathbb{R}^{n+m} \mid z \geq \mathbf{0}, z^T C = c^T\}$ and $P_2 := \{y \in \mathbb{R}^m \mid y \geq \mathbf{0}, y^T A \geq c^T\}$. We shall prove that $\{d^T z \mid z \in P_1\} = \{b^T y \mid y \in P_2\}$. We do this by showing that

(i) for all $z \in P_1$ there exists $y \in P_2$ such that $d^T z = b^T y$.

(ii) for all $y \in P_2$ there exists $z \in P_1$ such that $d^T z = b^T y$.

Let $z \in P_1$ Then $z = \begin{pmatrix} u \\ y \end{pmatrix}$, where $u \in \mathbb{R}^n, y \in \mathbb{R}^m$, $u \geq \mathbf{0}, y \geq \mathbf{0}$ and

$$c^T = z^T C = z^T \begin{pmatrix} -I_n \\ A \end{pmatrix} = u^T(-I_n) + y^T A = y^T A - u^T.$$

Thus, $y \geq \mathbf{0}$ and $y^T A = c^T + u^T \geq c^T$, hence $y \in P_2$. Moreover, $d^T z = (\mathbf{0}^T, b^T) \begin{pmatrix} u \\ y \end{pmatrix} = b^T y$.

Let now $y \in P_2$ and take $z := \begin{pmatrix} u \\ y \end{pmatrix}$, where $u^T := y^T A - c^T \geq \mathbf{0}$. Then $z \in P_1$ and $d^T z = b^T y$. □

(S2.2) Let $A \in \mathbb{R}^{m \times n}$ be a TU matrix, let $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. Assume that the primal LP $\max\{c^T x \mid Ax \leq b\}$ and dual LP $\min\{b^T y \mid y \geq \mathbf{0}, y^T A = c^T\}$ are bounded. Then they have integer optimal solutions.

Proof. By strong duality (Theorem 1.4.3), we have that

$$\max\{c^T x \mid Ax \leq b\} = \min\{b^T y \mid y \geq \mathbf{0}, y^T A = c^T\}.$$

By Theorem 1.8.3, $P = \{x \mid Ax \leq b\}$ is integer, so we can apply Theorem 1.7.3 to conclude that $\max\{c^T x \mid Ax \leq b\}$ has an integer optimal solution x^*. Remark now that the dual LP

$$\begin{aligned} \min\{b^T y \mid y \geq \mathbf{0}, y^T A = c^T\} &= \min\{b^T y \mid y \geq \mathbf{0}, A^T y = c\} \\ &= \min\{b^T y \mid Cy \leq d\} = -\max\{(-b)^T y \mid Cy \leq d\}, \end{aligned}$$

where $C = \begin{pmatrix} -I_m \\ A^T \\ -A^T \end{pmatrix}$ and $d = \begin{pmatrix} 0 \\ c \\ -c \end{pmatrix}$. Since C is obtained from A by using operations that preserve the TU property, C is also a TU matrix. As d is an integer vector, the dual polyhedron is integer and the dual LP has an integer optimal solution. □

(S2.3) Let $A \in \mathbb{R}^{m \times n}$ be a TU matrix, let b, b', d, d' be vectors in $(\mathbb{Z} \cup \{-\infty, +\infty\})^m$ with $b \leq b'$ and $d \leq d'$. Then

$$P = \{x \in \mathbb{R}^n \mid b \leq Ax \leq b', d \leq x \leq d'\}$$

is an integer polyhedron.

Proof. We have that $P = \{x \in \mathbb{R}^n \mid Cx \leq c\}$, where

$$C = \begin{pmatrix} A \\ -A \\ I_m \\ -I_m \end{pmatrix}, \quad c = \begin{pmatrix} b' \\ -b \\ d' \\ -d \end{pmatrix}.$$

Whenever a component of b, b', d, d' is $\pm\infty$, the corresponding constraint is dropped. Let C' and c' be obtained after dropping these constraints. Then $P = \{x \in \mathbb{R}^n \mid C'x \leq c'\}$ We have that C is TU as it is obtained from A by TU preserving operations and C' is a submatrix of C, hence C' is TU too. Since c' is integer, we can apply Theorem 1.8.3 to conclude that P is integer. □