FMI, CS, Master I
Techniques of Combinatorial Optimization
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Seminar 3

(S3.1) Let G be the complete graph K_3 on three vertices. Prove that its incidence matrix is not totally unimodular.

Proof. Let $V(K_3) = \{1, 2, 3\}$ and $E(K_3) = \{ij \mid 1 \le i < j \le 3\} = \{12, 13, 23\}$. Then its incidence matrix A has 3 rows corresponding to the vertices $1, 2, 3$ and 3 columns, corresponding to the edges $12, 13, 23$. Thus,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We get that $\det(A) = -2$. ☐

(S3.2) Let A be the incidence matrix of a cycle of length 5. Prove that A is a square matrix and compute its determinant.

Proof. Let $C_5 = v_0v_1v_2v_3v_4$ be the cycle of length 5, with vertices v_0, v_1, v_2, v_3, v_4 and edges $v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_0$. Then its incidence matrix is

$$A = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} v_0v_1 & v_1v_2 & v_2v_3 & v_3v_4 & v_4v_0 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

One can easily see that $det(A) = 2$. ☐

(S3.3) Verify if the following matrices are totally unimodular:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Proof. A is the incidence matrix of the bipartite graph $G = (X \cup Y, E)$, where

$$X = \{1, 2, 3\}, \quad Y = \{4, 5, 6\}, E = \{14, 15, 25, 26, 36, 34\}.$$

Thus, A is totally unimodular, by Theorem 2.1.3.

B is the incidence matrix of the graph $G = (V, E)$, where

$$V = \{1, 2, 3, 4, 5, 6, 7\}, \quad E = \{15, 16, 25, 27, 36, 37, 47, 75\}.$$

We remark that G contains the cycle $C = 2572$ of length 3. By Proposition 2.1.2, we get that G is not a bipartite graph. Apply Theorem 2.1.3 to conclude that A is not totally unimodular. $\square$

(S3.4) Let G be a graph. Prove that

$$\max\{|M| \mid M \text{ is a matching of } G\} \leq \min\{|S| \mid S \text{ is a vertex cover of } G\}.$$

Show that the complete graph K_3 is an example of a graph where strict inequality holds.

Proof. Let $M = \{e_1, \ldots, e_n\}$ be an arbitrary matching and S be an arbitrary vertex cover of G. Then for every $i = 1, \ldots, n$, e_i intersects S in some $v_i \in V$. Since the edges in M are disjoint, it follows that for different i's we get different v_i's. Thus $|S| \geq n = |M|$. The conclusion follows.

We have that $K_3 = (\{1, 2, 3\}, \{12, 23, 31\}$. Then

$$\max\{|M| \mid M \text{ is a matching of } G\} = 1, \quad \text{while} \quad \min\{|S| \mid S \text{ is a vertex cover of } G\} = 2.$$

$\square$

(S3.5) Let $G = (V, E)$ be a bipartite graph and $w : E \to \mathbb{N}$ be a weight function. The maximum weight of a matching in G is equal to the minimum value of $\sum_{v \in V} y_v$, where y ranges over all functions $y : V \to \mathbb{N}$ such that $y_u + y_v \geq w(e)$ for each edge $e = uv$ of G.

Proof. We have that

$$\begin{aligned}
\max\{w(M) \mid M \text{ matching in } G\} &= \max\{w^T x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} \\
&= \min\{y^T \mathbf{1} \mid y \geq \mathbf{0}, y^T A \geq w^T\} \text{ (by Proposition 2.2.2)} \\
&= \min\{y^T \mathbf{1} \mid y \geq \mathbf{0}, y^T A \geq w^T, y \in \mathbb{Z}^V\},
\end{aligned}$$

as a consequence of Proposition 1.8.5. Since $y \in \mathbb{Z}^V$ and $y \geq \mathbf{0}$, we get that $y \in \mathbb{N}^V$, hence $y : V \to \mathbb{N}$. Remark that $y^T \mathbf{1} = \sum_{v \in V} y_v$ and that for every edge $e = uv$ of G, $(y^T A)_e = y_u + y_v$. Thus, $y^T A \geq w^T$ if and only if $y_u + y_v \geq w(e)$ for each edge $e = uv$ of G. $\square$