FMI, CS, Master I
Techniques of Combinatorial Optimization
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Seminar 4

(S4.1) Figure 1 represents a flow network $N = (D, c, s, t)$.

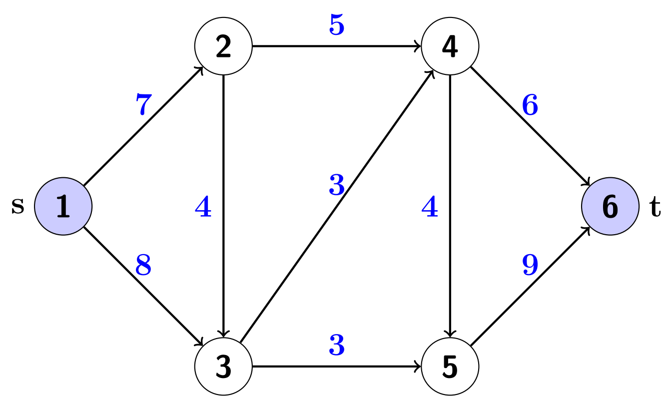

Figure 1: The flow network N

Write the corresponding digraph D and the capacity function c.

Proof. We have that $D = (V, A)$, where $V = \{1, 2, 3, 4, 5, 6\}$ and

$$A = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5), (4, 6), (5, 6)\}.$$

Furthermore, $c : A \to \mathbb{Z}_+$, $c(1, 2) = 7, c(1, 3) = 8, c(2, 4) = 5, c(2, 3) = c(4, 5) = 4, c(3, 4) = c(3, 5) = 3, c(4, 6) = 6, c(5, 6) = 9$.

□

(S4.2) Find vectors b, d and a matrix B such that

$$\max\{\text{value}(f) \mid f \text{ is an } s - t \text{ flow for } N\} = \max\{d^T f \mid Bf \le b\}.$$

Proof. Let M be the incidence matrix of D and for every $i = 1, \dots, 6$, let us denote by $\mathbf{m}_i$ the i-th line of M. Let M_0 be the matrix obtained from M by deleting the lines $\mathbf{m}_1$ and $\mathbf{m}_6$. Thus,

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{matrix} (1,2) & (1,3) & (2,3) & (2,4) & (3,4) & (3,5) & (4,5) & (4,6) & (5,6) \end{matrix} \\ \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array},$$

$$M_0 = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{matrix} (1,2) & (1,3) & (2,3) & (2,4) & (3,4) & (3,5) & (4,5) & (4,6) & (5,6) \end{matrix} \\ \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \end{pmatrix} \end{array}.$$

Then (see Section 3.1.1 in the lecture notes),

$$\begin{aligned} \max\{\text{value}(f) \mid f \text{ is an } s-t \text{ flow}\} &= \max\{\mathbf{m}_6 f \mid M_0 f = \mathbf{0}, \mathbf{0} \leq f \leq c\} \\ &= \max\{d^T f \mid Bf \leq b\}, \end{aligned}$$

where

$$d = \mathbf{m}_6^T, \quad B = \begin{pmatrix} M_0 \\ -M_0 \\ I \\ -I \end{pmatrix}, \quad b = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ c \\ \mathbf{0} \end{pmatrix}.$$

□

(S4.3) Figure 2 represents an s-t flow f for the network N.

(i) Verify that f is an s-t flow. What is the value of f?

(ii) Show that the set $\{(2,4), (3,4), (3,5)\}$ is an s-t cut and compute its capacity.

(iii) Prove that f is a maximum flow.

Proof. (i) $f : A \to \mathbb{Z}_+$ is defined by $f(1,2) = 7$, $f(1,3) = 4$, $f(2,3) = f(4,5) = 2$, $f(3,4) = f(3,5) = 3$, $f(2,4) = f(5,6) = 5$, $f(4,6) = 6$. It is obvious that $0 \leq f \leq c$. It remains to verify the flow conservation law at every vertex $v \neq 1, 6$:

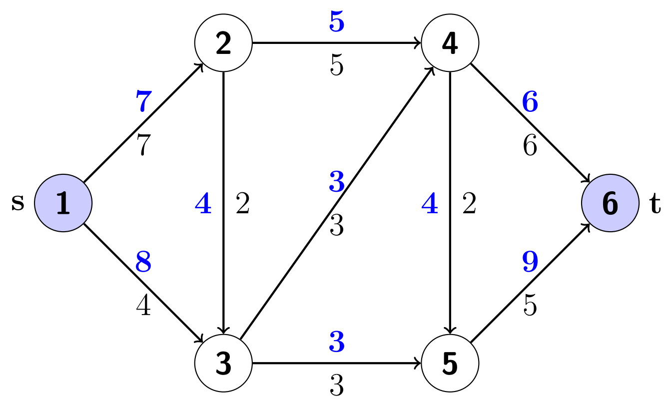

Figure 2: The flow network N with the flow f

(a) $v = 2$. $in_f(2) = 7, out_f(2) = 2 + 5 = 7$

(b) $v = 3$. $in_f(3) = 2 + 4 = 6, out_f(3) = 3 + 3 = 6$

(c) $v = 4$. $in_f(4) = 5 + 3 = 8, out_f(4) = 2 + 6 = 8$

(d) $v = 5$. $in_f(5) = 3 + 2 = 5, out_f(5) = 5$.

We have that $\text{value}(f) = out_f(s) - in_f(s) = out_f(s) = 7 + 4 = 11$.

(ii) Let $U = \{1, 2, 3\}$, hence $s \in U$, but $t \notin U$. We have that $\delta^{out}(U) = \{(2, 4), (3, 4), (3, 5)\}$ and its capacity is $c(\delta^{out}(U)) = 5 + 3 + 3 = 11$.

(iii) Since $value(f) = c(\delta^{out}(U))$, we can apply Corollary 3.0.9 to conclude that f is a maximum flow.

□

(S4.4) Let $N = (D, c, s, t)$ be a flow network and $f : A \to \mathbb{R}$ be an s-t flow. Prove that the value of f is equal to the net amount of flow entering t, that is prove that

$$\text{value}(f) = f(\delta^{in}(t)) - f(\delta^{out}(t).$$

Proof. Applying Lemma 3.0.7 and the flow conservation law for $v \neq s, t$ we get that

$$\begin{aligned} 0 &= \text{excess}_f(V) = \sum_{v \in V} \text{excess}_f(v) = \sum_{v \in V \setminus \{s,t\}} \text{excess}_f(v) + \text{excess}_f(s) + \text{excess}_f(t) \\ &= \text{excess}_f(s) + \text{excess}_f(t). \end{aligned}$$

Thus, $\text{value}(f) = -\text{excess}_f(s) = \text{excess}_f(t) = f(\delta^{in}(t)) - f(\delta^{out}(t).$ □

(S4.5) Let $N = (D, c, s, t)$ be a flow network with the property that all capacities are even (that is, $c(a)$ is even for every arc a of D). Prove that the maximum value of a flow is even.

Proof. Since all capacities are even, the capacity of every cut is even, hence the minimum capacity of a cut is even. Apply the Max-Flow Min-Cut theorem to conclude that the maximum value of a flow is even. □