FMI, CS, Master I
Techniques of Combinatorial Optimization
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Seminar 5

(S5.1) Figure 1 represents a flow network $N = (D, c, s, t)$.

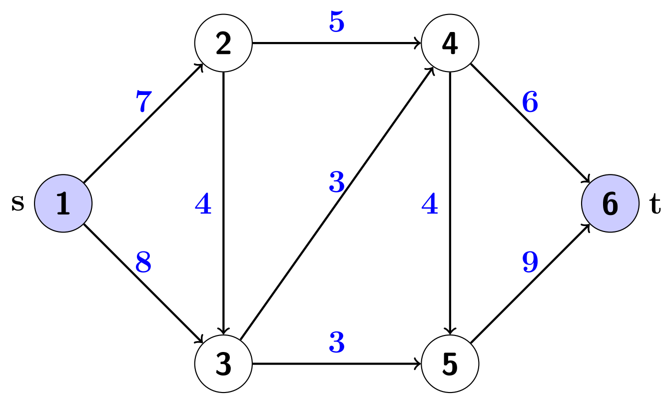

Figure 1: The flow network N

Give two iterations of the Ford-Fulkerson algorithm for N, considering the path $P = 1246$ for the first augmentation and $Q = 1356$ for the second augmentation.

Proof. The initial flow is $f := 0$, hence the residual network coincides with N.

Let us consider the s-t path $P = 1246$ as an f-augmenting path. Then

$$\gamma = \min_{e \in A(P)} c_f(e) = \min\{7, 5, 6\} = 5.$$

Thus, the algorithm augments f along P with 5 units, i.e. we replace f with $f_1 := f_P^\gamma$.

After the first augmentation, we get the following residual graph D_{f_1} and residual capacities c_{f_1}:

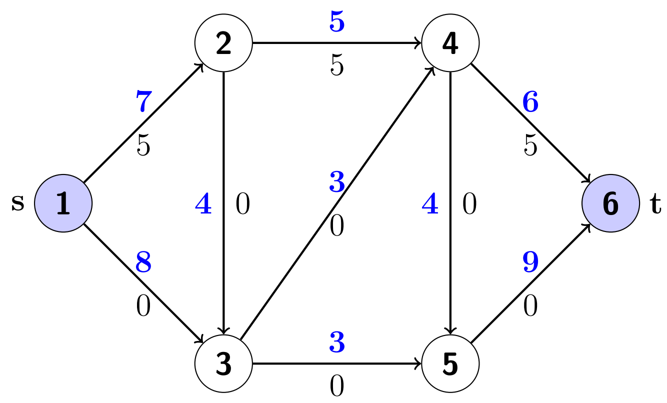

Figure 2: The flow network N with the flow f_1

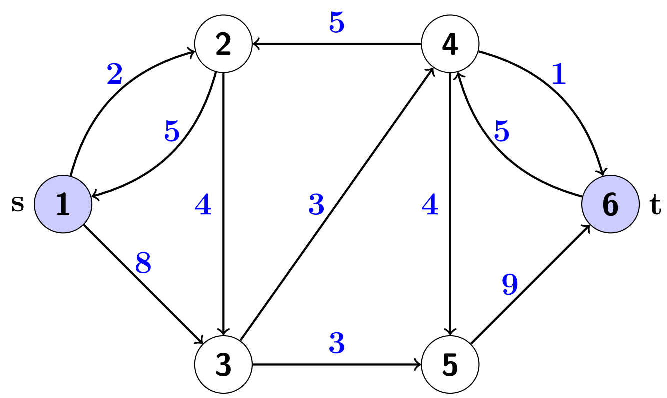

Figure 3: The residual graph D_{f_1}

Let us consider at the second iteration the s-t path $Q = 1356$ as an f_1-augmenting path. Then

$$\gamma = \min_{e \in A(P)} c_{f_1}(e) = \min\{8, 3, 9\} = 3.$$

Thus, the algorithm augments f along Q with 3 units, i.e. we replace f_1 with $f_2 := f_{1}{}_Q^{\gamma}$.

After the second augmentation, we get the following residual graph D_{f_2} and residual capacities c_{f_2}:

□

(S5.2) Figure 6 represents a flow network N and an s-t flow f for N.

(i) Represent the residual graph D_f and the residual capacities c_f.

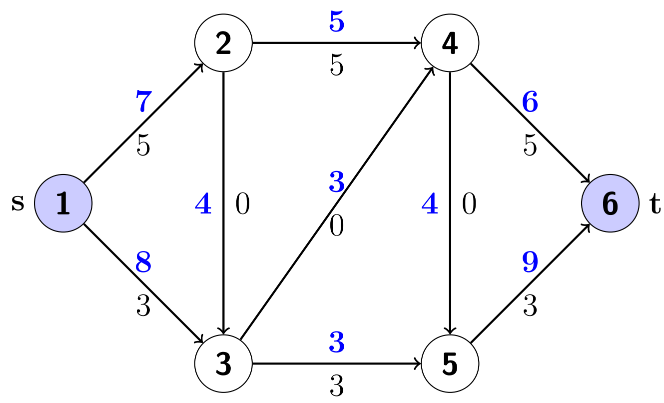

Figure 4: The flow network N with the flow f_2

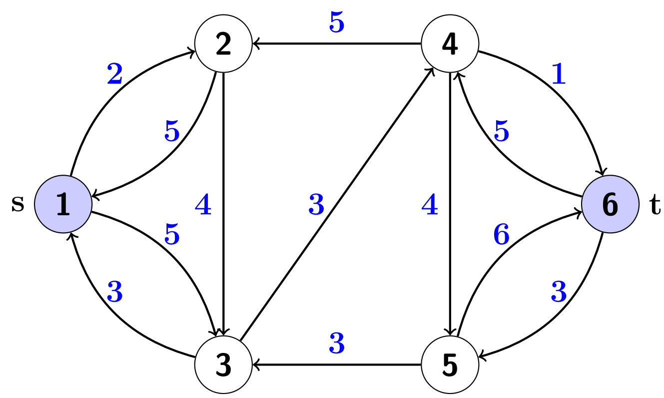

Figure 5: The residual graph D_{f_2}

(ii) Choose an f-augmenting path P of minimum length and compute the flow $g := f_P^{\gamma}$, where $\gamma = \min_{e \in A(P)} c_f(e)$.

(iii) Represent the residual graph D_g and the residual capacities c_g. Can you find an s-t path in D_g?

(iv) What is the maximum value of and s-t flow for N?

(v) Give an example of an s-t cut in N of minimum capacity.

Proof. (i) The residual graph D_f and residual capacities c_f are given in Figure 7.

(ii) The s-t path of minimum length is $P = 14256$, so we choose it as an f-augmenting

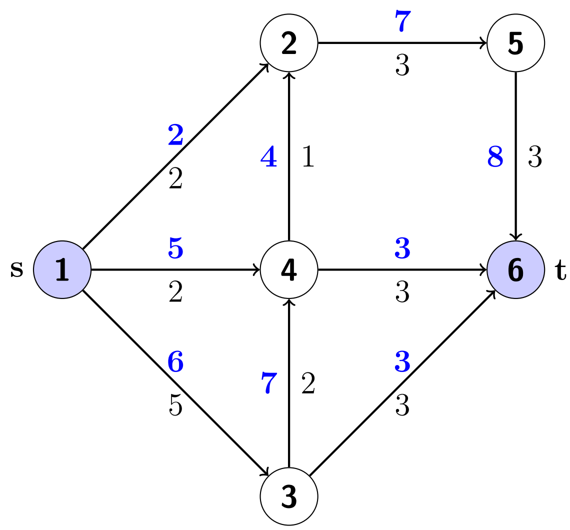

Figure 6: The flow network N with the flow f

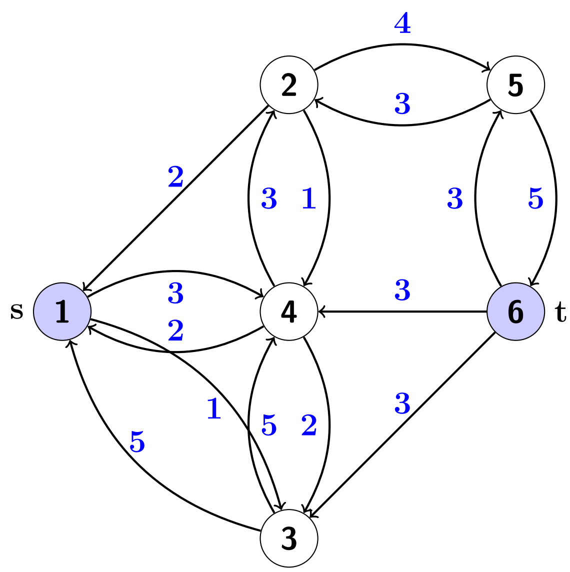

Figure 7: The residual graph D_f

path. Then

$$\gamma = \min_{e \in A(P)} c_f(e) = \min\{3, 4, 5\} = 3.$$

The flow network N with the flow $g := f_P^{\gamma}$ is given in Figure 8.

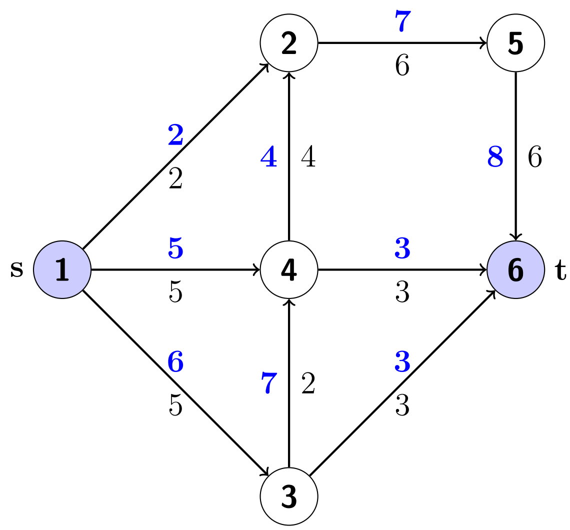

Figure 8: The flow network N with the flow g

(iii) The residual graph D_g and residual capacities c_g are given in Figure 9.

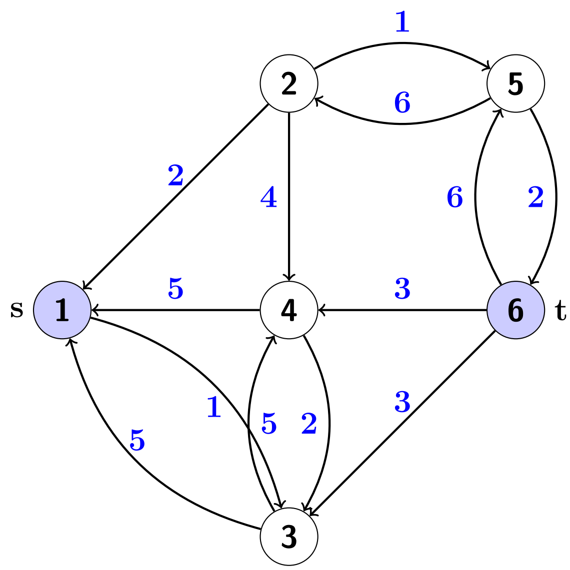

Figure 9: The residual graph D_g

It is obvious that there are no s-t paths in D_g.

(iv) By Theorem 3.2.4, it follows that g is a maximum flow. Hence, the maximal value of an s-t flow is value$(g) = 5 + 2 + 5 = 12$.

(v) By the Max-Flow Min-Cut Theorem 3.0.11, the minimum capacity of an s-t cut is 12. Apply Proposition 3.2.3. to get that an s-t cut having this capacity is

$$\{(1,2),(4,2),(4,6),(3,6)\} = \delta^{out}(U),$$

where $U = \{1,3,4\}$ is the set of vertices reachable in D_g from 1.

□

(S5.3) Prove Proposition 3.4.2..

Proof. Apply the Flow Decomposition Theorem 3.4.1. Then there exist $K, L \in \mathbb{Z}_+$, positive numbers $w_1, \ldots, w_K, \mu_1, \ldots, \mu_L$, s-t paths $P_1, \ldots, P_K$ and circuits $C_1, \ldots, C_L$ such that

$$f = \sum_{i=1}^{K} w_i \chi^{P_i} + \sum_{j=1}^{L} \mu_j \chi^{C_j} \quad \text{and} \quad \text{value}(f) = \sum_{i=1}^{K} w_i.$$

Furthermore, the w_i's, μ_j's are positive integers. Since f is a $\{0,1\}$-flow, we must have $w_i = \mu_j = 1$ for all i, j. Thus,

$$f = \sum_{i=1}^{K} \chi^{P_i} + \sum_{j=1}^{L} \chi^{C_j} \quad \text{and} \quad \text{value}(f) = K.$$

It remains to show that the family $\mathcal{F} = \{P_1, \ldots, P_K, C_1, \ldots, C_L\}$ is arc-disjoint. If $Q_1, Q_2 \in \mathcal{F}$ have an arc a in common, then $f(a) \geq \chi^{Q_1}(a) + \chi^{Q_2}(a) = 2$, which contradicts the fact that f is a $\{0,1\}$-flow. □

(S5.4) For any s-t path P in D, prove that χ^P satisfies the flow conservation law at every $v \neq s, t$ and that value$(\chi^P) = 1$.

Proof. If $P = st$, then $\chi^P(a) = 0$ for all $a \neq (s,t)$, hence $in_{\chi^P}(v) = out_{\chi^P}(v) = 0$ for all $v \neq s, t$. Assume that $P = sv_1 \ldots v_k t$ with $k \geq 1$. Let us denote $v_0 := s, v_{k+1} := t$. Then $\chi^P((s,v_1)) = \chi^P((v_1,v_2)) = \ldots = \chi^P((v_{k-1},v_k)) = \chi^P((v_k,t)) = 1$ and $\chi^P(a) = 0$ for all the other arcs a. For an arbitrary $v \neq s, t$ we have two cases:

(i) $v \notin P$. Then $in_{\chi^P}(v) = out_{\chi^P}(v) = 0$.

(ii) $v = v_i, i = 1, \ldots, k$. Then

$$in_{\chi^P}(v_i) = \sum_{a \in \delta^{in}(v_i)} \chi^P(a) = \chi^P((v_{i-1}, v_i)) + 0 = 1,$$
$$out_{\chi^P}(v_i) = \sum_{a \in \delta^{out}(v_i)} \chi^P(a) = \chi^P((v_i, v_{i+1})) + 0 = 1.$$

Finally,

$$\text{value}(\chi^P) = out_{\chi^P}(s) - in_{\chi^P}(s) = \chi^P((s, v_1)) - 0 = 1.$$

□