**FMI, CS, Master I**
**Techniques of Combinatorial Optimization**
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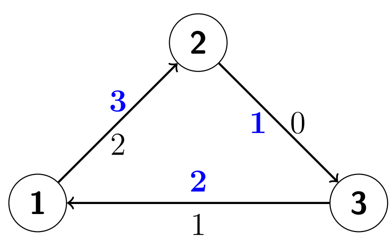
# Seminar 6

**(S6.1)** Give an example where feasible circulations do not exist.

*Proof.* We consider the following example




Thus,

(i) $D = (V, A)$, where $V = \{1, 2, 3\}, A = \{(1, 2), (2, 3), (3, 1)\}$,

(ii) $c((1, 2)) = 3, c((2, 3)) = 1, c((3, 1)) = 2$ and

(iii) $d((1, 2)) = 2, d((2, 3)) = 0, d((3, 1)) = 1$.

Let $f : A \to \mathbb{R}$ be feasible w.r.t. $d, c$. Then $in_f(2) = f((1, 2)) \geq d((1, 2)) = 2$, while $out_f(2) = f((2, 3)) \leq c((2, 3)) = 1$. Thus, $in_f(2) \neq out_f(2)$, so $f$ cannot be a circulation. □

**(S6.2)** Let $N = (D, s, t)$ be a unit capacity network, $k \geq 1$ and $P_1, \ldots, P_k$ be $k$ arc-disjoint $s$-$t$ paths in $D$. Then for all $k \geq 1$,

$$f := \chi^{P_1} + \ldots + \chi^{P_k}$$

is an $s$-$t$ $\{0, 1\}$-flow $f$ with value$(f) = k$.

*Proof.* For $k = 1$, we have that $f := \chi^{P_1}$ is an $s$-$t$ $\{0, 1\}$-flow of value 1, by (S5.4).

Let $k \geq 2$. By (S5.4) and Lemma 3.3.2, we have that $f$ satisfies the flow conservation law at every $v \neq s, t$ and value$(f) = k$. It remains to prove that $f$ takes values in $\{0, 1\}$. For any $a \in A$, we have one of the two cases:

(i) $a \notin P_1 \cup \ldots \cup P_k$, so $f(a) = 0$.

(ii) there exists a unique $i = 1, \ldots, k$ such that $a \in P_i$, so $f(a) = 1$.

Thus, $f : A \to \{0, 1\}$ is a flow. □

**(S6.3)** Let $D = (V, A)$ be a digraph. Prove that

(i) Each $s$-$t$ cut is an $s$-$t$ disconnecting arc set.

(ii) Each $s$-$t$ disconnecting arc set of minimum size is an $s$-$t$ cut.

(iii) The minimum size of an $s$-$t$ disconnecting arc set coincides with the minimum size of an $s$-$t$ cut.

*Proof.* (i) Let $\delta^{out}(U)$ be an $s$-$t$ cut, where $s \in U$ and $t \notin U$. Let $P = sv_1 \ldots v_k t$ be an $s$-$t$ path and denote $v_0 := s$ and $v_{k+1} := t$. We have two cases:

(a) there exists $i = 1, \ldots, k$ such that $v_i \notin U$ and $v_{i-1} \in U$. Then $(v_{i-1}, v_i) \in \delta^{out}(U)$.

(b) $v_i \in U$ for all $i = 1, \ldots, k$. Then $(v_k, t) \in \delta^{out}(U)$.

(ii) Let $B$ be an $s$-$t$ disconnecting arc set of minimum size. Define $U$ as the the set of vertices in $V$ accessible from $s$ by paths that contain no arcs of $B$. Then $s \in U$ and $t \notin U$, since $B$ is $s$-$t$ disconnecting. Hence, $\delta^{out}(U)$ is an $s$-$t$ cut, hence it is an $s$-$t$ disconnecting arc set, by (i).

**Claim:** $\delta^{out}(U) \subseteq B$

**Proof of Claim:** Let $a = (u, v) \in \delta^{out}(U)$. Since $u \in U$, there exists an $s$-$u$ path $P$ containing no arcs of $B$. If $a \notin B$, then $P + a$ is an $s$-$v$ path containing no arcs of $B$, hence $v \in U$, which is a contradiction with the fact that $(u, v) \in \delta^{out}(U)$. Thus, $a \in B$. ■

By the fact that $B$ is of minimum size, we must have $B = \delta^{out}(U)$.

(iii) Let $m$ be the first minimum and $m'$ be the second minimum. We have that $m \leq m'$ by (i) and that $m \geq m'$ by (ii).

□

**(S6.4)** Prove that the incidence matrix $M$ of a directed graph $D = (V, A)$ is totally unimodular.

*Proof.* Let $B$ be a square submatrix of $M$ of order $t$. We prove by induction on $t$ that $\det(B)$ is $-1$, 0 or 1. The case $t = 1$ is trivial. Let $t > 1$. We have the following cases:

(i) $B$ has a column with only zeros. Then obviously $\det(B) = 0$.

(ii) $B$ has a column with exactly one nonzero, which is $\pm 1$. Expand the determinant by this column and use the induction hypothesis to conclude that $\det(B) \in \{-1, 0, 1\}$.

(iii) Each column of $B$ contains two nonzeros, one of them 1 and the other $-1$. Then the sum of all lines of $B$ is $\mathbf{0}$, hence the lines of $B$ are linearly dependent. As a consequence, $\det(B) = 0$.

□