Logic for Multiagent Systems

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Propositional logic

Language

Definition 1.1

The language of propositional logic PL consists of:

- *a countable set* $V = \{v_n \mid n \in \mathbb{N}\}$ *of variables;*
- *the logic connectives* $\neg$ *(non),* $\rightarrow$ *(implies)*
- *parantheses:* (,).

- The set *Sym* of symbols of *PL* is

$$Sym := V \cup \{\neg, \rightarrow, (,)\}.$$

- We denote variables by $u, v, x, y, z \ldots$

Definition 1.2

The set Expr of expressions of PL is the set of all finite sequences of symbols of PL.

Definition 1.3

Let $\theta = \theta_0\theta_1 \dots \theta_{k-1}$ be an expression, where $\theta_i \in Sym$ for all $i = 0, \dots, k-1$.

- *If $0 \leq i \leq j \leq k-1$, then the expression $\theta_i \dots \theta_j$ is called the (i, j)-subexpression of θ.*
- *We say that an expression ψ appears in θ if there exists $0 \leq i \leq j \leq k-1$ such that ψ is the (i, j)-subexpression of θ.*
- *We denote by $Var(\theta)$ the set of variables appearing in θ.*

The definition of formulas is an example of an inductive definition.

Definition 1.4

The formulas of PL are the expressions of PL defined as follows:

(F0) *Any variable is a formula.*

(F1) *If* φ *is a formula, then* $(\neg\varphi)$ *is a formula.*

(F2) *If* φ *and* ψ *are formulas, then* $(\varphi \to \psi)$ *is a formula.*

(F3) *Only the expressions obtained by applying rules (F0), (F1), (F2) are formulas.*

Notations

The set of formulas is denoted by *Form*. Formulas are denoted by $\varphi, \psi, \chi, \ldots$.

Proposition 1.5

The set Form is countable.

Unique readability

If φ is a formula, then exactly one of the following hold:

- $\varphi = v$, where $v \in V$.
- $\varphi = (\neg\psi)$, where ψ is a formula.
- $\varphi = (\psi \to \chi)$, where ψ, χ are formulas.

Furthermore, φ can be written in a unique way in one of these forms.

Definition 1.6

Let φ be a formula. A subformula of φ is any formula ψ that appears in φ.

Proposition 1.7 (Induction principle on formulas)

Let Γ *be a set of formulas satisfying the following properties:*

- $V \subseteq \Gamma$.
- Γ *is closed to* $\neg$, *that is:* $\varphi \in \Gamma$ *implies* $(\neg\varphi) \in \Gamma$.
- Γ *is closed to* $\rightarrow$, *that is:* $\varphi, \psi \in \Gamma$ *implies* $(\varphi \rightarrow \psi) \in \Gamma$.

Then $\Gamma = Form$.

It is used to prove that all formulas have a property $\mathcal{P}$: we define Γ as the set of all formulas satisfying $\mathcal{P}$ and apply induction on formulas to obtain that $\Gamma = Form$.

Language

The derived connectives $\vee$ (or), $\wedge$ (and), $\leftrightarrow$ (if and only if) are introduced by the following abbreviations:

$$\begin{aligned} \varphi \vee \psi &:= ((\neg\varphi) \rightarrow \psi) \\ \varphi \wedge \psi &:= \neg(\varphi \rightarrow (\neg\psi))) \\ \varphi \leftrightarrow \psi &:= ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)) \end{aligned}$$

Conventions and notations

- The external parantheses are omitted, we put them only when necessary. We write $\neg\varphi$, $\varphi \rightarrow \psi$, but we write $(\varphi \rightarrow \psi) \rightarrow \chi$.
- To reduce the use of parentheses, we assume that
  - $\neg$ has higher precedence than $\rightarrow, \wedge, \vee, \leftrightarrow$;
  - $\wedge, \vee$ have higher precedence than $\rightarrow, \leftrightarrow$.
- Hence, the formula $(((\varphi \rightarrow (\psi \vee \chi)) \wedge ((\neg\psi) \leftrightarrow (\psi \vee \chi)))$ is written as $(\varphi \rightarrow \psi \vee \chi) \wedge (\neg\psi \leftrightarrow \psi \vee \chi)$.

Truth values

We use the following notations for the truth values:

1 for true and 0 for false.

Hence, the set of truth values is $\{0, 1\}$.

Define the following operations on $\{0, 1\}$ using truth tables.

$\neg : \{0, 1\} \to \{0, 1\}$,

p	$\neg p$
0	1
1	0

$\to : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$,

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Semantics

$\vee : \{0,1\} \times \{0,1\} \to \{0,1\},$

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

$\wedge : \{0,1\} \times \{0,1\} \to \{0,1\},$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

$\leftrightarrow: \{0,1\} \times \{0,1\} \to \{0,1\},$

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Semantics

Definition 1.8

An *evaluation* *(or* *interpretation*) *is a function* $e : V \to \{0,1\}$.

Theorem 1.9

For any evaluation $e : V \to \{0,1\}$ *there exists a unique function*

$$e^+ : Form \to \{0,1\}$$

satisfying the following properties:

- $e^+(v) = e(v)$ *for all* $v \in V$.
- $e^+(\neg\varphi) = \neg e^+(\varphi)$ *for any formula* φ.
- $e^+(\varphi \to \psi) = e^+(\varphi) \to e^+(\psi)$ *for any formulas* φ, ψ.

Proposition 1.10

For any formula φ *and all evaluations* $e_1, e_2 : V \to \{0,1\}$,

if $e_1(v) = e_2(v)$ *for all* $v \in Var(\varphi)$, *then* $e_1^+(\varphi) = e_2^+(\varphi)$.

Semantics

Let φ be a formula.

Definition 1.11

- *An evaluation $e : V \to \{0, 1\}$ is a model of φ if $e^+(\varphi) = 1$.*
  Notation: $e \vDash \varphi$.
- *φ is satisfiable if it has a model.*
- *If φ is not satisfiable, we also say that φ is unsatisfiable or contradictory.*
- *φ is a tautology if every evaluation is a model of φ.*
  Notation: $\vDash \varphi$.

Notation 1.12

The set of models of φ is denoted by $Mod(\varphi)$.

Remark 1.13

- φ is a tautology iff $\neg\varphi$ is unsatisfiable.
- φ is unsatisfiable iff $\neg\varphi$ is a tautology.

Proposition 1.14

Let $e : V \to \{0,1\}$ be an evaluation. Then for all formulas φ, ψ,

- $e \vDash \neg\varphi$ iff $e \not\vDash \varphi$.
- $e \vDash \varphi \to \psi$ iff ($e \vDash \varphi$ implies $e \vDash \psi$) iff ($e \not\vDash \varphi$ or $e \vDash \psi$).
- $e \vDash \varphi \vee \psi$ iff ($e \vDash \varphi$ or $e \vDash \psi$).
- $e \vDash \varphi \wedge \psi$ iff ($e \vDash \varphi$ and $e \vDash \psi$).
- $e \vDash \varphi \leftrightarrow \psi$ iff ($e \vDash \varphi$ iff $e \vDash \psi$).

Semantics

Definition 1.15

Let φ, ψ be formulas. We say that

- *φ is a semantic consequence of ψ if $Mod(\psi) \subseteq Mod(\varphi)$.*
  Notation: $\psi \vDash \varphi$.
- *φ and ψ are (logically) equivalent if $Mod(\psi) = Mod(\varphi)$.*
  Notation: $\varphi \sim \psi$.

Remark 1.16

Let φ, ψ be formulas.

- *$\psi \vDash \varphi$ iff $\vDash \psi \to \varphi$.*
- *$\psi \sim \varphi$ iff ($\psi \vDash \varphi$ and $\varphi \vDash \psi$) iff $\vDash \psi \leftrightarrow \varphi$.*

For all formulas φ, ψ, χ,

$$\models \quad \varphi \vee \neg\varphi$$
$$\models \quad \neg(\varphi \wedge \neg\varphi)$$
$$\models \quad \varphi \wedge \psi \to \varphi$$
$$\models \quad \varphi \to \varphi \vee \psi$$
$$\models \quad \varphi \to (\psi \to \varphi)$$
$$\models \quad (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$
$$\models \quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$\models \quad (\varphi \to \psi) \vee (\neg\varphi \to \psi)$$
$$\models \quad (\varphi \to \psi) \vee (\varphi \to \neg\psi)$$
$$\models \quad \neg\varphi \to (\neg\psi \leftrightarrow (\psi \to \varphi))$$
$$\models \quad (\varphi \to \psi) \to (((\varphi \to \chi) \to \psi) \to \psi)$$
$$\models \quad \neg\psi \to (\psi \to \varphi)$$

Semantics

$$
\begin{aligned}
&\models \psi \to (\neg\psi \to \varphi) \\
&\models (\varphi \to \neg\varphi) \to \neg\varphi \\
&\models (\neg\varphi \to \varphi) \to \varphi \\
\psi &\models \varphi \to \psi \\
\neg\varphi &\models \varphi \to \psi \\
\neg\psi \wedge (\varphi \to \psi) &\models \neg\varphi \\
(\varphi \to \psi) \wedge (\psi \to \chi) &\models \varphi \to \chi \\
\varphi \wedge (\varphi \to \psi) &\models \psi \\
\{\psi, \neg\psi\} &\models \varphi \\
\{\psi, \neg\varphi\} &\models \neg(\psi \to \varphi)
\end{aligned}
$$

Semantics

$$
\begin{array}{rcl}
\varphi & \sim & \neg\neg\varphi \\
\varphi \to \psi & \sim & \neg\psi \to \neg\varphi \\
\varphi \vee \psi & \sim & \neg(\neg\varphi \wedge \neg\psi) \\
\varphi \wedge \psi & \sim & \neg(\neg\varphi \vee \neg\psi) \\
\varphi \to (\psi \to \chi) & \sim & \varphi \wedge \psi \to \chi \\
\varphi \sim \varphi \wedge \varphi & \sim & \varphi \vee \varphi \\
\varphi \wedge \psi & \sim & \psi \wedge \varphi \\
\varphi \vee \psi & \sim & \psi \vee \varphi \\
\varphi \wedge (\psi \wedge \chi) & \sim & (\varphi \wedge \psi) \wedge \chi \\
\varphi \vee (\psi \vee \chi) & \sim & (\varphi \vee \psi) \vee \chi \\
\varphi \vee (\varphi \wedge \psi) & \sim & \varphi \\
\varphi \wedge (\varphi \vee \psi) & \sim & \varphi
\end{array}
$$

Semantics

$$
\begin{aligned}
\varphi \wedge (\psi \vee \chi) &\sim (\varphi \wedge \psi) \vee (\varphi \wedge \chi) \\
\varphi \vee (\psi \wedge \chi) &\sim (\varphi \vee \psi) \wedge (\varphi \vee \chi) \\
\varphi \to \psi \wedge \chi &\sim (\varphi \to \psi) \wedge (\varphi \to \chi) \\
\varphi \to \psi \vee \chi &\sim (\varphi \to \psi) \vee (\varphi \to \chi) \\
\varphi \wedge \psi \to \chi &\sim (\varphi \to \chi) \vee (\psi \to \chi) \\
\varphi \vee \psi \to \chi &\sim (\varphi \to \chi) \wedge (\psi \to \chi) \\
\varphi \to (\psi \to \chi) &\sim \psi \to (\varphi \to \chi) \\
&\sim (\varphi \to \psi) \to (\varphi \to \chi) \\
\neg\varphi \sim \varphi \to \neg\varphi &\sim (\varphi \to \psi) \wedge (\varphi \to \neg\psi) \\
\varphi \to \psi \sim \neg\varphi \vee \psi &\sim \neg(\varphi \wedge \neg\psi) \\
\varphi \vee \psi \sim \varphi \vee (\neg\varphi \wedge \psi) &\sim (\varphi \to \psi) \to \psi \\
\varphi \leftrightarrow (\psi \leftrightarrow \chi) &\sim (\varphi \leftrightarrow \psi) \leftrightarrow \chi
\end{aligned}
$$

Semantics

It is often useful to have a canonical tautology and a canonical unsatisfiable formula.

Remark 1.17

$v_0 \to v_0$ is a tautology and $\neg(v_0 \to v_0)$ is unsatisfiable.

Notation 1.18

Denote $v_0 \to v_0$ by $\top$ and call it the truth.
Denote $\neg(v_0 \to v_0)$ by $\bot$ and call it the false.

Remark 1.19

- *φ is a tautology iff $\varphi \sim \top$.*
- *φ is unsatisfiable iff $\varphi \sim \bot$.*

Semantics

Let Γ be a set of formulas.

Definition 1.20

An evaluation $e : V \to \{0,1\}$ is a model of Γ if it is a model of every formula from Γ.
Notation: $e \models \Gamma$.

Notation 1.21

The set of models of Γ is denoted by $Mod(\Gamma)$.

Definition 1.22

A formula φ is a semantic consequence of Γ if $Mod(\Gamma) \subseteq Mod(\varphi)$.
Notation: $\Gamma \models \varphi$.

Definition 1.23

- Γ *is satisfiable if it has a model.*
- Γ *is finitely satisfiable if every finite subset of* Γ *is satisfiable.*
- *If* Γ *is not satisfiable, we say also that* Γ *is unsatisfiable or contradictory.*

Proposition 1.24

The following are equivalent:

- Γ *is unsatisfiable.*
- $\Gamma \models \perp$.

Theorem 1.25 (Compactness Theorem)

Γ *is satisfiable iff* Γ *is finitely satisfiable.*

Syntax

We use a deductive system of Hilbert type for LP.

Logical axioms

The set Axm of (logical) axioms of LP consists of:

(A1) $\varphi \to (\psi \to \varphi)$

(A2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(A3) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$,

where φ, ψ and χ are formulas.

The deduction rule

For any formulas φ, ψ, from φ and $\varphi \to \psi$ infer ψ (modus ponens or (MP)):

$$\frac{\varphi,\ \varphi \to \psi}{\psi}$$

Syntax

Let Γ be a set of formulas. The definition of Γ-theorems is another example of an inductive definition.

Definition 1.26

The *Γ-theorems* *of PL are the formulas defined as follows:*

- *(T0) Every logical axiom is a Γ-theorem.*
- *(T1) Every formula of Γ is a Γ-theorem.*
- *(T2) If φ and $\varphi \to \psi$ are Γ-theorems, then ψ is a Γ-theorem.*
- *(T3) Only the formulas obtained by applying rules (T0), (T1), (T2) are Γ-theorems.*

If φ is a Γ-theorem, then we also say that φ is deduced from the hypotheses Γ.

Syntax

Notations

$\Gamma \vdash \varphi$	:⇔	φ is a Γ-theorem
$\vdash \varphi$	:⇔	$\emptyset \vdash \varphi$.

Definition 1.27

A formula φ is called a theorem of LP if $\vdash \varphi$.

By a reformulation of the conditions (T0), (T1), (T2) using the notation $\vdash$, we get

Remark 1.28

- *If φ is an axiom, then $\Gamma \vdash \varphi$.*
- *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*
- *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.*

Definition 1.29

A Γ*-proof (or proof from the hypotheses* Γ*) is a sequence of formulas* $\theta_1, \ldots, \theta_n$ *such that for all* $i \in \{1, \ldots, n\}$*, one of the following holds:*

- θ_i *is an axiom.*
- $\theta_i \in \Gamma$.
- *there exist* $k, j < i$ *such that* $\theta_k = \theta_j \to \theta_i$.

Definition 1.30

Let φ *be a formula. A* Γ*-proof of* φ *or a proof of* φ *from the hypotheses* Γ *is a* Γ*-proof* $\theta_1, \ldots, \theta_n$ *such that* $\theta_n = \varphi$.

Proposition 1.31

For any formula φ,

$$\Gamma \vdash \varphi \quad \text{iff} \quad \text{there exists a } \Gamma\text{-proof of } \varphi.$$

Theorem 1.32 (Deduction Theorem)

Let $\Gamma \cup \{\varphi, \psi\}$ *be a set of formulas. Then*

$$\Gamma \cup \{\varphi\} \vdash \psi \quad \textit{iff} \quad \Gamma \vdash \varphi \to \psi.$$

Proposition 1.33

For any formulas φ, ψ, χ,

$$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$

$$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$$

Proposition 1.34

Let $\Gamma \cup \{\varphi, \psi, \chi\}$ *be a set of formulas.*

$$\Gamma \vdash \varphi \to \psi \text{ and } \Gamma \vdash \psi \to \chi \quad \Rightarrow \quad \Gamma \vdash \varphi \to \chi$$

$$\Gamma \cup \{\neg\psi\} \vdash \neg(\varphi \to \varphi) \quad \Rightarrow \quad \Gamma \vdash \psi$$

$$\Gamma \cup \{\psi\} \vdash \varphi \text{ and } \Gamma \cup \{\neg\psi\} \vdash \varphi \quad \Rightarrow \quad \Gamma \vdash \varphi.$$

Consistent sets

Let Γ be a set of formulas.

Definition 1.35

Γ is called consistent if there exists a formula φ such that $\Gamma \not\vdash \varphi$. Γ is said to be inconsistent if it is not consistent, that is $\Gamma \vdash \varphi$ for any formula φ.

Proposition 1.36

- *$\emptyset$ is consistent.*
- *The set of theorems is consistent.*

Proposition 1.37

The following are equivalent:

- *Γ is inconsistent.*
- *$\Gamma \vdash \bot$.*

Theorem 1.38 (Completeness Theorem (version 1))

Let Γ be a set of formulas. Then

$$\Gamma \text{ is consistent} \iff \Gamma \text{ is satisfiable.}$$

Theorem 1.39 (Completeness Theorem (version 2))

Let Γ be a set of formulas. For any formula φ,

$$\Gamma \vdash \varphi \iff \Gamma \models \varphi.$$

First-order logic

Definition 2.1

A first-order language $\mathcal{L}$ consists of:

- *a countable set $V = \{v_n \mid n \in \mathbb{N}\}$ of variables;*
- *the connectives $\neg$ and $\rightarrow$;*
- *parantheses (,);*
- *the equality symbol $=$;*
- *the universal quantifier $\forall$;*
- *a set $\mathcal{R}$ of relation symbols;*
- *a set $\mathcal{F}$ of function symbols;*
- *a set $\mathcal{C}$ of constant symbols;*
- *an arity function $\mathrm{ari} : \mathcal{F} \cup \mathcal{R} \rightarrow \mathbb{N}^*$.*

- $\mathcal{L}$ is uniquely determined by the quadruple $\tau := (\mathcal{R}, \mathcal{F}, \mathcal{C}, \mathrm{ari})$.
- τ is called the signature of $\mathcal{L}$ or the similaritaty type of $\mathcal{L}$.

First-order languages

Let $\mathcal{L}$ be a first-order language.

- The set $Sym_{\mathcal{L}}$ of symbols of $\mathcal{L}$ is

$$Sym_{\mathcal{L}} := V \cup \{\neg, \rightarrow, (,), =, \forall\} \cup \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$$

- The elements of $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ are called non-logical symbols.
- The elements of $V \cup \{\neg, \rightarrow, (,), =, \forall\}$ are called logical symbols.

- We denote variables by $x, y, z, v, \ldots$, relation symbols by $P, Q, R \ldots$, function symbols by $f, g, h, \ldots$ and constant symbols by $c, d, e, \ldots$.

- For every $m \in \mathbb{N}^*$ we denote:

$\mathcal{F}_m$:= the set of function symbols of arity m;

$\mathcal{R}_m$:= the set of relation symbols of arity m.

Definition 2.2

The set $Expr_{\mathcal{L}}$ of expressions of $\mathcal{L}$ is the set of all finite sequences of symbols of $\mathcal{L}$.

Definition 2.3

Let $\theta = \theta_0\theta_1\ldots\theta_{k-1}$ be an expression of $\mathcal{L}$, where $\theta_i \in Sym_{\mathcal{L}}$ for all $i = 0, \ldots, k-1$.

- *If $0 \leq i \leq j \leq k-1$, then the expression $\theta_i \ldots \theta_j$ is called the (i,j)-subexpression of θ.*
- *We say that an expression ψ appears in θ if there exists $0 \leq i \leq j \leq k-1$ such that ψ is the (i,j)-subexpression of θ.*
- *We denote by $Var(\theta)$ the set of variables appearing in θ.*

Definition 2.4

The terms of $\mathcal{L}$ are the expressions defined as follows:

(T0) *Every variable is a term.*

(T1) *Every constant symbol is a term.*

(T2) *If $m \geq 1$, $f \in \mathcal{F}_m$ and $t_1, \ldots, t_m$ are terms, then $ft_1 \ldots t_m$ is a term.*

(T3) *Only the expressions obtained by applying rules (T0), (T1), (T2) are terms.*

Notations:

- The set of terms is denoted by $Term_{\mathcal{L}}$.
- Terms are denoted by $t, s, t_1, t_2, s_1, s_2, \ldots$.
- $Var(t)$ is the set of variables that appear in the term t.

Definition 2.5

A term t is called closed if $Var(t) = \emptyset$.

Proposition 2.6 (Induction on terms)

Let Γ *be a set of terms satisfying the following properties:*

- Γ *contains the variables and the constant symbols.*
- *If* $m \geq 1$, $f \in \mathcal{F}_m$ *and* $t_1, \ldots, t_m \in \Gamma$, *then* $ft_1 \ldots t_m \in \Gamma$.

Then $\Gamma = Term_{\mathcal{L}}$.

It is used to prove that all terms have a property $\mathcal{P}$: we define Γ as the set of all terms satisfying $\mathcal{P}$ and apply induction on terms to obtain that $\Gamma = Term_{\mathcal{L}}$.

Definition 2.7

The atomic formulas of $\mathcal{L}$ are the expressions having one of the following forms:

- $(s = t)$, *where* s, t *are terms;*
- $(Rt_1 \ldots t_m)$, *where* $R \in \mathcal{R}_m$ *and* $t_1, \ldots, t_m$ *are terms.*

Definition 2.8

The formulas of $\mathcal{L}$ are the expressions defined as follows:

(F0) *Every atomic formula is a formula.*

(F1) *If* φ *is a formula, then* $(\neg\varphi)$ *is a formula.*

(F2) *If* φ *and* ψ *are formulas, then* $(\varphi \to \psi)$ *is a formula.*

(F3) *If* φ *is a formula, then* $(\forall x \varphi)$ *is a formula for every variable* x.

(F4) *Only the expressions obtained by applying rules (F0), (F1), (F2), (F3) are formulas.*

Notations

- The set of formulas is denoted by $Form_{\mathcal{L}}$.
- Formulas are denoted by $\varphi, \psi, \chi, \ldots$.
- $Var(\varphi)$ is the set of variables that appear in the formula φ.

Unique readability

If φ is a formula, then exactly one of the following hold:

- $\varphi = (s = t)$, where s, t are terms.
- $\varphi = (Rt_1 \ldots t_m)$, where $R \in \mathcal{R}_m$ and $t_1, \ldots, t_m$ are terms.
- $\varphi = (\neg\psi)$, where ψ is a formula.
- $\varphi = (\psi \to \chi)$, where ψ, χ are formulas.
- $\varphi = (\forall x \psi)$, where x is a variable and ψ is a formula.

Furthermore, φ can be written in a unique way in one of these forms.

Proposition 2.9 (Induction principle on formulas)

Let Γ be a set of formulas satisfying the following properties:

- *Γ contains all atomic formulas.*
- *Γ is closed to $\neg, \rightarrow$ and $\forall x$ (for any variable x), that is:*

$$\text{if } \varphi, \psi \in \Gamma, \text{ then } (\neg\varphi), (\varphi \rightarrow \psi), (\forall x \varphi) \in \Gamma.$$

Then $\Gamma = Form_{\mathcal{L}}$.

It is used to prove that all formulas have a property $\mathcal{P}$: we define Γ as the set of all formulas satisfying $\mathcal{P}$ and apply induction on formulas to obtain that $\Gamma = Form_{\mathcal{L}}$.

Derived connectives

Connectives $\vee$, $\wedge$, $\leftrightarrow$ and the existential quantifier $\exists$ are introduced by the following abbreviations:

$$
\begin{aligned}
\varphi \vee \psi &:= ((\neg\varphi) \to \psi) \\
\varphi \wedge \psi &:= \neg(\varphi \to (\neg\psi))) \\
\varphi \leftrightarrow \psi &:= ((\varphi \to \psi) \wedge (\psi \to \varphi)) \\
\exists x\varphi &:= (\neg\forall x(\neg\varphi))
\end{aligned}
$$