FMI, Computer Science, Master
Advanced Logic for Computer Science

Seminar 1

(S1.1) Consider the first-order language $\mathcal{L}_{ar} = (\dot{<}; \dot{+}, \dot{\times}, \dot{S}; \dot{0})$ (the language of arithmetics) and the $\mathcal{L}_{ar}$-structure $\mathcal{N} = (\mathbb{N}, <, +, \cdot, S, 0)$.

(i) Let $x, y \in V$ with $x \neq y$ and $t = \dot{S}x \dot{\times} \dot{S}\dot{S}y = \dot{\times}(\dot{S}x, \dot{S}\dot{S}y)$. Evaluate $t^{\mathcal{N}}(e)$, where $e : V \to \mathbb{N}$ is an assignment verifying $e(x) = 3$ and $e(y) = 7$.

(ii) Let $\varphi = x \dot{<} \dot{S}y \to (x \dot{<} y \vee x = y) = \dot{<}(x, \dot{S}y) \to (\dot{<}(x, y) \vee x = y)$. Prove that $\mathcal{N} \vDash \varphi[e]$ for all $e : V \to \mathbb{N}$.

Notation. *Let $\mathcal{L}$ be a first-order language. For any variables x, y with $x \neq y$, $\mathcal{L}$-structure $\mathcal{A}$, $e : V \to A$ and $a, b \in A$, we have that:*

$$(e_{y \leftarrow b})_{x \leftarrow a} = (e_{x \leftarrow a})_{y \leftarrow b}.$$

In this case, we denote their common value with$e_{x \leftarrow a, y \leftarrow b}$. Thus,

$$e_{x \leftarrow a, y \leftarrow b} : V \to A, \quad e_{x \leftarrow a, y \leftarrow b}(v) = \begin{cases} e(v) & \textit{dacă } v \neq x \textit{ and } v \neq y \\ a & \textit{dacă } v = x \\ b & \textit{dacă } v = y. \end{cases}$$

(S1.2) Let $\mathcal{L}$ be a first-order language. Prove that for any formulas φ, ψ and any distinct variables x, y,

(i) $\neg \exists x \varphi \mathrel{\text{⫤}\!\vDash} \forall x \neg \varphi$;

(ii) $\forall x(\varphi \wedge \psi) \mathrel{\text{⫤}\!\vDash} \forall x \varphi \wedge \forall x \psi$;

(iii) $\exists y \forall x \varphi \vDash \forall x \exists y \varphi$;

(iv) $\forall x(\varphi \to \psi) \vDash \forall x \varphi \to \forall x \psi$.

(S1.3) Let x, y be distinct variables. Give examples of first-order languages $\mathcal{L}$ and formulas φ, ψ of $\mathcal{L}$ such that:

(i) $\forall x(\varphi \vee \psi) \nvDash \forall x \varphi \vee \forall x \psi$;

(ii) $\exists x \varphi \wedge \exists x \psi \nvDash \exists x(\varphi \wedge \psi)$;

(iii) $\forall x \exists y \varphi \nvDash \exists y \forall x \varphi$.