FMI, Computer Science, Master
Advanced Logic for Computer Science

Seminar 2

(S2.1) Let $\mathcal{L}$ be a first-order language. Prove that for any formulas φ, ψ of $\mathcal{L}$ and any variable $x \notin FV(\varphi)$,

$$\forall x(\varphi \wedge \psi) \quad \mathrel{\text{⫤⊨}} \quad \varphi \wedge \forall x\psi \tag{1}$$
$$\exists x(\varphi \vee \psi) \quad \mathrel{\text{⫤⊨}} \quad \varphi \vee \exists x\psi \tag{2}$$
$$\varphi \quad \mathrel{\text{⫤⊨}} \quad \exists x\varphi \tag{3}$$
$$\forall x(\varphi \to \psi) \quad \mathrel{\text{⫤⊨}} \quad \varphi \to \forall x\psi \tag{4}$$
$$\exists x(\psi \to \varphi) \quad \mathrel{\text{⫤⊨}} \quad \forall x\psi \to \varphi. \tag{5}$$

Proof. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $e : V \to A$.
We prove (1):

$$\begin{aligned}
\mathcal{A} \vDash (\forall x(\varphi \wedge \psi))[e] &\iff \text{for all } a \in A,\ \mathcal{A} \vDash (\varphi \wedge \psi)[e_{x\leftarrow a}] \\
&\iff \text{for all } a \in A,\ \big(\mathcal{A} \vDash \varphi[e_{x\leftarrow a}] \text{ and } \mathcal{A} \vDash \psi[e_{x\leftarrow a}]\big) \\
&\iff \text{for all } a \in A,\ \big(\mathcal{A} \vDash \varphi[e] \text{ and } \mathcal{A} \vDash \psi[e_{x\leftarrow a}]\big) \text{ (by P. 1.26.(ii))} \\
&\iff \mathcal{A} \vDash \varphi[e] \text{ and for all } a \in A,\ \mathcal{A} \vDash \psi[e_{x\leftarrow a}] \\
&\iff \mathcal{A} \vDash \varphi[e] \text{ and } \mathcal{A} \vDash \forall x\psi[e] \\
&\iff \mathcal{A} \vDash (\varphi \wedge \forall x\psi)[e].
\end{aligned}$$

We prove (2):

$$\begin{aligned}
\mathcal{A} \vDash (\exists x(\varphi \vee \psi))[e] &\iff \text{there exists } a \in A \text{ such that } \mathcal{A} \vDash (\varphi \vee \psi)[e_{x\leftarrow a}] \\
&\iff \text{there exists } a \in A \text{ such that } \big(\mathcal{A} \vDash \varphi[e_{x\leftarrow a}] \text{ or } \mathcal{A} \vDash \psi[e_{x\leftarrow a}]\big) \\
&\iff \text{there exists } a \in A \text{ such that } \big(\mathcal{A} \vDash \varphi[e] \text{ or } \mathcal{A} \vDash \psi[e_{x\leftarrow a}]\big) \text{ (by P. 1.26.(ii))} \\
&\iff \mathcal{A} \vDash \varphi[e] \text{ or there exists } a \in A \text{ such that } \mathcal{A} \vDash \psi[e_{x\leftarrow a}] \\
&\iff \mathcal{A} \vDash \varphi[e] \text{ or } \mathcal{A} \vDash \exists x\psi[e] \\
&\iff \mathcal{A} \vDash (\varphi \vee \exists x\psi)[e].
\end{aligned}$$

We prove (3):

$$\begin{aligned}
\mathcal{A} \vDash \exists x\varphi[e] &\iff \text{there exists } a \in A \text{ such that } \mathcal{A} \vDash \varphi[e_{x\leftarrow a}] \\
&\iff \text{there exists } a \in A \text{ such that } \mathcal{A} \vDash \varphi[e] \text{ (by P. 1.26.(ii))} \\
&\iff \mathcal{A} \vDash \varphi[e].
\end{aligned}$$

We prove (4):

$$
\begin{aligned}
\mathcal{A} \vDash (\forall x(\varphi \to \psi))[e] &\iff \text{for all } a \in A,\ \mathcal{A} \vDash (\varphi \to \psi)[e_{x\leftarrow a}] \\
&\iff \text{for all } a \in A,\ \big(\mathcal{A} \nvDash \varphi[e_{x\leftarrow a}] \text{ or } \mathcal{A} \vDash \psi[e_{x\leftarrow a}]\big) \\
&\iff \text{for all } a \in A,\ \big(\mathcal{A} \nvDash \varphi[e] \text{ or } \mathcal{A} \vDash \psi[e_{x\leftarrow a}]\ \big) \text{ (by P. 1.26.(ii))} \\
&\iff \mathcal{A} \nvDash \varphi[e] \text{ or for all } a \in A,\ \mathcal{A} \vDash \psi[e_{x\leftarrow a}] \\
&\iff \mathcal{A} \nvDash \varphi[e] \text{ or } \mathcal{A} \vDash \forall x\psi[e] \\
&\iff \mathcal{A} \vDash (\varphi \to \forall x\psi)[e].
\end{aligned}
$$

We prove (5):

$$
\begin{aligned}
\mathcal{A} \vDash \exists x(\psi \to \varphi)[e] &\iff \text{there exists } a \in A \text{ such that } \mathcal{A} \vDash (\psi \to \varphi)[e_{x\leftarrow a}] \\
&\iff \text{there exists } a \in A \text{ such that } \big(\mathcal{A} \nvDash \psi[e_{x\leftarrow a}] \text{ or } \mathcal{A} \vDash \varphi[e_{x\leftarrow a}]\big) \\
&\iff \text{there exists } a \in A \text{ such that } \big(\mathcal{A} \nvDash \psi[e_{x\leftarrow a}] \text{ or } \mathcal{A} \vDash \varphi[e]\big) \\
&\qquad \text{(by P. 1.26.(ii))} \\
&\iff \big(\text{there exists } a \in A \text{ such that } \mathcal{A} \nvDash \psi[e_{x\leftarrow a}]\big) \text{ or } \mathcal{A} \vDash \varphi[e] \\
&\iff \big(\text{it is not true that for all } a \in A \text{ we have that } \mathcal{A} \vDash \psi[e_{x\leftarrow a}]\big) \\
&\qquad \text{or } \mathcal{A} \vDash \varphi[e] \\
&\iff \big(\text{it is not true that } \mathcal{A} \vDash \forall x\psi[e]\big) \text{ or } \mathcal{A} \vDash \varphi[e] \\
&\iff \mathcal{A} \nvDash \forall x\psi[e] \text{ or } \mathcal{A} \vDash \varphi[e] \\
&\iff \mathcal{A} \vDash (\forall x\psi \to \varphi)[e].
\end{aligned}
$$

□

(S2.2) Let $\mathcal{L}$ be a first-order language that contains

- two unary relation symbols R, S and two binary relation symbols P, Q;
- a unary function symbol f and a binary function symbol g;
- two constant symbols c, d.

Find prenex normal forms for the following formulas of $\mathcal{L}$:

$$
\begin{aligned}
\varphi_1 &= \forall x(f(x) = c) \land \neg\forall z(g(y, z) = d) \\
\varphi_2 &= \forall y(\forall x P(x, y) \to \exists z Q(x, z)) \\
\varphi_3 &= \exists x\forall y P(x, y) \lor \neg\exists y(S(y) \to \forall z R(z)) \\
\varphi_4 &= \exists z(\exists x Q(x, z) \lor \exists x R(x)) \to \neg(\neg\exists x R(x) \land \forall x\exists z Q(z, x))
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\forall x(f(x) = c) \land \neg\forall z(g(y, z) = d) &\mathrel{\text{⊨⫤}} \forall x\big(f(x) = c \land \exists z\neg(g(y, z) = d)\big) \\
&\mathrel{\text{⊨⫤}} \forall x\exists z\big(f(x) = c \land \neg(g(y, z) = d)\big)
\end{aligned}
$$

$$
\begin{aligned}
\forall y(\forall x P(x, y) \to \exists z Q(x, z)) &\mathrel{\text{⊨⫤}} \forall y\exists z(\forall x P(x, y) \to Q(x, z)) \\
&\mathrel{\text{⊨⫤}} \forall y\exists z(\forall u P(u, y) \to Q(x, z)) \\
&\mathrel{\text{⊨⫤}} \forall y\exists z\exists u(P(u, y) \to Q(x, z)).
\end{aligned}
$$

$$
\begin{aligned}
\exists x\forall y P(x,y)\vee\neg\exists y(S(y)\to\forall z R(z)) \;&\mathrel{\text{⫤}\!\vDash}\; \exists x\big(\forall y P(x,y)\vee\neg\exists y\forall z(S(y)\to R(z))\big)\\
&\mathrel{\text{⫤}\!\vDash}\; \exists x\big(\forall y P(x,y)\vee\forall y\exists z\neg(S(y)\to R(z))\big)\\
&\mathrel{\text{⫤}\!\vDash}\; \exists x\big(\forall u P(x,u)\vee\forall y\exists z\neg(S(y)\to R(z))\big)\\
&\mathrel{\text{⫤}\!\vDash}\; \exists x\forall u\forall y\exists z\big(P(x,u)\vee\neg(S(y)\to R(z))\big)
\end{aligned}
$$

$$
\begin{aligned}
\exists z(\exists x Q(x,z)\vee\exists x R(x))\to\neg(\neg\exists x R(x)\wedge\forall x\exists z Q(z,x)) \;&\mathrel{\text{⫤}\!\vDash}\\
\exists z\exists x(Q(x,z)\vee R(x))\to(\neg\neg\exists x R(x)\vee\neg\forall x\exists z Q(z,x)) \;&\mathrel{\text{⫤}\!\vDash}\\
\exists z\exists x(Q(x,z)\vee R(x))\to(\exists x R(x)\vee\exists x\forall z\neg Q(z,x)) \;&\mathrel{\text{⫤}\!\vDash}\\
\exists z\exists x(Q(x,z)\vee R(x))\to\exists x(R(x)\vee\forall z\neg Q(z,x)) \;&\mathrel{\text{⫤}\!\vDash}\\
\exists z\exists x(Q(x,z)\vee R(x))\to\exists x\forall z(R(x)\vee\neg Q(z,x)) \;&\mathrel{\text{⫤}\!\vDash}\\
\exists z\exists x(Q(x,z)\vee R(x))\to\exists u\forall v(R(u)\vee\neg Q(v,u)) \;&\mathrel{\text{⫤}\!\vDash}\\
\forall z\forall x\exists u\forall v\big((Q(x,z)\vee R(x))\to(R(u)\vee\neg Q(v,u))\big)
\end{aligned}
$$

□

(S2.3) Axiomatize the following classes of graphs:

(i) complete graphs;

(ii) graphs with at least one path of length 3;

(iii) graphs with at least one cycle of length 3;

(iv) graphs with the property that any vertex has exactly one incident edge.

Proof. We use the notations from the lectures. We take $\mathcal{L}_{Graf} = (\dot{E})$. Graph theory is $Th((IREFL), (SIM))$. We denote by $\mathcal{K}$ the class of graphs that will be axiomatized.

(i) We add the sentence

$$\varphi_1 := \forall x\forall y(\neg(x = y) \to \dot{E}(x, y)).$$

Then $\mathcal{K} = Mod(Th((IREFL), (SIM), \varphi_1))$.

(ii) We add the sentence

$$\varphi_2 := \exists v_1\exists v_2\exists v_3\exists v_4 \left(\bigwedge_{1\le i<j\le 4} \neg(v_i = v_j) \wedge \dot{E}(v_1, v_2) \wedge \dot{E}(v_2, v_3) \wedge \dot{E}(v_3, v_4)\right).$$

Then $\mathcal{K} = Mod(Th((IREFL), (SIM), \varphi_2))$.

(iii) We add the sentence

$$\varphi_3 := \exists v_1 \exists v_2 \exists v_3 \left(\bigwedge_{1 \le i < j \le 3} \neg(v_i = v_j) \wedge \dot{E}(v_1, v_2) \wedge \dot{E}(v_2, v_3) \wedge \dot{E}(v_3, v_1) \right).$$

Then $\mathcal{K} = Mod(Th((IREFL), (SIM), \varphi_3))$.

(iv) We add the sentence

$$\varphi_4 := \forall x \exists y \dot{E}(x, y) \wedge \forall x \forall y \forall z (\dot{E}(x, y) \wedge \dot{E}(x, z) \to y = z).$$

Then $\mathcal{K} = Mod(Th((IREFL), (SIM), \varphi_4))$.

□

(S2.4) Let $\mathcal{L}$ be a first-order language, φ, ψ be formulas and x be a variable. Prove that:

(i) $\vDash \varphi$ implies $\vDash \forall x \varphi$;

(ii) $\vDash \forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$.

Proof. (i) Assume that $\vDash \varphi$. We have to prove that $\vDash \forall x \varphi$, that is, for any $\mathcal{L}$-structure $\mathcal{A}$ and any $\mathcal{A}$-assignment $e : V \to A$, we have that $\mathcal{A} \vDash (\forall x \varphi)[e]$.
Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $e : V \to A$. We get that $\mathcal{A} \vDash (\forall x \varphi)[e]$ iff for all $a \in A$, $\mathcal{A} \vDash \varphi[e_{x \leftarrow a}]$. But this is true, taking into account the fact that $\vDash \varphi$, hence $\mathcal{A} \vDash \varphi[e']$, with $e' := e_{x \leftarrow a}$.

(ii) Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $e : V \to A$ be an $\mathcal{A}$-assignment. We have to prove that

$$\mathcal{A} \vDash (\forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi))[e].$$

We assume that

$$(*) \quad \mathcal{A} \vDash (\forall x(\varphi \to \psi))[e]$$

and we wish to get that

$$\mathcal{A} \vDash (\forall x \varphi \to \forall x \psi))[e].$$

Suppose that

$$(**) \quad \mathcal{A} \vDash (\forall x \varphi)[e].$$

We have to prove that $\mathcal{A} \vDash (\forall x \psi))[e]$.

Let $a \in A$. Applying (*), we get that $\mathcal{A} \vDash (\varphi \to \psi)[e_{x \leftarrow a}]$, and, by (**), we have that $\mathcal{A} \vDash \varphi[e_{x \leftarrow a}]$. From $\mathcal{A} \vDash (\varphi \to \psi)[e_{x \leftarrow a}]$ and $\mathcal{A} \vDash \varphi[e_{x \leftarrow a}]$, it follows that $\mathcal{A} \vDash \psi[e_{x \leftarrow a}]$. Thus, $\mathcal{A} \vDash (\forall x \psi))[e]$.

□