FMI, Computer Science, Master
Advanced Logic for Computer Science

Seminar 5

(S5.1) Let $\mathcal{M} = (W, R, V)$ be a model for ML_0 and w a state in $\mathcal{M}$. Prove that for every formula φ and any $n \geq 1$,

$$\begin{array}{lll} (1) & \mathcal{M}, w \Vdash \Diamond^n \varphi \iff & \text{there exists } v \in W \text{ such that } R^n wv \text{ and } \mathcal{M}, v \Vdash \varphi \\ (2) & \mathcal{M}, w \Vdash \Box^n \varphi \iff & \text{for every } v \in W, R^n wv \text{ implies } \mathcal{M}, v \Vdash \varphi. \end{array}$$

Proof. We prove (1) by induction on n.
$n = 1$: Apply Definition 2.13.
$n \Rightarrow n + 1$: We have that

$$\begin{array}{lll} \mathcal{M}, w \Vdash \Diamond^{n+1} \varphi & \text{iff} & \mathcal{M}, w \Vdash \Diamond^n \Diamond \varphi \\ & \text{iff} & \text{there exists } u \in W \text{ such that } R^n wu \text{ and } \mathcal{M}, u \Vdash \Diamond \varphi \\ & & \text{by the induction hypothesis} \\ & \text{iff} & \text{there exist } u, v \in W \text{ such that } R^n wu, Ruv \text{ and } \mathcal{M}, v \Vdash \varphi \\ & \text{iff} & \text{there exists } v \in W \text{ such that } R^{n+1} wv \text{ and } \mathcal{M}, v \Vdash \varphi. \end{array}$$

(2) is proved similarly.

□

(S5.2) Prove that for every $p, q \in PROP$ the formula

$$\Box(p \to q) \to (\Box p \to \Box q)$$

is valid in the class of all frames for ML_0.

Proof. Let $\mathcal{F} = (W, R)$ be an arbitrary frame, w a state in $\mathcal{F}$ and $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$. We have to show that

$$\mathcal{M}, w \Vdash \Box(p \to q) \to (\Box p \to \Box q).$$

Suppose that

$$(*) \quad \mathcal{M}, w \Vdash \Box(p \to q).$$

We have to show that $\mathcal{M}, w \Vdash \Box p \to \Box q$. Assume, furthermore, that

$$(**) \quad \mathcal{M}, w \Vdash \Box p.$$

It remains to prove that $\mathcal{M}, w \Vdash \Box q$. Let $v \in W$ be such that Rwv. Applying (*) and (**) we obtain that $\mathcal{M}, v \Vdash p \to q$ and $\mathcal{M}, v \Vdash p$. It follows immediately that $\mathcal{M}, v \Vdash q$. Thus, $\mathcal{M}, w \Vdash \Box q$. □

(S5.3) Prove that for any formula φ,

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$$

is valid in the class of all frames for ML_0.

Proof. Let $\mathcal{F} = (W, R)$ be an arbitrary frame, w a state in $\mathcal{F}$ and $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$. We have that

$$\begin{array}{lcl}
\mathcal{M}, w \Vdash \neg\Box\neg\varphi & \iff & \mathcal{M}, w \nVdash \Box\neg\varphi \\
& \iff & \text{it is not true that for every } v \in W, \\
& & Rwv \text{ implies } \mathcal{M}, v \Vdash \neg\varphi \\
& \iff & \text{there exists } v \in W \text{ such that it is not true that} \\
& & Rwv \text{ implies } \mathcal{M}, v \Vdash \neg\varphi \\
& \iff & \text{there exists } v \in W \text{ such that it is not true that} \\
& & Rwv \text{ is false or } \mathcal{M}, v \Vdash \neg\varphi \\
& \iff & \text{there exists } v \in W \text{ such that } Rwv \text{ and } \mathcal{M}, v \nVdash \neg\varphi \\
& \iff & \text{there exists } v \in W \text{ such that } Rwv \text{ and } \mathcal{M}, v \Vdash \varphi \\
& \iff & \mathcal{M}, w \Vdash \Diamond\varphi.
\end{array}$$

Hence, $\mathcal{M}, w \Vdash \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$. □

(S5.4) Let $p \in PROP$. Prove that the formula

$$\Box p \to \Diamond p$$

is not valid in the class of all frames for ML_0.

Proof. Let $\mathcal{F} = (W, R)$, where $W = \{1, 2\}, R = \{(1, 1), (1, 2)\}$ and $\mathcal{M} = (\mathcal{F}, V)$ be an arbitrary model based on $\mathcal{F}$.
We have that

$$\begin{array}{lcl}
\mathcal{M}, 2 \Vdash \Box p & \iff & \text{for every } n \in W,\ R2n \text{ implies } \mathcal{M}, n \Vdash p, \\
\mathcal{M}, 2 \Vdash \Diamond p & \iff & \text{there exists } n \in W \text{ such that } R2n \text{ and } \mathcal{M}, n \Vdash p.
\end{array}$$

Since there exists no $n \in W$ such that $R2n$, we have that $\mathcal{M}, 2 \Vdash \Box p$ and $\mathcal{M}, 2 \nVdash \Diamond p$, so $\mathcal{M}, 2 \nVdash \Box p \to \Diamond p$. It follows that $\mathcal{F}, 2 \nVdash \Box p \to \Diamond p$, hence $\Box p \to \Diamond p$ is not valid in $\mathcal{F}$. □

(S5.5) Let $p, q \in PROP$. Verify if the following formulas are valid in the class of all frames for ML_0:

(i) $p \to \Diamond p$.

(ii) $\Box p \wedge \Diamond q \to \Diamond(p \wedge q)$.

Proof. (i) The answer is NO. Let $\mathcal{M}_0 = (W_0, R_0, V_0)$, where

$$W_0 = \{a, b\}, \quad R_0 = \{(a, b)\}, \quad V_0(p) = \{a\}.$$

Then $\mathcal{M}, a \Vdash p$, but $\mathcal{M}, a \nVdash \Diamond p$, since b is the only state R_0-accesible from a and $b \notin V_0(p)$, hence $\mathcal{M}, b \nVdash p$. Thus, $\mathcal{M}, a \nVdash p \to \Diamond p$.

(ii) The answer is YES. Let $\mathcal{F} = (W, R)$ be an arbitrary frame, w a state in $\mathcal{F}$ and $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$. We have to show that

$$\mathcal{M}, w \Vdash \Box p \wedge \Diamond q \to \Diamond(p \wedge q).$$

Assume that $\mathcal{M}, w \Vdash \Box p \wedge \Diamond q$, that is $\mathcal{M}, w \Vdash \Box p$ and $\mathcal{M}, w \Vdash \Diamond q$. As $\mathcal{M}, w \Vdash \Diamond q$, there exists $v \in W$ such that Rwv and $\mathcal{M}, v \Vdash q$. As $\mathcal{M}, w \Vdash \Box p$ and Rwv, we have that $\mathcal{M}, v \Vdash p$. It follows that $v \in W$ is such that Rwv and $\mathcal{M}, v \Vdash p \wedge q$. Thus, $\mathcal{M}, v \Vdash \Diamond(p \wedge q)$.

□