FMI, Computer Science, Master
Advanced Logic for Computer Science

Seminar 6

(S6.1) Let Λ be a normal logic. Prove that for any formulas φ, ψ,

(i) $\vdash_\Lambda \varphi \to \psi$ implies $\vdash_\Lambda \Box\varphi \to \Box\psi$.

(ii) $\vdash_\Lambda \varphi \leftrightarrow \psi$ implies $\vdash_\Lambda \Box\varphi \leftrightarrow \Box\psi$.

Proof. (i)

(1)	$\vdash_\Lambda \varphi \to \psi$	hypothesis
(2)	$\vdash_\Lambda \Box(\varphi \to \psi)$	generalization: (1)
(3)	$\vdash_\Lambda \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	(K)
(4)	$\vdash_\Lambda \Box\varphi \to \Box\psi$	(MP): (2), (3).

(ii) Assume that $\vdash_\Lambda \varphi \leftrightarrow \psi$. Then $\vdash_\Lambda \varphi \to \psi$ and $\vdash_\Lambda \psi \to \varphi$. Applying (i) twice, we get that $\vdash_\Lambda \Box\varphi \to \Box\psi$ and $\vdash_\Lambda \Box\psi \to \Box\varphi$, hence $\vdash_\Lambda (\Box\varphi \to \Box\psi) \wedge (\Box\psi \to \Box\varphi)$. Thus, $\vdash_\Lambda \Box\varphi \leftrightarrow \Box\psi$.

□

(S6.2) Prove that for any formulas φ, ψ,

$$\vdash_{\boldsymbol{K}} \Box\varphi \vee \Box\psi \to \Box(\varphi \vee \psi).$$

Proof. We use the following notations:

$\chi_1 := \Box\varphi \to \Box(\varphi \vee \psi)$, $\chi_2 := \Box\psi \to \Box(\varphi \vee \psi)$ and $\chi_3 := \Box\varphi \vee \Box\psi \to \Box(\varphi \vee \psi)$.

(1)	$\vdash_{\boldsymbol{K}} \varphi \to \varphi \vee \psi$	tautology
(2)	$\vdash_{\boldsymbol{K}} \chi_1$	(S6.1).(i): (1)
(3)	$\vdash_{\boldsymbol{K}} \psi \to \varphi \vee \psi$	tautology
(4)	$\vdash_{\boldsymbol{K}} \chi_2$	(S6.1).(i): (3)
(5)	$\vdash_{\boldsymbol{K}} \chi_1 \wedge \chi_2$	Proposition 2.56: (2), (4) and the tautology $\sigma \to \sigma$ with $\sigma := \chi_1 \wedge \chi_2$
(6)	$\vdash_{\boldsymbol{K}} \chi_1 \wedge \chi_2 \to \chi_3$	tautology: $(\sigma_1 \to \sigma_3) \wedge (\sigma_2 \to \sigma_3) \to (\sigma_1 \vee \sigma_2 \to \sigma_3)$, with $\sigma_1 := \Box\varphi$, $\sigma_2 := \Box\psi$, $\sigma_3 := \Box(\varphi \vee \psi)$
(7)	$\vdash_{\boldsymbol{K}} \chi_3$	(MP): (5), (6).

□

(S6.3) Let $\Gamma \cup \{\varphi, \psi\}$ be a set of formulas. Prove that

$$\text{if } \Gamma \vdash_\Lambda \varphi \text{ and } \Gamma \vdash_\Lambda \varphi \to \psi, \text{ then } \Gamma \vdash_\Lambda \psi.$$

Proof. Since $\Gamma \vdash_\Lambda \varphi$, there exist $\theta_1, \ldots, \theta_n \in \Gamma \, (n \geq 0)$ such that

$$\vdash_\Lambda (\theta_1 \wedge \ldots \wedge \theta_n) \to \varphi.$$

Since $\Gamma \vdash_\Lambda \varphi \to \psi$, there exist $\chi_1, \ldots, \chi_p \in \Gamma \, (p \geq 0)$ such that

$$\vdash_\Lambda (\chi_1 \wedge \ldots \wedge \chi_p) \to (\varphi \to \psi).$$

We have the following cases:

(i) $n = p = 0$. Then $\vdash_\Lambda \varphi$ and $\vdash_\Lambda \varphi \to \psi$. Applying (MP), we get that $\vdash_\Lambda \psi$. Hence, $\Gamma \vdash_\Lambda \psi$.

(ii) $n \geq 1$ and $p \geq 1$. Let us denote $\theta := \theta_1 \wedge \ldots \wedge \theta_n$, $\chi := \chi_1 \wedge \ldots \wedge \chi_p$.

We have that

(1)	$\vdash_\Lambda \theta \to \varphi$	hypothesis
(2)	$\vdash_\Lambda \chi \to (\varphi \to \psi)$	hypothesis
(3)	$\vdash_\Lambda \big(\chi \to (\varphi \to \psi)\big) \to \big(\varphi \to (\chi \to \psi)\big)$	tautology
(4)	$\vdash_\Lambda \varphi \to (\chi \to \psi)$	(MP): (2), (3)
(5)	$\vdash_\Lambda \theta \to (\chi \to \psi)$	P. 2.56: (1), (4) and the tautology $(\sigma_1 \to \sigma_2) \wedge (\sigma_2 \to \sigma_3) \to (\sigma_1 \to \sigma_3)$ with $\sigma_1 := \theta$, $\sigma_2 := \varphi$, $\sigma_3 := \chi \to \psi$
(6)	$\vdash_\Lambda \theta \wedge \chi \to \psi$	P. 2.56: (5) and the tautology $(\theta \to (\chi \to \psi)) \to (\theta \wedge \chi \to \psi)$.

We have proved that $\vdash_\Lambda (\theta_1 \wedge \ldots \wedge \theta_n \wedge \chi_1 \wedge \ldots \wedge \chi_p) \to \psi$. Hence, $\Gamma \vdash_\Lambda \psi$.

(iii) $n = 0$ and $p \geq 1$. Let us denote $\chi := \chi_1 \wedge \ldots \wedge \chi_p$.

We have that

(1)	$\vdash_\Lambda \varphi$	hypothesis
(2)	$\vdash_\Lambda \chi \to (\varphi \to \psi)$	hypothesis
(3)	$\vdash_\Lambda \big(\chi \to (\varphi \to \psi)\big) \to \big(\varphi \to (\chi \to \psi)\big)$	tautology
(4)	$\vdash_\Lambda \varphi \to (\chi \to \psi)$	(MP): (2), (3)
(5)	$\vdash_\Lambda \chi \to \psi$	(MP): (1), (4).

We have proved that $\vdash_\Lambda (\chi_1 \wedge \ldots \wedge \chi_p) \to \psi$. Hence, $\Gamma \vdash_\Lambda \psi$.

(iv) $n \geq 1$ and $p = 0$. Similarly.

□

(S6.4) Let Λ be a normal logic. Prove that for any formulas φ, ψ,

(i) $\vdash_\Lambda \varphi \to \psi$ implies $\vdash_\Lambda \Diamond\varphi \to \Diamond\psi$.

(ii) $\vdash_\Lambda \varphi \leftrightarrow \psi$ implies $\vdash_\Lambda \Diamond\varphi \leftrightarrow \Diamond\psi$.

Proof. (i) We use in the sequel the following results which follow from classical propositional reasoning:

(*) $(\sigma_1 \to \sigma_2) \to (\neg\sigma_2 \to \neg\sigma_1)$ is a tautology

(**) If $\vdash_\Lambda \sigma_1 \leftrightarrow \sigma_2$, $\vdash_\Lambda \sigma_3 \leftrightarrow \sigma_4$ and $\vdash_\Lambda \sigma_2 \to \sigma_4$, then $\vdash_\Lambda \sigma_1 \to \sigma_3$ (substitution of equivalents).

(1)	$\vdash_\Lambda \varphi \to \psi$	hypothesis
(2)	$\vdash_\Lambda (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	(*) with $\sigma_1 := \varphi$, $\sigma_2 := \psi$
(3)	$\vdash_\Lambda \neg\psi \to \neg\varphi$	(MP): (1), (2)
(4)	$\vdash_\Lambda \Box\neg\psi \to \Box\neg\varphi$	(S6.1).(i): (3)
(5)	$\vdash_\Lambda (\Box\neg\psi \to \Box\neg\varphi) \to (\neg\Box\neg\varphi \to \neg\Box\neg\psi)$	(*) with $\sigma_1 := \Box\neg\psi$, $\sigma_2 := \Box\neg\varphi$
(6)	$\vdash_\Lambda \neg\Box\neg\varphi \to \neg\Box\neg\psi$	(MP): (4), (5)
(7)	$\vdash_\Lambda \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$	(Dual)
(8)	$\vdash_\Lambda \Diamond\psi \leftrightarrow \neg\Box\neg\psi$	(Dual)
(9)	$\vdash_\Lambda \Diamond\varphi \to \Diamond\psi$	(**): (7), (8), (6).

(ii) Assume that $\vdash_\Lambda \varphi \leftrightarrow \psi$. Then $\vdash_\Lambda \varphi \to \psi$ and $\vdash_\Lambda \psi \to \varphi$. Applying (i) twice, we get that $\vdash_\Lambda \Diamond\varphi \to \Diamond\psi$ and $\vdash_\Lambda \Diamond\psi \to \Diamond\varphi$, hence $\vdash_\Lambda (\Diamond\varphi \to \Diamond\psi) \wedge (\Diamond\psi \to \Diamond\varphi)$. Thus, $\vdash_\Lambda \Diamond\varphi \leftrightarrow \Diamond\psi$.

$\square$

(S6.5) Prove that for any formulas φ, ψ,

(i) $\vdash_{\boldsymbol{K}} \varphi \to \psi$ implies $\vdash_{\boldsymbol{K}} \Box\Diamond\varphi \to \Box\Diamond\psi$.

(ii) $\vdash_{\boldsymbol{K}} \Diamond\varphi \vee \Diamond\psi \to \Diamond(\varphi \vee \psi)$.

Proof. (i) We have that

(1)	$\vdash_{\boldsymbol{K}} \varphi \to \psi$	hypothesis
(2)	$\vdash_{\boldsymbol{K}} \Diamond\varphi \to \Diamond\psi$	(S6.4).(i): (1)
(3)	$\vdash_{\boldsymbol{K}} \Box\Diamond\varphi \to \Box\Diamond\psi$	(S6.1).(i): (2).

(ii) Replace in the solution of (S6.2) $\Box$ with $\Diamond$ and use (S6.4).(i) instead of (S6.1).(i). We give the details in the sequel.

We use the following notations:

$$\chi_1 := \Diamond\varphi \to \Diamond(\varphi \vee \psi),\ \chi_2 := \Diamond\psi \to \Diamond(\varphi \vee \psi) \text{ and } \chi_3 := \Diamond\varphi \vee \Diamond\psi \to \Diamond(\varphi \vee \psi).$$

(1)	$\vdash_{\boldsymbol{K}} \varphi \to \varphi \vee \psi$	tautology
(2)	$\vdash_{\boldsymbol{K}} \chi_1$	(S6.4).(i): (1)
(3)	$\vdash_{\boldsymbol{K}} \psi \to \varphi \vee \psi$	tautology
(4)	$\vdash_{\boldsymbol{K}} \chi_2$	(S6.4).(i): (3)
(5)	$\vdash_{\boldsymbol{K}} \chi_1 \wedge \chi_2$	Proposition 2.56: (2), (4) and the tautology $\sigma \to \sigma$ with $\sigma := \chi_1 \wedge \chi_2$
(6)	$\vdash_{\boldsymbol{K}} \chi_1 \wedge \chi_2 \to \chi_3$	tautology: $(\sigma_1 \to \sigma_3) \wedge (\sigma_2 \to \sigma_3) \to (\sigma_1 \vee \sigma_2 \to \sigma_3)$, with $\sigma_1 := \Diamond\varphi$, $\sigma_2 := \Diamond\psi$, $\sigma_3 := \Diamond(\varphi \vee \psi)$
(7)	$\vdash_{\boldsymbol{K}} \chi_3$	(MP): (5), (6).

$\square$