FMI, Computer Science, Master
Advanced Logic for Computer Science

Seminar 7

(S7.1) Let Λ be a normal logic and Σ be a Λ-MCS. Prove that $\Lambda \subseteq \Sigma$.

Proof. Suppose by contradiction that $\Lambda \not\subseteq \Sigma$. Then there exists φ such that $\vdash_\Lambda \varphi$ and $\varphi \notin \Sigma$. By Proposition 2.74.(ii), we get that $\Sigma \cup \{\varphi\}$ is Λ-inconsistent. Applying Proposition 2.65.(ii), it follows that $\Sigma \vdash_\Lambda \neg\varphi$. We have obtained that $\Sigma \vdash_\Lambda \varphi$ (since $\vdash_\Lambda \varphi$) and $\Sigma \vdash_\Lambda \neg\varphi$. Apply now Proposition 2.64.(ii) to get that Σ is Λ-inconsistent, which is a contradiction. $\square$

(S7.2) Let Λ be a normal logic. Prove that for all $w, v \in W^\Lambda$, the following are equivalent:

(i) $R^\Lambda wv$;

(ii) for any formula ψ,

$$\Box\psi \in w \text{ implies } \psi \in v.$$

Proof. (i)$\Rightarrow$(ii) Let ψ be a formula. We prove the contrapositive of (ii). Suppose that $\psi \notin v$. Since v is a Λ-MCS, we have, by Proposition 2.77.(iii), that $\neg\psi \in v$. Since $R^\Lambda wv$, we get that $\Diamond\neg\psi \in w$. We apply again Proposition 2.77.(iii) to obtain that $\neg\Diamond\neg\psi \notin w$, that is $\Box\psi \notin w$.
(ii)$\Rightarrow$(i) Let ψ be a formula such that $\psi \in v$. Since v is a Λ-MCS, we have, by Proposition 2.77.(iii), that $\neg\psi \notin v$. Aplying the contrapositive of (ii), we get that $\Box\neg\psi \notin w$, hence, by Proposition 2.77.(iii) we obtain that $\neg\Box\neg\psi \in w$. We apply now (Dual) and Proposition 2.77.(i) to conclude that $\Diamond\psi \in w$. $\square$

(S7.3) Let us consider the following formula in ML_0:

$$(B) \quad p \to \Box\Diamond p, \qquad \text{where } p \in PROP$$

and let $\boldsymbol{B}$ be the normal logic generated by (B). Prove the following:

(i) (B) is valid in the class of symmetric frames.

(ii) The canonical frame $\mathcal{F}^{\boldsymbol{B}} = (W^{\boldsymbol{B}}, R^{\boldsymbol{B}})$ is symmetric.

(iii) $\boldsymbol{B}$ is strongly complete with respect to the class of symmetric frames.

(iv) $\boldsymbol{B}$ is sound and weakly complete with respect to the class of symmetric frames.

Proof. (i) Let $\mathcal{F} = (W, R)$ be a symmetric frame, w a state in $\mathcal{F}$ and $\mathcal{M} = (\mathcal{F}, V)$ a model based on $\mathcal{F}$. Suppose that $\mathcal{M}, w \Vdash p$ and let $v \in W$ be such that Rwv. We have to prove that $\mathcal{M}, v \Vdash \Diamond p$, that is

$$\text{there exists } u \in W \text{ such that } Rvu \text{ and } \mathcal{M}, u \Vdash p.$$

Take $u := w$. Then Rvw (since Rwv and R is symmetric) and $\mathcal{M}, w \Vdash p$ (by assumption).

(ii) Let $w, v \in W^{\boldsymbol{B}}$ be such that $R^{\boldsymbol{B}}wv$. We have to prove that $R^{\boldsymbol{B}}vw$, that is

$$\text{for any formula } \varphi,\ \varphi \in w \text{ implies } \Diamond\varphi \in v.$$

Let φ be a formula such that $\varphi \in w$. Since w is a $\boldsymbol{B}$-MCS, we can apply Proposition 2.77.(ii) to get that $\boldsymbol{B} \subseteq w$. In particular, $\varphi \to \Box\Diamond\varphi \in w$. By modus ponens (Proposition 2.77.(i)), we get that $\Box\Diamond\varphi \in w$. Since $R^{\boldsymbol{B}}wv$, we conclude, by an application of (S7.2), that $\Diamond\varphi \in v$.

(iii) We apply Proposition 2.71. Let Γ be a $\boldsymbol{B}$-consistent set. By Theorem 2.82, Γ is satisfiable in $\mathcal{M}^{\boldsymbol{B}}$. By (ii), we have that $\mathcal{F}^{\boldsymbol{B}}$ is a symmetric frame.

(iv) Soundness follows from (i) and Theorem 2.42. Weak completeness is a particular case of (iii).

□

(S7.4) Let $ML := ML(PROP, \tau)$ be a modal language (where $\tau = (O, \rho)$), $\mathcal{M} = (W, \{R_\Delta \mid \Delta \in O\}, V)$ be a model and w a state in $\mathcal{M}$. Suppose that $\Delta \in O_m, m \geq 1$ and that ∇ is its dual operator. Then for any formulas $\varphi_1, \ldots, \varphi_m$,

$$\mathcal{M}, w \Vdash \nabla\varphi_1 \ldots \varphi_m \quad \text{iff} \quad \text{for any } v_1, \ldots, v_m \in W,\ R_\Delta wv_1 \ldots v_m \text{ implies } \mathcal{M}, v_i \Vdash \varphi_i \text{ for some } i = 1, \ldots, m.$$

Proof. We have that

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\begin{array}{lll}
\mathcal{M}, w \Vdash \nabla\varphi_1 \ldots \varphi_m & \text{iff} & \mathcal{M}, w \Vdash \neg\Delta(\neg\varphi_1) \ldots (\neg\varphi_m) \\
& \text{iff} & \mathcal{M}, w \nVdash \Delta(\neg\varphi_1) \ldots (\neg\varphi_m) \\
& \text{iff} & \text{it is not true that there exist } v_1, \ldots, v_m \in W \text{ such that} \\
& & \big(R_\Delta w v_1 \ldots v_m \text{ and (for all } i = 1, \ldots, m\text{, we have that } \mathcal{M}, v_i \Vdash \neg\varphi_i) \ \big) \\
& \text{iff} & \text{for all } v_1, \ldots, v_m \in W\text{, it is not true that} \\
& & \big(R_\Delta w v_1 \ldots v_m \text{ and (for all } i = 1, \ldots, m\text{, we have that } \mathcal{M}, v_i \Vdash \neg\varphi_i) \ \big) \\
& \text{iff} & \text{for all } v_1, \ldots, v_m \in W\text{, } (R_\Delta w v_1 \ldots v_m \text{ does not hold) or} \\
& & \text{(it is not true that for all } i = 1, \ldots, m\text{, we have that } \mathcal{M}, v_i \Vdash \neg\varphi_i) \\
& \text{iff} & \text{for all } v_1, \ldots, v_m \in W\text{, } (R_\Delta w v_1 \ldots v_m \text{ does not hold) or} \\
& & \text{(there exists } i = 1, \ldots, m \text{ such that } \mathcal{M}, v_i \nVdash \neg\varphi_i) \\
& \text{iff} & \text{for all } v_1, \ldots, v_m \in W\text{, } (R_\Delta w v_1 \ldots v_m \text{ does not hold) or} \\
& & \text{(there exists } i = 1, \ldots, m \text{ such that } \mathcal{M}, v_i \Vdash \varphi_i) \\
& \text{iff} & \text{for all } v_1, \ldots, v_m \in W\text{, } R_\Delta w v_1 \ldots v_m \\
& & \text{implies that there exists } i = 1, \ldots, m \text{ such that } \mathcal{M}, v_i \Vdash \varphi_i \\
& \text{iff} & \text{for any } v_1, \ldots, v_m \in W, \\
& & R_\Delta w v_1 \ldots v_m \text{ implies } \mathcal{M}, v_i \Vdash \varphi_i \text{ for some } i = 1, \ldots, m.
\end{array}
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