FMI, Computer Science, Master
Advanced Logic for Computer Science

Exam

First Name: ______________________

Last Name: __________________

P1	**P2**	**P3**	**P4**	**P5**	**P6**	**P7**	**P8**	**Extra point**
___/2	___/2	___/1,5	___/1,5	___/2	___/2	___/1	___/2	1

TOTAL
_____/15

1 First-order logic

(P1) [2 points]

(i) Prove that for every first-order language $\mathcal{L}$ and any formulas φ, ψ of $\mathcal{L}$, we have that

$$\forall x(\varphi \vee \psi) \vDash \exists x\varphi \vee \exists x\psi \text{ for any variable } x.$$

(ii) Give an example of a first-order language $\mathcal{L}$ and formulas φ, ψ of $\mathcal{L}$ such that:

$$\forall x\varphi \to \forall x\psi \nvDash \forall x(\varphi \to \psi), \text{ where } x \text{ is a variable.}$$

Proof. (i) Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $e : V \to A$ be an $\mathcal{A}$-assignment. We have that

$$\begin{aligned}
\mathcal{A} \vDash (\forall x(\varphi \vee \psi))[e] \quad &\Longleftrightarrow \quad \text{for any } a \in A,\ \mathcal{A} \vDash (\varphi \vee \psi)[e_{x\leftarrow a}] \\
&\Longleftrightarrow \quad \text{for any } a \in A, \\
&\qquad\quad \mathcal{A} \vDash \varphi[e_{x\leftarrow a}] \text{ or } \mathcal{A} \vDash \psi[e_{x\leftarrow a}] \\
&\Longrightarrow \quad \text{there exists } a \in A \text{ such that} \\
&\qquad\quad \mathcal{A} \vDash \varphi[e_{x\leftarrow a}] \text{ or } \mathcal{A} \vDash \psi[e_{x\leftarrow a}] \\
&\Longleftrightarrow \quad (\text{there exists } a \in A \text{ such that } \mathcal{A} \vDash \varphi[e_{x\leftarrow a}]) \text{ or} \\
&\qquad\quad (\text{there exists } a \in A \text{ such that } \mathcal{A} \vDash \psi[e_{x\leftarrow a}]) \\
&\Longleftrightarrow \quad \mathcal{A} \vDash (\exists x\varphi)[e] \text{ or } \mathcal{A} \vDash (\exists x\psi)[e] \\
&\Longleftrightarrow \quad \mathcal{A} \vDash (\exists x\varphi \vee \exists x\psi)[e].
\end{aligned}$$

(ii) Consider $\mathcal{L}_{ar} = (\dot{<}, \dot{+}, \dot{\times}, \dot{S}, \dot{0})$, the $\mathcal{L}_{ar}$-structure $\mathcal{N} := (\mathbb{N}, <, +, \cdot, S, 0)$ and $e : V \to \mathbb{N}$ be an arbitrary assignment. Let

$$\varphi := x = \dot{0}, \quad \psi := \dot{S}x = \dot{0}.$$

We have that $\mathcal{N} \vDash (\forall x\varphi \to \forall x\psi)[e] \iff \mathcal{N} \nvDash (\forall x\varphi)[e]$ or $\mathcal{N} \vDash (\forall x\psi)[e]$.

$\mathcal{N} \vDash (\forall x\varphi)[e] \iff$ for any $n \in \mathbb{N}$, we have that $n = 0$, which is obviously false. Hence, $\mathcal{N} \nvDash (\forall x\varphi)[e]$. It follows that

$$\mathcal{N} \vDash (\forall x\varphi \to \forall x\psi)[e].$$

We have that $\mathcal{N} \vDash (\forall x(\varphi \to \psi))[e] \iff$ for any $n \in \mathbb{N}$, if $n = 0$ then $n + 1 = 0$, which is obviously false. It follows that

$$\mathcal{N} \nvDash (\forall x(\varphi \to \psi))[e].$$

□

(P2) [2 points] Let $\mathcal{L}$ be a first-order language that contains

- two unary relation symbols S, T and one binary relation symbols P;
- a unary function symbol g;
- two constant symbols a, d.

(i) Find prenex normal forms for the following formulas of $\mathcal{L}$:

$$\begin{aligned}\varphi &:= \neg\exists x P(x,a) \wedge \forall y \neg S(y),\\ \psi &:= \exists x(S(x) \to \forall y(g(y) = d)) \to \neg(\forall x T(x) \vee \forall y S(y)).\end{aligned}$$

(ii) Find Skolem normal forms for the following sentences of $\mathcal{L}$:

$$\begin{aligned}\chi &:= \exists y \forall x \exists v(S(y) \vee P(x,v) \to (T(v) \to S(y)))\\ \delta &:= \forall x \exists u \forall y \exists v\left((S(u) \to P(v,y)) \vee (S(v) \to T(x))\right).\end{aligned}$$

Proof. (i) We have that

$$\begin{aligned}\varphi &\mathrel{\text{⫤⊨}} \forall x \neg P(x,a) \wedge \forall y \neg S(y))\\ &\mathrel{\text{⫤⊨}} \forall x(\neg P(x,a) \wedge \forall y \neg S(y))\\ &\mathrel{\text{⫤⊨}} \forall x \forall y(\neg P(x,a) \wedge \neg S(y))\end{aligned}$$

$$
\begin{aligned}
\psi \;& \mathrel{\dashv\vDash}\; \exists x\forall y(S(x)\to g(y)=d)\to\neg(\forall x T(x)\vee\forall y S(y))\\
& \mathrel{\dashv\vDash}\; \exists x\forall y(S(x)\to g(y)=d)\to(\neg\forall x T(x)\wedge\neg\forall y S(y))\\
& \mathrel{\dashv\vDash}\; \exists x\forall y(S(x)\to g(y)=d)\to(\exists x\neg T(x)\wedge\exists y\neg S(y))\\
& \mathrel{\dashv\vDash}\; \exists x\forall y(S(x)\to g(y)=d)\to\exists x(\neg T(x)\wedge\exists y\neg S(y))\\
& \mathrel{\dashv\vDash}\; \exists x\forall y(S(x)\to g(y)=d)\to\exists x\exists y(\neg T(x)\wedge\neg S(y))\\
& \mathrel{\dashv\vDash}\; \forall x\Big(\forall y(S(x)\to g(y)=d)\to\exists x\exists y(\neg T(x)\wedge\neg S(y))\Big)\\
& \mathrel{\dashv\vDash}\; \forall x\exists y\Big((S(x)\to g(y)=d)\to\exists x\exists y(\neg T(x)\wedge\neg S(y))\Big)\\
& \mathrel{\dashv\vDash}\; \forall x\exists y\Big((S(x)\to g(y)=d)\to\exists u\exists v(\neg T(u)\wedge\neg S(v))\Big)\\
& \mathrel{\dashv\vDash}\; \forall x\exists y\exists u\Big((S(x)\to g(y)=d)\to\exists v(\neg T(u)\wedge\neg S(v))\Big)\\
& \mathrel{\dashv\vDash}\; \forall x\exists y\exists u\exists v\Big((S(x)\to g(y)=d)\to(\neg T(u)\wedge\neg S(v))\Big).
\end{aligned}
$$

(ii) We obtain that

$$
\begin{aligned}
\chi^1 &= \forall x\exists v(S(e)\vee P(x,v)\to(T(v)\to S(e)))\\
&\quad \text{where } e \text{ is a new constant symbol}\\
\chi^2 &= \forall x(S(e)\vee P(x,h(x))\to(T(g(x))\to S(e)))\\
&\quad \text{where } h \text{ is a new unary function symbol.}
\end{aligned}
$$

As χ^2 is a universal sentence, it follows that χ^2 is a Skolem normal form for χ.

$$
\begin{aligned}
\delta^1 &= \forall x\forall y\exists v\,((S(l(x))\to P(v,y))\vee(S(v)\to T(x)))\\
&\quad \text{where } l \text{ is a new unary function symbol}\\
\delta^2 &= \forall x\forall y\,((S(l(x))\to R(n(x,y),y))\vee(S(n(x,y))\to T(x)))\\
&\quad \text{where } n \text{ is a new binary function symbol.}
\end{aligned}
$$

As δ^2 is a universal sentence, it follows that δ^2 is a Skolem normal form for δ.

□

(P3) [1,5 points] Let $\mathcal{L}$ be a first-order language and Δ be a set of sentences satisfying

(*) for all $p\in\mathbb{N}$, Δ has a finite model of cardinality $\geq p$.

Prove that the class of finite models of Δ is not axiomatizable.

Proof. Let us denote with $\mathcal{T}$ the class of finite models of Δ. Suppose by contradiction that $\mathcal{T}$ is axiomatizable and let $\Gamma\subseteq Sen_{\mathcal{L}}$ be such that $\mathcal{T}=Mod(\Gamma)$. Let

$$\Sigma := \Gamma\cup\{\exists^{\geq n}\mid n\geq 1\}.$$

We prove that Σ is satisfiable with the help of the Compactness Theorem. Let Σ_0 be a finite subset of Σ. Then

$$\Sigma_0 \subseteq \Gamma \cup \{\exists^{\geq n_1}, \dots, \exists^{\geq n_k}\} \text{ for some } k \in \mathbb{N}.$$

By (*), there exists $\mathcal{A} \in \mathcal{T}$ such that $|A| \geq \max\{n_1, \dots, n_k\}$. Then $\mathcal{A} \vDash \exists^{\geq n_i}$ for all $i = 1, \dots, k$ and $\mathcal{A} \vDash \Gamma$, since $\mathcal{T} = Mod(\Gamma)$. We get that $\mathcal{A} \vDash \Gamma \cup \{\exists^{\geq n_1}, \dots, \exists^{\geq n_k}\}$, so $\mathcal{A} \vDash \Sigma_0$. Thus, Σ_0 is satisfiable.
Applying the Compactness Theorem, it follows that Σ has a model $\mathcal{B}$.
Since $\mathcal{B} \vDash \Gamma$, we have that $\mathcal{B}$ is finite.
Since $\mathcal{B} \vDash \{\exists^{\geq n} \mid n \geq 1\}$, we have that $\mathcal{B}$ is infinite.
We have obtained a contradiction. □

(P4) [1,5 points] Let $\mathcal{L}$ be a first-order language and $\mathcal{K}$ be a finitely axiomatizable class of $\mathcal{L}$-structures. Prove the following:

(i) $\mathcal{K}$ is axiomatized by a single sentence.

(ii) The class $\mathcal{K}^c$ (of $\mathcal{L}$-structures that are not members of $\mathcal{K}$) is finitely axiomatizable.

Proof. (i) Let $\Gamma = \{\varphi_1, \dots, \varphi_n\}$ be a finite set of sentences such that $\mathcal{K} = Mod(\Gamma)$. Take

$$\varphi := \varphi_1 \wedge \dots \wedge \varphi_n.$$

Then, for every $\mathcal{L}$-structure $\mathcal{A}$, we have that

$$\mathcal{A} \vDash \varphi \iff \mathcal{A} \vDash \varphi_i \text{ for every } i = 1, \dots, n \iff \mathcal{A} \vDash \Gamma.$$

Thus, $Mod(\varphi) = Mod(\Gamma) = \mathcal{K}$.

(ii) By (i), $\mathcal{K}$ is axiomatized by a single sentence φ, hence $\mathcal{K} = Mod(\varphi)$. It follows immediately that for any $\mathcal{L}$-structure $\mathcal{A}$,

$$\mathcal{A} \in \mathcal{K}^c \iff \mathcal{A} \notin \mathcal{K} \iff \mathcal{A} \not\vDash \varphi \iff \mathcal{A} \vDash \neg\varphi.$$

Thus, $\mathcal{K}^c = Mod(\neg\varphi)$.

□

2 Modal logics

(P5) [2 points] Let $p, q \in PROP$. Verify if the following formulas are valid in the class of all frames for ML_0:

(i) $\Diamond p \to \Box p$.

(ii) $\Box q \wedge \Diamond p \to \Diamond(p \wedge q)$.

Proof. (i) The answer is NO. Let $\mathcal{M}_0 = (W_0, R_0, V_0)$, where

$$W_0 = \{a, b\}, \quad R_0 = \{(a,a), (a,b)\}, \quad V_0(p) = \{b\}.$$

Then $\mathcal{M}, a \Vdash \Diamond p$, since R_0ab and $b \in V_0(p)$, hence $\mathcal{M}, b \Vdash p$. On the other hand, $\mathcal{M}, a \not\Vdash \Box p$, since R_0aa and $a \notin V_0(p)$, hence $\mathcal{M}, a \not\Vdash p$. Thus, $\mathcal{M}, a \not\Vdash \Diamond p \to \Box p$.

(ii) The answer is YES. Let $\mathcal{F} = (W, R)$ be an arbitrary frame, w a state in $\mathcal{F}$ and $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$. We have to show that

$$\mathcal{M}, w \Vdash \Box q \wedge \Diamond p \to \Diamond(p \wedge q).$$

Assume that $\mathcal{M}, w \Vdash \Box q \wedge \Diamond p$, that is $\mathcal{M}, w \Vdash \Box q$ and $\mathcal{M}, w \Vdash \Diamond p$. As $\mathcal{M}, w \Vdash \Diamond p$, there exists $v \in W$ such that Rwv and $\mathcal{M}, v \Vdash p$. As $\mathcal{M}, w \Vdash \Box q$ and Rwv, we have that $\mathcal{M}, v \Vdash q$. It follows that $v \in W$ is such that Rwv and $\mathcal{M}, v \Vdash p \wedge q$. Thus, $\mathcal{M}, v \Vdash \Diamond(p \wedge q)$.

□

(P6) [2 points] Prove the following for any formulas φ, ψ of ML_0:

(i) $\vdash_{\boldsymbol{K}} \varphi \to \psi$ implies $\vdash_{\boldsymbol{K}} \Diamond\Box\varphi \to \Diamond\Box\psi$.

(ii) $\vdash_{\boldsymbol{K}} \Diamond\Diamond\varphi \vee \Diamond\Diamond\psi \to \Diamond\Diamond(\varphi \vee \psi)$.

Proof. (i) We have that

(1)	$\vdash_{\boldsymbol{K}} \varphi \to \psi$	hypothesis
(2)	$\vdash_{\boldsymbol{K}} \Box\varphi \to \Box\psi$	(S6.1).(i): (1)
(3)	$\vdash_{\boldsymbol{K}} \Diamond\Box\varphi \to \Diamond\Box\psi$	(S6.4).(i): (2).

(ii) We have that

(1)	$\vdash_{\boldsymbol{K}} \Diamond\Diamond\varphi \vee \Diamond\Diamond\psi \to \Diamond(\Diamond\varphi \vee \Diamond\psi)$	(S6.5).(ii) with $\varphi := \Diamond\varphi$ and $\psi := \Diamond\psi$
(2)	$\vdash_{\boldsymbol{K}} \Diamond\varphi \vee \Diamond\psi \to \Diamond(\varphi \vee \psi)$	(S6.5).(ii)
(3)	$\vdash_{\boldsymbol{K}} \Diamond(\Diamond\varphi \vee \Diamond\psi) \to \Diamond\Diamond(\varphi \vee \psi)$	(S6.4).(i): (2)
(4)	$\vdash_{\boldsymbol{K}} \Diamond\Diamond\varphi \vee \Diamond\Diamond\psi \to \Diamond\Diamond(\varphi \vee \psi)$	P. 2.56: (1), (3) and the tautology $(\sigma_1 \to \sigma_2) \wedge (\sigma_2 \to \sigma_3) \to (\sigma_1 \to \sigma_3)$ with $\sigma_1 := \Diamond\Diamond\varphi \vee \Diamond\Diamond\psi$, $\sigma_2 := \Diamond(\Diamond\varphi \vee \Diamond\psi)$ and $\sigma_3 := \Diamond\Diamond(\varphi \vee \psi)$

□

(P7) [1 point] Let Λ be a normal logic and $\Gamma \cup \{\varphi, \psi\}$ be a set of formulas of Λ. Prove that

if $\Gamma \vdash_\Lambda \varphi$ and ψ is deducible in propositional logic from φ, then $\Gamma \vdash_\Lambda \psi$.

Proof. Since $\Gamma \vdash_\Lambda \varphi$, there exist $\theta_1, \ldots, \theta_n \in \Gamma\,(n \geq 0)$ such that

$$\vdash_\Lambda (\theta_1 \wedge \ldots \wedge \theta_n) \to \varphi.$$

As ψ is deducible in propositional logic from φ, we get that $\varphi \to \psi$ is a tautology, hence $\vdash_\Lambda \varphi \to \psi$.
We have the following cases:

(i) $n = 0$. Then $\vdash_\Lambda \varphi$ and $\vdash_\Lambda \varphi \to \psi$. Applying (MP), we get that $\vdash_\Lambda \psi$.

(ii) $n \geq 1$. Let us denote $\theta := \theta_1 \wedge \ldots \wedge \theta_n$.

We have that

(1)	$\vdash_\Lambda \theta \to \varphi$	hypothesis
(2)	$\vdash_\Lambda \varphi \to \psi$	hypothesis
(3)	$\vdash_\Lambda \theta \to \psi$	P. 2.56: (1), (2) and the tautology $(\sigma_1 \to \sigma_2) \wedge (\sigma_2 \to \sigma_3) \to (\sigma_1 \to \sigma_3)$.

We have proved that $\vdash_\Lambda (\theta_1 \wedge \ldots \wedge \theta_p) \to \psi$. Hence, $\Gamma \vdash_\Lambda \psi$.

□

(P8) [2 points] Let Λ be a normal logic and Γ be a Λ-MCS. Prove that $\Lambda \subseteq \Gamma$.

Proof. Suppose by contradiction that $\Lambda \not\subseteq \Gamma$. Then there exists φ such that $\vdash_\Lambda \varphi$ and $\varphi \notin \Gamma$. By Proposition 2.74.(ii), we get that $\Gamma \cup \{\varphi\}$ is Λ-inconsistent. Applying Proposition 2.65.(ii), it follows that $\Gamma \vdash_\Lambda \neg\varphi$. We have obtained that $\Gamma \vdash_\Lambda \varphi$ (since $\vdash_\Lambda \varphi$) and $\Gamma \vdash_\Lambda \neg\varphi$. Apply now Proposition 2.64.(ii) to get that Γ is Λ-inconsistent, which is a contradiction. □