FMI, Computer Science, Master
Advanced Logic for Computer Science

Exam

First Name: ______________________

Last Name: __________________

P1	P2	P3	P4	P5	P6	P7	P8	Extra point
___/2	___/2	___/1,5	___/1,5	___/2	___/2	___/1	___/2	1

TOTAL
_____/15

1 First-order logic

(P1) [2 points]

(i) Prove that for every first-order language $\mathcal{L}$ and any formulas φ, ψ of $\mathcal{L}$, we have that

$$\forall x(\varphi \vee \psi) \vDash \exists x\varphi \vee \exists x\psi \text{ for any variable } x.$$

(ii) Give an example of a first-order language $\mathcal{L}$ and formulas φ, ψ of $\mathcal{L}$ such that:

$$\forall x\varphi \rightarrow \forall x\psi \nvDash \forall x(\varphi \rightarrow \psi), \text{ where } x \text{ is a variable.}$$

(P2) [2 points] Let $\mathcal{L}$ be a first-order language that contains

- two unary relation symbols S, T and one binary relation symbols P;
- a unary function symbol g;
- two constant symbols a, d.

(i) Find prenex normal forms for the following formulas of $\mathcal{L}$:

$$\begin{aligned} \varphi &:= \neg\exists x P(x,a) \wedge \forall y \neg S(y), \\ \psi &:= \exists x(S(x) \to \forall y(g(y) = d)) \to \neg(\forall x T(x) \vee \forall y S(y)). \end{aligned}$$

(ii) Find Skolem normal forms for the following sentences of $\mathcal{L}$:

$$\begin{aligned} \chi &:= \exists y \forall x \exists v (S(y) \vee P(x,v) \to (T(v) \to S(y))) \\ \delta &:= \forall x \exists u \forall y \exists v \left((S(u) \to P(v,y)) \vee (S(v) \to T(x))\right). \end{aligned}$$

(P3) [1,5 points] Let $\mathcal{L}$ be a first-order language and Δ be a set of sentences satisfying

(*) for all $p \in \mathbb{N}$, Δ has a finite model of cardinality $\geq p$.

Prove that the class of finite models of Δ is not axiomatizable.

(P4) [1,5 points] Let $\mathcal{L}$ be a first-order language and $\mathcal{K}$ be a finitely axiomatizable class of $\mathcal{L}$-structures. Prove the following:

(i) $\mathcal{K}$ is axiomatized by a single sentence.

(ii) The class $\mathcal{K}^c$ (of $\mathcal{L}$-structures that are not members of $\mathcal{K}$) is finitely axiomatizable.

2 Modal logics

(P5) [2 points] Let $p, q \in PROP$. Verify if the following formulas are valid in the class of all frames for ML_0:

(i) $\Diamond p \to \Box p$.

(ii) $\Box q \wedge \Diamond p \to \Diamond(p \wedge q)$.

(P6) [2 points] Prove the following for any formulas φ, ψ of ML_0:

(i) $\vdash_{\boldsymbol{K}} \varphi \to \psi$ implies $\vdash_{\boldsymbol{K}} \Diamond\Box\varphi \to \Diamond\Box\psi$.

(ii) $\vdash_{\boldsymbol{K}} \Diamond\Diamond\varphi \vee \Diamond\Diamond\psi \to \Diamond\Diamond(\varphi \vee \psi)$.

(P7) [1 point] Let Λ be a normal logic and $\Gamma \cup \{\varphi, \psi\}$ be a set of formulas of Λ. Prove that

if $\Gamma \vdash_\Lambda \varphi$ and ψ is deducible in propositional logic from φ, then $\Gamma \vdash_\Lambda \psi$.

(P8) [2 points] Let Λ be a normal logic and Γ be a Λ-MCS. Prove that $\Lambda \subseteq \Gamma$.