FMI, Computer Science, Master
Logic for Multiagent Systems

Exam

(P1) [1.5 points] Let $\mathcal{L}$ be a first-order language that contains

- two unary relation symbols S, T;
- a unary function symbol f;
- a constant symbol c.

Find prenex normal forms for the following formulas of $\mathcal{L}$:

$$\begin{aligned} \varphi &:= \forall x S(x) \wedge \neg \exists y S(y), \\ \psi &:= \neg \forall y \left(f(y) = c \rightarrow \exists x S(x) \right) \rightarrow \left(\exists x T(x) \vee \forall y T(y) \right). \end{aligned}$$

Proof. $\varphi^* = \forall x \forall y (S(x) \wedge \neg S(y))$ is a prenex normal form for φ.
$\psi^* = \forall y \exists x \exists u \forall v \left(\neg \left(f(y) = c \rightarrow S(x) \right) \rightarrow (T(u) \vee T(v)) \right)$ is a prenex normal form φ. □

(P2) [2 points] Let $p, q \in PROP$. Verify if the following formulas are valid in the class of all frames for ML_0:

(i) $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$;

(ii) $\Box p \rightarrow p$.

Proof. (i) The answer is YES. Let $\mathcal{F} = (W, R)$ be an arbitrary frame, w a state in $\mathcal{F}$ and $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$. Assume that $\mathcal{M}, w \Vdash \Box(p \wedge q)$. Then for all $v \in W$, Rwv implies $\mathcal{M}, v \Vdash p \wedge q$, hence

$$(*) \quad \text{for all } v \in W,\ Rwv \text{ implies } (\mathcal{M}, v \Vdash p \text{ and } \mathcal{M}, v \Vdash q).$$

Let us show now that $\mathcal{M}, w \Vdash \Box p \wedge \Box q$.

Let $v \in W$ be such that Rwv. By (*), we get that $\mathcal{M}, v \Vdash p$ and $\mathcal{M}, v \Vdash q$. Thus, $\mathcal{M}, w \Vdash \Box p$ and $\mathcal{M}, w \Vdash \Box q$, so $\mathcal{M}, w \Vdash \Box p \wedge \Box q$.

(ii) The answer is NO (it is valid in the class of reflexive frames). Let $\mathcal{M}_0 = (W_0, R_0, V_0)$, where

$$W_0 = \{a, b\}, \quad R_0 = \{(a, b)\}, \quad V_0(p) = \{b\}.$$

Then $\mathcal{M}_0, a \Vdash \Box p$, since $R_0 ab$ and $\mathcal{M}_0, b \Vdash p$. On the other hand, $\mathcal{M}_0, a \nVdash p$.

□

(P3) [1.5 points] Prove the following for any formulas φ, ψ of ML_0:

(i) $\vdash_{\boldsymbol{K}} \Box\varphi \to (\Box\psi \to \Box\varphi)$;

(ii) $\vdash_{\boldsymbol{K}} \psi \to \varphi$ implies $\vdash_{\boldsymbol{K}} \Diamond\Box\psi \to \Diamond\Box\varphi$.

Proof. (i) We have that

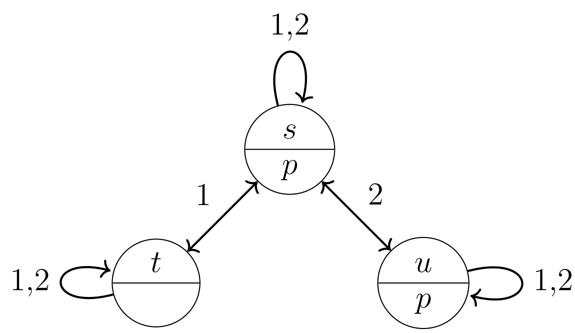
(1)	$\vdash_{\boldsymbol{K}} \varphi \to (\psi \to \varphi)$	(TAUT)
(2)	$\vdash_{\boldsymbol{K}} \Box\varphi \to \Box(\psi \to \varphi)$	Example 4.38: (1)
(3)	$\vdash_{\boldsymbol{K}} \Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$	(K)
(4)	$\vdash_{\boldsymbol{K}} \Box\varphi \to (\Box\psi \to \Box\varphi)$	(TAUT): (2), (3)

(ii) We have that

(1)	$\vdash_{\boldsymbol{K}} \psi \to \varphi$	hypothesis
(2)	$\vdash_{\boldsymbol{K}} \Box\psi \to \Box\varphi$	Example 4.38: (1)
(3)	$\vdash_{\boldsymbol{K}} \Diamond\Box\psi \to \Diamond\Box\varphi$	Example 4.39: (2)

□

(P4) [1.5 points] Consider the model $\mathcal{M} = (W, \mathcal{K}_1, \mathcal{K}_2, V)$ for epistemic logic represented as follows:



Verify if the following are true:

(i) $\mathcal{M}, t \Vdash K_1 p$;

(ii) $\mathcal{M}, t \Vdash \neg K_2 \neg K_1 p$.

Proof. (i) Since $t\mathcal{K}_1 t$ and $\mathcal{M}, t \nVdash p$, it follows that $\mathcal{M}, t \nVdash K_1 p$.

(ii) We have that

$$\begin{array}{lll} \mathcal{M}, t \Vdash \neg K_2 \neg K_1 p & \text{iff} & \mathcal{M}, t \nVdash K_2 \neg K_1 p \\ & \text{iff} & \text{it is not true that for every world } w,\ t\mathcal{K}_2 w \text{ implies } \mathcal{M}, w \Vdash \neg K_1 p \\ & \text{iff} & \text{there exists a world } w \text{ such that } t\mathcal{K}_2 w \text{ and } \mathcal{M}, w \nVdash \neg K_1 p \\ & \text{iff} & \text{there exists a world } w \text{ such that } t\mathcal{K}_2 w \text{ and } \mathcal{M}, w \Vdash K_1 p. \end{array}$$

Take $w := t$. Then $t\mathcal{K}_1 t$ and we have already proved at (i) that $\mathcal{M}, t \Vdash K_1 p$.

□

(P5) [1.5 points] Let $\mathcal{M}_c$ be the model describing the simple card game. Prove the following:

(i) $\mathcal{M}_c, (A, B) \models 1A \wedge 2B$;

(ii) $\mathcal{M}_c, (A, B) \models K_1 \neg K_2 1A$.

Proof. (i) We have that

$$\begin{aligned} \mathcal{M}_c, (A, B) \models 1A \wedge 2B \quad & \text{iff} \quad \mathcal{M}_c, (A, B) \models 1A \text{ and } M_c, (A, B) \models 2B \\ & \text{iff} \quad (A, B) \in V(1A) \text{ and } (A, B) \in V(2B), \end{aligned}$$

which is true.

(ii) We have that

$$\begin{aligned} \mathcal{M}_c, (A, B) \models K_1 \neg K_2 1A \quad & \text{iff} \quad \mathcal{M}_c, (A, B) \models \neg K_2 1A \text{ and } \mathcal{M}_c, (A, C) \models \neg K_2 1A \\ & \text{iff} \quad \mathcal{M}_c, (A, B) \not\models K_2 1A \text{ and } \mathcal{M}_c, (A, C) \not\models K_2 1A. \end{aligned}$$

As $(A, B)\mathcal{K}_2(C, B)$ and $\mathcal{M}_c, (C, B) \not\models 1A$ (as $(C, B) \notin V(1A)$), it follows that $\mathcal{M}_c, (A, B) \not\models K_2 1A$.
As $(A, C)\mathcal{K}_2(B, C)$ and $\mathcal{M}_c, (B, C) \not\models 1A$ (as $(B, C) \notin V(1A)$), it follows that $\mathcal{M}_c, (A, C) \not\models K_2 1A$.
Thus, $\mathcal{M}_c, (A, B) \models K_1 \neg K_2 1A$.

□

(P6) [1 point] Let $\mathcal{M}$ be the model for the muddy children puzzle with $n = 3$. Prove the following:

(i) $\mathcal{M}, (1, 0, 1) \Vdash K_1 \neg p_2$;

(ii) $\mathcal{M}, (1, 0, 1) \Vdash K_2 p_3$.

Proof. (i) We have that

$$\begin{aligned} \mathcal{M}, (1, 0, 1) \Vdash K_1 \neg p_2 \quad & \text{iff} \quad \mathcal{M}, (1, 0, 1) \Vdash \neg p_2 \text{ and } \mathcal{M}, (0, 0, 1) \Vdash \neg p_2 \\ & \text{iff} \quad \mathcal{M}, (1, 0, 1) \not\Vdash p_2 \text{ and } \mathcal{M}, (0, 0, 1) \not\Vdash p_2 \\ & \text{iff} \quad (1, 0, 1) \notin V(p_2) \text{ and } (0, 0, 1) \notin V(p_2), \end{aligned}$$

which is true.

(ii) We have that

$$\begin{aligned} \mathcal{M}, (1, 0, 1) \Vdash K_2 p_3 \quad & \text{iff} \quad \mathcal{M}, (1, 0, 1) \Vdash p_3 \text{ and } \mathcal{M}, (1, 1, 1) \Vdash p_3 \\ & \text{iff} \quad (1, 0, 1) \in V(p_3) \text{ and } (1, 1, 1) \in V(p_3), \end{aligned}$$

which is true.

□