# Baer extensions of BL-algebras

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**Abstract**

In this paper we define Baer BL-algebras as BL-algebras with the property that co-annihilator filters are generated by central elements. We use sheaf-theoretic techniques to construct a Baer extension of any BL-algebra, that is to embed any nontrivial BL-algebra $A$ into a Baer BL-algebra $A^*$. The embedding turns to be an isomorphism if $A$ is itself a Baer BL-algebra.

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## Introduction

In 1998, Hájek [14] introduced a very general many-valued logic, called *Basic Logic* (or *BL*), with the idea to formalize the many-valued semantics induced by a continuous $t$-norm on the unit real interval $[0, 1]$. This Basic Logic turns to be a fragment common to three important many-valued logics: $\aleph_0$-valued Łukasiewicz logic, Gödel logic and Product logic. The Lindenbaum-Tarski algebras for Basic Logic are called *BL-algebras*. Apart from their logical interest, BL-algebras have important algebraic properties and they have been intensively studied from an algebraic point of view.

The aim of this paper is to construct a Baer extension of any BL-algebra, that is to embed any nontrivial BL-algebra $A$ into a Baer BL-algebra. Baer BL-algebras are BL-algebras with the property that co-annihilator filters are generated by central elements. The definition is similar to the one of Baer rings, extensively studied (see [15, 17, 18, 7, 21, 22, 23] or, for some recent papers, [3, 4, 5]), or Baer MV-algebras, defined in [12] (also studied under the name of strongly stonian MV-algebras in [1]).

In order to construct the Baer extension of a BL-algebra, we use sheaf-theoretic techniques inspired by Keimel's construction for rings and semigroups [16], which is similar to the methods used by Kist in [17]. Thus, for any BL-algebra $A$ we construct a Hausdorff BL-sheaf space having as a base space the complete Boolean algebra $Co-An(A)$ of co-annihilator filters of $A$, and it turns out that the BL-algebra of global sections, which we will denote by $A^*$, is a Baer extension of $A$. Moreover, if $A$ is a Baer BL-algebra, then $A \cong A^*$.

# 1 BL-algebras. Definitions and first properties

A *BL-algebra* [14] is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \rightarrow$ and two constants $0, 1$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid, and for all $a, b, c \in A$,

$$c \leq a \rightarrow b \quad \text{iff} \quad a \odot c \leq b \tag{1.1}$$

$$a \wedge b = a \odot (a \rightarrow b) \tag{1.2}$$

$$(a \rightarrow b) \vee (b \rightarrow a) = 1. \tag{1.3}$$

In order to simplify the notation, a BL-algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ will be referred by its support set $A$.

A BL-algebra $A$ is nontrivial iff $0 \neq 1$. For any BL-algebra $A$, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. A *BL-chain* is a totally ordered BL-algebra, i.e. a BL-algebra such that its lattice order is total.

For any $a \in A$, we define $a^- = a \rightarrow 0$. We denote the set of natural numbers by $\omega$. We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \omega - \{0\}$.

The following properties hold in any BL-algebra $A$ and will be used in the sequel:

$$a \odot b \leq a \wedge b \leq a, b \tag{1.4}$$

$$a \rightarrow b = 1 \quad \text{iff} \quad a \leq b \tag{1.5}$$

$$(a \vee b) \odot (a \vee c) \leq a \vee (b \odot c) \tag{1.6}$$

$$a \odot a^- = 0 \tag{1.7}$$

$$\tag{1.8}$$

Let $A$ be a BL-algebra. A *filter* of $A$ is a nonempty set $F \subseteq A$ such that for all $a, b \in A$,
(i) $a, b \in F$ implies $a \odot b \in F$;
(ii) $a \in F$ and $a \leq b$ imply $b \in F$.

Trivial examples of filters are $\{1\}$ and $A$. A filter $F$ of $A$ is *proper* iff $F \neq A$. Any filter of $A$ is also a filter of the lattice $L(A)$.

A proper filter $P$ of $A$ is called *prime* provided that it is prime as a filter of $L(A)$, that is

$$a \vee b \in P \quad \text{implies} \quad a \in P \text{ or } b \in P.$$

In the sequel, we shall denote by $\mathcal{F}(A)$ the set of filters of $A$, and by $Spec(A)$ the set of prime filters of $A$.

If $X \subseteq A$, then the filter of $A$ generated by $X$ will be denoted by $< X >$. We have that $< \emptyset >= \{1\}$ and, if $X \neq \emptyset$,

$$\begin{aligned} < X > &= \{y \in A \mid x_1 \odot \ldots \odot x_n \leq y \\ &\quad \text{for some } n \in \omega - \{0\} \text{ and some } x_1, \ldots, x_n \in X\}. \end{aligned}$$

For any $a \in A$, $< a >$ denotes the principal filter of $A$ generated by $\{a\}$. Then,

$$< a >= \{b \in A \mid a^n \leq b \text{ for some } n \in \omega - \{0\}\}.$$

It follows immediately that $< 1 >= \{1\}$ and $< 0 >= A$.

**Proposition 1.1.** *$(\mathcal{F}(A), \subseteq)$ is a complete lattice. For every family $\{F_i\}_{i \in I}$ of filters of $A$, we have that*

$$\begin{aligned} \bigwedge_{i \in I} F_i &= \bigcap_{i \in I} F_i, \\ \bigvee_{i \in I} F_i &= < \bigcup_{i \in I} F_i > . \end{aligned}$$

It is easy to see that if $F, G$ are filters of $A$, then

$$F \vee G \;=\; < F \cup G > \;=\; \{a \in A \mid b \odot c \leq a \text{ for some } b \in F, c \in G\}. \tag{1.9}$$

**Proposition 1.2.** *(i) $a \leq b$ implies $< b > \subseteq < a >$;*
*(ii) $< a \vee b >=< a > \cap < b >$;*
*(iii) $< a > \vee < b >=< a \wedge b >=< a \odot b >$;*
*(iv) if $F$ is a filter of $A$, then $a \in F$ iff $< a > \subseteq F$;*
*(v) $< a >= \{1\}$ iff $a = 1$.*

*Proof.* (i) Let $c \in < b >$. Then there is $n \in \omega - \{0\}$ such that $c \geq b^n \geq a^n$, hence $c \in < a >$.
(ii) By [9, Lemma 4.11].
(iii) Applying (i), we obtain that $< a > \vee < b > \subseteq < a \wedge b > \subseteq < a \odot b >$, since $a \odot b \leq a \wedge b \leq a, b$. It remains to prove that $< a \odot b > \subseteq < a > \vee < b >$. Let $x \in < a \odot b >$; that is there is $n \in \omega - \{0\}$ such that $a^n \odot b^n = (a \odot b)^n \leq x$. Applying now the fact that $a^n \in < a >, b^n \in < b >$, and (1.9), it follows that $x \in < a > \vee < b >$.
(iv),(v) Obviously. □

With any filter $F$ of $A$ we can associate a congruence relation $\equiv (mod\, F)$ on $A$ by defining

$$a \equiv b (mod\, F) \text{ iff } a \to b \in F \text{ and } b \to a \in F \text{ iff } (a \to b) \odot (b \to a) \in F.$$

For any $a \in A$, let $a/F$ be the equivalence class $a/_{\equiv(mod\, F)}$. If we denote by $A/F$ the quotient set $A/_{\equiv(mod\, F)}$, then $A/F$ becomes a BL-algebra with the natural operations induced from those of $A$.

Let $B(A)$ be the Boolean algebra of all complemented elements in the distributive lattice $L(A)$. We shall refer to $B(A)$ as the *center* of $A$ and to elements of $B(A)$ as*central elements* of $A$.

**Proposition 1.3.** [10, Proposition 1.9]
*Let $e \in A$. The following are equivalent:*
*(i) $e \in B(A)$;*
*(ii) $e \odot e = e$ and $e = e^{--}$;*
*(iii) $e \odot e = e$ and $e^{-} \to e = e$;*
*(iv) $e \vee e^{-} = 1$.*

**Proposition 1.4.** *Suppose that $a, b \in A$ and $e, f \in B(A)$. Then*
*(i) $< e >= \{a \in A \mid e \leq a\}$;*
*(ii) $< e >= e \vee A = \{e \vee a \mid a \in A\}$;*
*(iii) $e = f$ iff $< e >=< f >$;*
*(iv) $e \odot a = e \wedge a$;*
*(v) $e \wedge e^{-} = 0$;*
*(vi) $a \vee e^{-} = 1$ iff $e \leq a$ iff $a \in< e >$.*

*Proof.* (i) Apply Proposition 1.3(ii).
(ii) "$\subseteq$" If $a \in< e >$, then $e \leq a$, so $a = e \vee a$, that is $a \in e \vee A$.
"$\supseteq$" Apply (i).
(iii) If $< e >=< f >$, then, by (i) $e \in< f >$, so $e \geq f$, and $f \in< e >$, so $e \leq f$.
(iv) By [10, Lemma 1.11].
(v) Apply (iv) and (1.7).
(vi) If $e \leq a$, then $e \wedge a = e$, hence $1 = e^{-} \vee e = e^{-} \vee (e \wedge a) = (e^{-} \vee e) \wedge (e^{-} \vee a) = 1 \wedge (e^{-} \vee a) = e^{-} \vee a$. Conversely, if $a \vee e^{-} = 1$, then $e = e \wedge 1 = e \wedge (a \vee e^{-}) = (e \wedge a) \vee (e \wedge e^{-}) = (e \wedge a) \vee 0 = e \wedge a$, so $e \leq a$. □

A BL-algebra $A$ is called *directly indecomposable* iff A is nontrivial and whenever $A \cong A_1 \times A_2$ then either $A_1$ or $A_2$ is trivial.

**Proposition 1.5.** [10, Proposition 1.12]
*A BL-algebra $A$ is directly indecomposable iff $B(A) = \{0, 1\}$.*

**Proposition 1.6.** [10, Proposition 1.13]
*Any BL-chain is directly indecomposable.*

# 2 Co-annihilators

Let $F$ be a filter of $A$ and $a \in A$. The *co-annihilator of $a$ relative to $F$* is the set

$$(F, a) = \{x \in A \mid a \vee x \in F\}.$$

**Proposition 2.1.** *Let $F, G$ be filters of $A$ and $a, b \in A$. Then*
*(i) $(F, a)$ is a filter of $A$;*
*(ii) $F \subseteq (F, a)$;*
*(iii) $a \leq b$ implies $(F, a) \subseteq (F, b)$;*

*(iv)* $F \subseteq G$ *implies* $(F, a) \subseteq (G, a)$*;*
*(v)* $(F, a) = A$ *iff* $a \in F$*;*
*(vi)* $(F, a) \cap (F, b) = (F, a \wedge b) = (F, a \odot b)$*;*
*(vii)* $(F, a) \cap (G, a) = (F \cap G, a)$ *and* $(F, a) \cup (G, a) = (F \cup G, a)$*;*
*(viii)* $((F, a), b) = ((F, b), a) = (F, a \vee b)$.

*Proof.* (i) We have that $a \vee 1 = 1 \in F$, hence $1 \in (F, a)$. If $x \leq y$ and $x \in (F, a)$, then $x \vee a \in F$ and $x \vee a \leq y \vee a$. Hence, $y \vee a \in F$, i.e. $y \in (F, a)$. Assume that $x, y \in (F, a)$, i.e. $x \vee a, y \vee a \in F$. Since, by (1.6), $(x \odot y) \vee a \geq (x \vee a) \odot (y \vee a) \in F$, it follows that $(x \odot y) \vee a \in F$, so $x \odot y \in (F, a)$.
(ii) Let $x \in F$. Then $x \vee a \geq x \in F$, hence $x \vee a \in F$. That is, $x \in (F, a)$.
(iii) Let $x \in (F, a)$. Then $x \vee a \leq x \vee b$ and $x \vee a \in F$. It follows that $x \vee b \in F$, i.e $x \in (F, b)$.
(iv) Let $x \in (F, a)$. Then $x \vee a \in F \subseteq G$. Hence, $x \in (G, a)$.
(v) If $(F, a) = A$, then $0 \in (F, a)$, hence $a = 0 \vee a \in F$. If $a \in F$, then for any $x \in A$, $a \leq x \vee a$, hence $x \vee a \in F$. That is, for any $x \in A$, we have that $x \in (F, a)$.
(vi) Since $a \odot b \leq a \wedge b \leq a, b$, by (iii), it follows that $(F, a \odot b) \subseteq (F, a \wedge b) \subseteq (F, a) \cap (F, b)$. Conversely, let $x \in (F, a) \cap (F, b)$, hence $x \vee a \in F$ and $x \vee b \in F$. By (1.6), we get that $x \vee (a \odot b) \geq (x \vee a) \odot (x \vee b) \in F$. That is, $x \in (F, a \odot b)$.
(vii) Obviously, using (iii).
(viii) We have that $x \in ((F, a), b)$ iff $x \vee b \in (F, a)$ iff $(x \vee b) \vee a \in F$ iff $x \vee (a \vee b) \in F$ iff $x \in (F, a \vee b)$, and, similarly, $x \in ((F, b), a)$ iff $x \in (F, a \vee b)$. □

For any $a, b \in A$, we shall denote by $(b, a)$ the co-annihilator $(< b >, a)$.

**Proposition 2.2.** *Let* $a, b \in A$. *Then*
*(i)* $(a, a) = A$*;*
*(ii)* $(b, a) = (b, a \wedge b) = (b, a \odot b)$*;*
*(iii)* $(b, a) = (a \vee b, a)$.

*Proof.* (i) Apply Proposition 2.1(v).
(ii) By Proposition 2.1(vi), we get that $(b, a \wedge b) = (b, a \odot b) = (b, a) \cap (b, b) = (b, a) \cap A = (b, a)$.
(iii) Applying Proposition 1.2(ii) and Proposition 2.1(vii), it follows that $(a \vee b, a) = (< a \vee b >, a) = (< a > \cap < b >, a) = (a, a) \cap (b, a) = (b, a)$. □

For any non-empty subset $X$ of $A$, the *co-annihilator of* $X$ is the set

$$^{\perp}X = \{a \in A \mid a \vee x = 1 \text{ for any } x \in X\}.$$

It is easy to see that $^{\perp}A = \{1\}$ and $^{\perp}\emptyset = {}^{\perp}\{1\} = A$.

**Proposition 2.3.** *Let* $\emptyset \neq X, Y \subseteq A$, $(X_i)_{i \in I} \subseteq A$ *and* $F$ *be a filter of* $A$. *Then,*
*(i)* $^{\perp}X$ *is a filter of* $A$*;*
*(ii) If* $X \subseteq Y$*, then* $^{\perp}Y \subseteq {}^{\perp}X$ *and* $^{\perp\perp}X \subseteq {}^{\perp\perp}Y$*;*
*(iii)* $X \subseteq {}^{\perp\perp}X$*;*

*(iv)* $^{\perp}X = {}^{\perp\perp\perp}X$*;*
*(v)* $^{\perp}X = {}^{\perp} < X >$*;*
*(vi)* $< X > \cap^{\perp} X = \{1\}$*;*
*(vii)* $F \cap {}^{\perp}F = \{1\}$*;*
*(viii)* $^{\perp}F$ *is a prime filter iff* $F$ *is a chain and* $F \neq \{1\}$*;*
*(ix)* $\bigcap_{i\in I} {}^{\perp}X_i = {}^{\perp}(\bigcup_{i\in I} X_i)$.

*Proof.* (i)-(viii) By [9, Proposition 4.38, 4.39, 4.40, 4.42].
(ix) Let $a \in A$. Then $a \in \bigcap_{i\in I} {}^{\perp}X_i$ iff $a \in {}^{\perp}X_i$ for all $i \in I$ iff $a \vee x = 1$ for all $x \in X_i, i \in I$ iff $a \vee x = 1$ for all $x \in \bigcup_{i\in I} X_i$ iff $a \in {}^{\perp}(\bigcup_{i\in I} X_i)$. □

Let us recall some facts from lattice theory (see [13]). Let $(L, \vee, \wedge, 0)$ be a lattice with 0. An element $a^* \in L$ is a *pseudocomplement* of $a \in L$ if

$$a \wedge a^* = 0 \text{ and } a \wedge x = 0 \quad \text{imply} \quad x \leq a^*.$$

A bounded lattice $L$ is called *pseudocomplemented* iff every element has a pseudocomplement.

**Proposition 2.4.** *The lattice* $\mathcal{F}(A)$ *is pseudocomplemented. For any filter* $F$*, its pseudocomplement is* $^{\perp}F$*.*

*Proof.* By Proposition 2.3(vii), $F \wedge_{\mathcal{F}(A)} {}^{\perp}F = F \cap {}^{\perp}F = \{1\}$. Let $G$ be a filter of $A$ such that $F \wedge_{\mathcal{F}(A)} G = F \cap G = \{1\}$. We shall prove that $G \subseteq {}^{\perp}F$. Let $a \in G$. For any $x \in F$, we have that $a \vee x \in F \cap G = \{1\}$, since $a \vee x \geq x \in F$ and $a \vee x \geq a \in G$. Hence, $a \vee x = 1$ for any $x \in F$, so $a \in {}^{\perp}F$. That is, $^{\perp}F$ is the pseudocomplement of $F$. □

We define

$$Co - An(A) = \{{}^{\perp}F \mid F \in \mathcal{F}(A)\}.$$

The elements of $Co - An(A)$ will be called *co-annihilator filters* of $A$ .

**Remark 2.5.** $Co - An(A) = \{{}^{\perp}X \mid X \subseteq A\}$

*Proof.* Apply Proposition 2.3(v). □

**Proposition 2.6.** *Let* $F, G$ *be filters of* $A$*.*
*(i)* $F \in Co - An(A)$ *iff* $^{\perp\perp}F = F$*;*
*(ii) if* $F, G \in Co - An(A)$*, then* $F \cap G \in Co - An(A)$*;*
*(iii)* $\{1\}, A \in Co - An(A)$*;*
*(iv) for* $F, G \in Co - An(A)$*, define* $F \vee_{Co-An(A)} G = {}^{\perp}({}^{\perp}F \cap {}^{\perp}G)$*. Then*

$$(Co - An(A), \cap, \vee_{Co-An(A)}, {}^{\perp}, \{1\}, A)$$

*is a Boolean algebra.*
*(v)* $Co - An(A)$ *is a complete Boolean algebra;*
*(vi) for any family* $(F_i)_{i\in I} \subseteq Co - An(A)$*,*

$$^{\perp\perp}(\bigcap_{i\in I} F_i) = \bigcap_{i\in I} {}^{\perp\perp}F_i;$$

*(vii) if* $F, G \in Co - An(A)$*, then* $F \vee G \subseteq F \vee_{Co-An(A)} G$*.*

*Proof.* (i)-(iv) By well-known results from lattice theory (see, e.g., [13]).
(v) Let $(F_i)_{i\in I} \subseteq Co-An(A)$. Hence, for all $i \in I$, there is $X_i \subseteq A$ such that $F_i = {}^{\perp}X_i$. Applying Proposition 2.3(ix), it follows that $\bigcap_{i\in I} F_i = \bigcap_{i\in I} {}^{\perp}X_i = {}^{\perp}(\bigcup_{i\in I} X_i) \in Co-An(A)$. Thus, $Co-An(A)$ is closed to arbitrary intersections, so it is a complete Boolean algebra.
(vi) Applying (vi) and (i), we get that ${}^{\perp\perp}(\bigcap_{i\in I} F_i) = \bigcap_{i\in I} F_i = \bigcap_{i\in I} {}^{\perp\perp}F_i$.
(vii) Obviously, since $F, G \subseteq F \vee_{Co-An(A)} G$. □

**Remark 2.7.** *If $A$ is a BL-chain, then the only co-annihilator filters are $A$ and $\{1\}$, so $Co-An(A)$ is isomorphic with $L_2$, the two-elements Boolean algebra.*

We shall denote by

$$^{\perp}a = {}^{\perp}\{a\}.$$

It is easy to see that ${}^{\perp}1 = A$, ${}^{\perp}0 = \{1\}$, ${}^{\perp\perp}0 = A$, and ${}^{\perp\perp}1 = \{1\}$.

**Remark 2.8.** *For any $X \subseteq A$, ${}^{\perp}X = \bigcap_{x\in X} {}^{\perp}x$.*

**Proposition 2.9.** *Let $a, b \in A$ and $e \in B(A)$. Then*
*(i)* ${}^{\perp}a = (1, a) = \{x \in A \mid x \vee a = 1\}$;
*(ii)* $a \leq b$ *implies* ${}^{\perp}a \subseteq {}^{\perp}b$;
*(iii)* ${}^{\perp}a = A$ *iff* $a = 1$;
*(iv)* ${}^{\perp}a \cap {}^{\perp}b = {}^{\perp}(a \wedge b) = {}^{\perp}(a \odot b)$;
*(v)* ${}^{\perp}e =< e^{-} >$.

*Proof.* (i) $(1, a) = (< 1 >, a) = (\{1\}, a) = \{x \in A \mid x \vee a = 1\} = {}^{\perp}a$.
(ii) $a \leq b$ implies $< b >\subseteq< a >$. Hence, applying Proposition 2.3(v), (ii), we get that ${}^{\perp}a = {}^{\perp} < a >\subseteq {}^{\perp} < b >= {}^{\perp}b$.
(iii) ${}^{\perp}a = A$ iff $0 \in {}^{\perp}a$ iff $0 \vee a = 1$ iff $a = 1$.
(iv) We have that $x \in {}^{\perp}a \cap {}^{\perp}b$ iff $x \vee a = x \vee b = 1$ iff $(x \vee a) \wedge (x \vee b) = 1$ iff $x \vee (a \wedge b) = 1$ iff $x \in {}^{\perp}(a \wedge b)$. Hence, ${}^{\perp}a \cap {}^{\perp}b = {}^{\perp}(a \wedge b)$. Since $a \odot b \leq a \wedge b$, by (ii), it follows that ${}^{\perp}(a \odot b) \subseteq {}^{\perp}(a \wedge b)$. Conversely, if $x \vee (a \wedge b) = 1$, then $x \vee a = x \vee b = 1$, so $(x \vee a) \odot (x \vee b) = 1$. By (1.6), it follows that $x \vee (a \odot b) = 1$, that is $x \in {}^{\perp}(a \odot b)$.
(v) We have that $a \in {}^{\perp}e$ iff $a \vee e = 1$ iff $a \vee e^{--} = 1$ iff $a \in< e^{-} >$, by Proposition 1.4(vi).

□

**Proposition 2.10.** *Let $a, b \in A$. Then*
*(i)* ${}^{\perp\perp}a = \{x \in A \mid x \vee y = 1$ *for any* $y \in A$ *such that* $y \vee a = 1\}$;
*(ii)* $a \in {}^{\perp\perp}a$;
*(iii)* $a \leq b$ *implies* ${}^{\perp\perp}b \subseteq {}^{\perp\perp}a$;
*(iv)* $b \in {}^{\perp}a$ *implies* ${}^{\perp\perp}a \subseteq {}^{\perp}b$;
*(v)* ${}^{\perp\perp}a \cap {}^{\perp\perp}b = {}^{\perp\perp}(a \vee b)$.

*Proof.* (i) By the definition.
(ii) It is an immediate consequence of (i).
(iii) Apply Proposition 2.9(ii) and Proposition 2.3(ii).
(iv) By Proposition 2.3(ii).

(v) By Proposition 2.3(v), Proposition 1.2(iii), and Proposition 2.6(vi), it follows that $^{\perp\perp}(a \vee b) = ^{\perp\perp} < a \vee b >= ^{\perp\perp}(< a > \cap < b >) = ^{\perp\perp} < a > \cap^{\perp\perp} < b >= ^{\perp\perp}a \cap ^{\perp\perp}b$.

$\square$

**Proposition 2.11.** *Let $a, b \in A$. Then*
*(i) $^{\perp}a \vee_{Co-An(A)} {}^{\perp}b = {}^{\perp}(a \vee b)$;*
*(ii) $^{\perp}a \vee {}^{\perp}b \subseteq {}^{\perp}(a \vee b)$.*

*Proof.* (i) Applying Proposition 2.6(v), Proposition 2.10(v), and Proposition 2.3(iv), we get that $^{\perp}a \vee_{Co-An(A)} {}^{\perp}b = {}^{\perp}({}^{\perp\perp}a \cap {}^{\perp\perp}b) = {}^{\perp\perp\perp}(a \vee b) = {}^{\perp}(a \vee b)$.
(ii) Apply Proposition 2.6(vii) and (i). $\square$

We shall call a co-annihilator filter of the form $^{\perp}a$, $a \in A$, a *co-annulet* and denote with $Co - An_0(A)$ the set of co-annulets of $A$.

**Proposition 2.12.** *$Co - An_0(A)$ is a bounded sublattice of the Boolean algebra $Co - An(A)$.*

*Proof.* Apply Proposition 2.9(iv) and Proposition 2.11(i) to get that $Co - An_0(A)$ is a sublattice of $Co - An(A)$. Finally, $A = {}^{\perp}1 \in Co - An_0(A)$, and $\{1\} = {}^{\perp}0 \in Co - An_0(A)$. $\square$

# 3 Sheaf spaces and sheaf representations of BL-algebras

The notion of a BL-sheaf space (or sheaf space of BL-algebras) is defined following the general line of sheaf spaces of universal algebras [8]. The following properties are presented in detail in [19, 11], but for the sake of completeness we recall them here.

A *sheaf space of BL-algebras* (or a *BL-sheaf space*) is a triple $(F, p, X)$ such that the following properties are satisfied:
(i) $F$ and $X$ are topological spaces;
(ii) $p : F \to X$ is a local homeomorphism from F onto X;
(iii) for each $x \in X$, $p^{-1}(\{x\}) = F_x$ is a nontrivial BL-algebra with operations denoted by $\vee_x, \wedge_x, \odot_x, \to_x, 0_x, 1_x$;
(iv) the functions $(a, b) \mapsto a \vee_x b$, $(a, b) \mapsto a \wedge_x b$, $(a, b) \mapsto a \odot_x b$, $(a, b) \mapsto a \to_x b$ from the set $\{(a, b) \in F \times F \mid p(a) = p(b)\}$ into $F$ are continuous, where $x = p(a) = p(b)$;
(v) the functions $\underline{0}, \underline{1} : X \to F$, which assign to each $x$ in $X$ the zero $0_x$ and the unit $1_x$ of $F_x$ respectively, are continuous.

$X$ is known as the *base space*, $F$ as the *total space* and $F_x$ is called the *stalk* of $F$ at $x \in X$.

If $Y \subseteq X$, then a *section* $\sigma$ over $Y$ is a continuous map $\sigma : Y \to F$ satisfying $(p \circ \sigma)(y) = y$ for all $y \in Y$. The set of all sections over $Y$ form a nontrivial BL-algebra with the operations defined pointwise, that will be denoted by $\Gamma(Y, F)$. The elements of $\Gamma(X, F)$ are called *global sections*.

For every $\sigma, \tau \in \Gamma(Y, F)$, we shall use the following notation:

$$[\sigma = \tau] = \{y \in Y \mid \sigma(y) = \tau(y)\}.$$

**Proposition 3.1.** *Let $(F, p, X)$ be a BL-sheaf space.*
*(i) for any $Y \subseteq X$ and $\sigma, \tau \in \Gamma(Y, F)$, the subset $[\sigma = \tau]$ is open in $Y$;*
*(ii) the family $\{\sigma(U) \mid U$ is open in $X, \sigma \in \Gamma(U, F)\}$ is a basis for the topology of $F$;*
*(iii) if $F$ is Hausdorff then $[\sigma = \tau]$ is clopen in $X$ for all $\sigma, \tau \in \Gamma(X, F)$.*

Following Mulvey [20], by a *sheaf representation* (or simply *representation*) of a nontrivial BL-algebra $A$ will be meant a BL-morphism

$$\varphi : A \to \Gamma(X, F)$$

from $A$ to the BL-algebra $\Gamma(X, F)$ of global sections of a BL-sheaf space $(F, p, X)$.

For each $x \in X$, we define

$$\varphi_x : A \to F_x, \quad \varphi_x(a) = \varphi(a)(x) \text{ for all } a \in A,$$
$$K_x = Ker(\varphi_x) = \{a \in A \mid \varphi(a)(x) = 1_x\}.$$

It is easy to see that $\varphi_x$ is a BL-morphism, so $K_x$ is a proper filter of $A$ for every $x \in X$. Moreover, $Ker(\varphi) = \bigcap_{x \in X} K_x$, hence $\varphi$ is a monomorphism iff $\bigcap_{x \in X} K_x = \{1\}$.

A *filter space* of a BL-algebra $A$ is a family $\{T_x\}_{x \in X}$ of proper filters of $A$, indexed by a topological space $X$. We say that a filter space $\{T_x\}_{x \in X}$ *canonically determines* a representation of $A$ if there is a representation $\varphi : A \to \Gamma(X, F)$ such that $T_x = K_x$ for all $x \in X$.

**Theorem 3.2.** [11, Theorem 2]
*Let $A$ be a nontrivial BL-algebra and $\{T_x\}_{x \in X}$ a filter space of $A$ such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in $X$ for all $a \in A$. Then $\{T_x\}_{x \in X}$ canonically determines a representation of $A$.*

The BL-sheaf space $(F_A, p_A, X)$ and the representation $\varphi : A \to \Gamma(X, F_A)$ are constructed in the following way, given in [8] for universal algebra. Let $F_A$ be the disjoint union of the sets $\{A/T_x\}_{x \in X}$ and $p_A : F_A \to X$ the canonical projection, so $p_A^{-1}(\{x\}) = A/T_x$ for all $x \in X$. For all $x \in X$, $T_x$ is a proper filter of $A$, so $A/T_x$ is a nontrivial BL-algebra. For each $a \in A$, define the map $[a] : X \to F_A$ by $[a](x) = a/T_x$. Endow $F_A$ with the topology generated by the family $\{[a](U) \mid a \in A$ and $U$ is open in $X\}$. Applying [8, Corollary 2], we get that $(F_A, p_A, X)$ is a sheaf space of BL-algebras and the function $\varphi : A \to \Gamma(X, F_A)$, defined by $\varphi(a) = [a]$ for all $a \in A$, is a representation of $A$. It is easy to see that $K_x = T_x$ for all $x \in X$.

# 4 Baer extensions of BL-algebras

A BL-algebra $A$ is called *Baer* if every co-annihilator filter of $A$ is a principal filter of $A$ generated by an element from the center $B(A)$.

**Remark 4.1.** *Let $A$ be a BL-algebra. The following are equivalent:*
*(i) $A$ is Baer;*
*(ii) for all $F \in Co-An(A)$, there is $e \in B(A)$ such that $F =< e >= e \vee A$;*
*(iii) for all $F \in Co-An(A)$, there is a unique $e \in B(A)$ such that $F =< e >= e \vee A$;*
*(iv) for all $X \in A$, there is a unique $e \in B(A)$ such that ${}^{\perp}X =< e >= e \vee A$.*

*Proof.* $(i) \Leftrightarrow (ii)$ By the definition and Proposition 1.4(ii).
$(ii) \Leftrightarrow (iii)$ Apply Proposition 1.4(iii).
$(iii) \Leftrightarrow (iv)$ By Remark 2.5. □

Let $A$ be a BL-algebra. A Baer BL-algebra $A^*$ is called a *Baer extension of $A$* if $A$ is isomorphic to a BL-subalgebra of $A^*$.

By Proposition 2.6(v), $Co-An(A)$ is a complete Boolean algebra. Let $Spec(Co-An(A))$ be the set of its prime filters. Then $Spec(Co-An(A))$ is a Boolean space and the clopen sets of the basis are all the sets of the form

$$N_H = \{\underline{m} \in Spec(Co-An(A)) \mid H \in \underline{m}\},$$

where $H \in Co-An(A)$.

Let us recall that a topological space $X$ is called *extremally disconnected* if the closure $\overline{U}$ of any open subset $U$ of $X$ is also an open subset of $X$. Since $Co-An(A)$ is a complete Boolean algebra, it follows that $Spec(Co-An(A))$ is extremally disconnected (see, e.g., [2, Proposition 10.3.6, p.209]).

For any $\underline{m} \in Spec(Co-An(A))$, we define

$$P_{\underline{m}} = \cup\{H \in Co-An(A) \mid H \notin \underline{m}\} = \cup\{H \in Co-An(A) \mid \underline{m} \notin N_H\}.$$

**Proposition 4.2.** *Let $\underline{m} \in Spec(Co-An(A))$ and $a \in A$. Then*
*(i) $a \in P_{\underline{m}}$ iff ${}^{\perp\perp}a \notin \underline{m}$;*
*(ii) $P_{\underline{m}}$ is a prime filter of $A$.*

*Proof.* (i) Suppose that $a \in P_{\underline{m}}$. Then there is $H \in Co-An(A)$ such that $a \in H$ and $H \notin \underline{m}$. By Proposition 2.3(ii), from $\{a\} \subseteq H$ we get that ${}^{\perp\perp}a \subseteq {}^{\perp\perp}H = H$. If ${}^{\perp\perp}a \in \underline{m}$, since ${}^{\perp\perp}a, H \in Co-An(A)$, and $\underline{m}$ is a filter of $Co-An(A)$ we get that $H \in \underline{m}$, that is a contradiction. Hence, ${}^{\perp\perp}a \notin \underline{m}$. Conversely, suppose that ${}^{\perp\perp}a \notin \underline{m}$. If we take $H = {}^{\perp\perp}a$, then $H \in Co-An(A)$, $H \notin \underline{m}$, and $a \in H$, by Proposition 2.10(ii). Hence, $a \in P_{\underline{m}}$.
(ii) Since $\underline{m}$ is a proper filter of $Co-An(A)$, we have that $\{1\} \notin \underline{m}$. Hence, ${}^{\perp\perp}1 = \{1\} \notin \underline{m}$. By (i), we get that $1 \in P_{\underline{m}}$. Let $a, b \in P_{\underline{m}}$. Hence, there are $H_1, H_2 \in Co-An(A)$ such that $H_1, H_2 \notin \underline{m}$, $a \in H_1, b \in H_2$. Let $H = H_1 \vee_{Co-An(A)} H_2$. Then, $H \in Co-An(A), a, b \in H$ and $H \notin \underline{m}$, since $\underline{m}$ is prime in $Co-An(A)$. From $a, b \in H$ and the fact that $H$ is a filter of $A$, we get that $a \odot b \in H$. Hence, $H \in Co-An(A)$ is such that $a \odot b \in H$ and $H \notin \underline{m}$, that is, $a \odot b \in P_{\underline{m}}$. Suppose now that $a, b$ are in $A$ such that $a \leq b$ and $a \in P_{\underline{m}}$. By (i) we have that ${}^{\perp\perp}a \notin \underline{m}$. From $a \leq b$ and Proposition 2.10(iii) it follows that ${}^{\perp\perp}b \subseteq {}^{\perp\perp}a$. Since $\underline{m}$ is a filter of $Co-An(A)$, we get that ${}^{\perp\perp}b \notin \underline{m}$. Hence, applying again (i), $b \in P_{\underline{m}}$. Thus, we have got that $P_{\underline{m}}$ is a filter of $A$. We have

that ${}^{\perp\perp}0 = A \in \underline{m}$, since $\underline{m}$ is a filter of $Co - An(A)$. Hence, by (i), $0 \notin P_{\underline{m}}$. That is, $P_{\underline{m}}$ is proper. Let us prove that $P_{\underline{m}}$ is a prime filter of $A$. Let $a, b \in A$ such that $a \vee b \in P_{\underline{m}}$, so, by (i), ${}^{\perp\perp}(a \vee b) \notin \underline{m}$. By Proposition 2.10(v), we get that ${}^{\perp\perp}a \cap {}^{\perp\perp}b \notin \underline{m}$. Since $\underline{m}$ is a filter of $Co - An(A)$, it follows that ${}^{\perp\perp}a \notin \underline{m}$ or ${}^{\perp\perp}b \notin \underline{m}$. Hence, again by (i), $a \in P_{\underline{m}}$ or $b \in P_{\underline{m}}$. □

**Proposition 4.3.**

$$\bigcap\{P_{\underline{m}} \mid \underline{m} \in Spec(Co - An(A))\} = \{1\}.$$

*Proof.* Let $a \neq 1$ in $A$. Then ${}^{\perp}a \neq A$, by Proposition 2.9(iii). Since $A$ is the greatest element of the Boolean algebra $Co - An(A)$, there is a prime filter $\underline{n}$ of $Co - An(A)$ such that ${}^{\perp}a \notin \underline{n}$. Since ${}^{\perp}a \vee_{Co-An(A)} {}^{\perp\perp}a = A$ and $\underline{n}$ is prime, it follows that ${}^{\perp\perp}a \in \underline{n}$, hence, by Proposition 4.2(i), $a \notin P_{\underline{n}}$. □

**Corollary 4.4.** *$A$ is isomorphic to a subdirect product of the family $\{A/P_{\underline{m}}\}_{\underline{m} \in Spec(Co-An(A))}$.*

*Proof.* Apply the above proposition and [6, Lemma II.8.2, p. 56]. □

**Proposition 4.5.** *For any $a, b \in A$, the set $\{\underline{m} \in Spec(Co - An(A)) \mid a/P_{\underline{m}} = b/P_{\underline{m}}\}$ is clopen in $Spec(Co - An(A))$.*

*Proof.* Let $U = \{\underline{m} \in Spec(Co - An(A)) \mid a/P_{\underline{m}} = b/P_{\underline{m}}\}$. We have that $\underline{m} \in U$ iff $(a \to b) \odot (b \to a) \in P_{\underline{m}}$ iff ${}^{\perp\perp}((a \to b) \odot (b \to a)) \notin \underline{m}$ iff ${}^{\perp}((a \to b) \odot (b \to a)) \in \underline{m}$ iff $\underline{m} \in N_{{}^{\perp}((a\to b)\odot(b\to a))}$. Hence, $U = N_{{}^{\perp}((a\to b)\odot(b\to a))}$, that is $U$ is clopen in $Spec(Co - An(A))$. □

**Proposition 4.6.** *The family $\{P_{\underline{m}}\}_{\underline{m} \in Spec(Co-An(A))}$ canonically determines a sheaf representation of $A$.*

*Proof.* Apply Proposition 4.5 and Theorem 3.2. □

Let $(F_A, p_A, Spec(Co - An(A)))$ be the sheaf space of BL-algebras and

$$\varphi : A \to \Gamma(Spec(Co - An(A)), F_A)$$

the sheaf representation determined by the family $\{P_{\underline{m}}\}_{\underline{m} \in Spec(Co-An(A))}$. Then $(F_A)_{\underline{m}} = A/P_{\underline{m}}$ for all $\underline{m} \in Spec(Co - An(A))$, $p_A : F_A \to Spec(Co - An(A))$ is the canonical projection and $\varphi(a) = [a]$ for all $a \in A$, where $[a] \in \Gamma(Spec(Co - An(A)), F_A)$ is defined by $[a](\underline{m}) = a/P_{\underline{m}}$.

By Proposition 4.3, it follows that $\varphi$ is a monomorphism of BL-algebras, embedding $A$ into the BL-algebra of global sections of the BL-sheaf space $(F_A, p_A, Spec(Co - An(A)))$.

**Proposition 4.7.** *$F_A$ is a Hausdorff space.*

*Proof.* Let $a/P_{\underline{m}} \neq b/P_{\underline{n}} \in F_A$. We have two cases:

(1) $\underline{m} \neq \underline{n}$. Since $Spec(Co-An(A))$ is Hausdorff, there are $U, V$ open in $Spec(Co-An(A))$ such that $\underline{m} \in U, \underline{n} \in V$ and $U \cap V = \emptyset$. Then, $[a](U) \cap [b](V) = \emptyset, a/P_{\underline{m}} \in [a](U)$ and $b/P_{\underline{m}} \in [b](V)$.

(2) $\underline{m} = \underline{n}$. Let $U = \{\underline{q} \in Spec(Co-An(A)) \mid a/P_{\underline{q}} \neq b/P_{\underline{q}}\}$. Applying Proposition 4.5, we get that $U$ is open in $Spec(Co-An(A))$, $\underline{m} \in U$, hence $a/P_{\underline{m}}, b/P_{\underline{m}} \in [b](U)$. Obviously, $[a](U) \cap [b](U) = \emptyset$.

□

In the sequel, we shall denote by $A^*$ the BL-algebra $\Gamma(Spec(Co-An(A)), F_A)$.

For $U \subseteq Spec(Co-An(A))$, define $e_U : Spec(Co-An(A)) \to F_A$ by

$$e_U(\underline{m}) = \begin{cases} 0/P_{\underline{m}} & \text{if } \underline{m} \notin U \\ 1/P_{\underline{m}} & \text{if } \underline{m} \in U \end{cases}$$

**Remark 4.8.** *For any $U \subseteq Spec(Co-An(A))$, $p_A \circ e_U = 1_{Spec(Co-An(A))}$.*

**Proposition 4.9.** *Let $U \subseteq Spec(Co-An(A))$. If $U$ is a clopen subset of $Spec(Co-An(A))$, then $e_U \in A^*$.*

*Proof.* A basic open set in $F_A$ is of the form $[a](V)$, where $a \in A$ and $V$ is open in $Spec(Co-An(A))$. It is clear that

$$e_U^{-1}([a](V)) = (V \cap U \cap [\,[a] = \underline{1}\,]) \cup (V \cap U^c \cap [\,[a] = \underline{0}\,]),$$

where $U^c := Spec(Co-An(A)) \setminus U$. Applying Proposition 3.1(i) and the facts that $U, U^c$ are open, it follows that $e_U^{-1}([a](V))$ is open in $Spec(Co-An(A))$. Hence, $e_U$ is continuous. □

**Proposition 4.10.** *If $U$ is clopen in $Spec(Co-An(A))$, then $e_U \in B(A^*)$ and the complement of $e_U$ is $e_{U^c}$, where $U^c$ is the set $Spec(Co-An(A)) - U$.*

*Proof.* Since $U$ is clopen, we have that $U^c$ is also clopen. Hence, $e_{U^c} \in A^*$ too. Obviously, $e_U \vee e_{U^c} = \underline{1}$ and $e_U \wedge e_{U^c} = \underline{0}$. That is, $e_U \in B(A^*)$ and the complement of $e_U$ is $e_{U^c}$. □

**Theorem 4.11.** *$A^*$ is a Baer extension of the BL-algebra $A$.*

*Proof.* Since $\varphi : A \to A^*$ is a monomorphism, it remains to prove that $A^*$ is a Baer BL-algebra.

For any $\sigma \in A^*$, let $U_\sigma = \{\underline{m} \mid \sigma(\underline{m}) \neq 1/P_{\underline{m}}\}$. Then, by Proposition 3.1(iii), $U_\sigma$ is clopen in $Spec(Co-An(A))$. For any $X \subseteq A^*$, let $U_X = \overline{\bigcup_{\sigma \in X} U_\sigma}$. Since $Spec(Co-An(A))$ is extremally disconnected, it follows that $U_X$ is a clopen subset of $Spec(Co-An(A))$, hence $e_{U_X} \in B(A^*)$.

In the sequel, we shall prove that for any $X \subseteq A^*$,

$$^{\perp}X = < e_{U_X} > .$$

First, let us prove

$$(*) \quad {}^{\perp}X = \{\sigma \in A^* \mid U_\sigma \cap U_X = \emptyset\}.$$

For any $\sigma \in A^*$, we have that

$\sigma \in {}^{\perp}X$ iff $\sigma \vee_{A^*} \tau = \underline{1}$ for all $\tau \in X$
iff for all $\tau \in X$ and all $\underline{m} \in Spec(Co - An(A))$, $\sigma(\underline{m}) \vee_{A/P_{\underline{m}}} \tau(\underline{m}) = 1/P_{\underline{m}}$
iff for all $\tau \in X$ and all $\underline{m} \in Spec(Co - An(A))$, $\sigma(\underline{m}) = 1/P_{\underline{m}}$ or $\tau(\underline{m}) = 1/P_{\underline{m}}$,

since $A/P_{\underline{m}}$ is a BL-chain, $P_{\underline{m}}$ being a prime filter of $A$. Hence,

$\sigma \notin {}^{\perp}X$ iff there exist $\tau \in X$ and $\underline{m} \in Spec(Co - An(A))$ such that $\sigma(\underline{m}) \neq 1/P_{\underline{m}}$ and $\tau(\underline{m}) \neq 1/P_{\underline{m}}$
iff there exists $\underline{m} \in Spec(Co - An(A))$ such that $\underline{m} \in U_\sigma$ and there exists $\tau \in X$ such that $\underline{m} \in U_\tau$
iff there exists $\underline{m} \in Spec(Co - An(A))$ such that $\underline{m} \in U_\sigma$ and $\underline{m} \in \bigcup_{\tau \in X} U_\tau$
iff there exists $\underline{m} \in Spec(Co - An(A))$ such that $\underline{m} \in U_\sigma \cap \bigcup_{\tau \in X} U_\tau$
iff $U_\sigma \cap \bigcup_{\tau \in X} U_\tau \neq \emptyset$
iff $U_\sigma \cap U_X \neq \emptyset$.

Hence, we have got $(*)$. It remains to prove that

$$< e_{U_X} >= \{\sigma \in A^* \mid U_\sigma \cap U_X = \emptyset\}.$$

Let $\sigma \in A^*$. Then $\sigma \in < e_{U_X} >$ iff $e_{U_X} \leq \sigma$ iff $e_{U_X}(\underline{m}) \leq \sigma(\underline{m})$ for all $\underline{m} \in Spec(Co - An(A))$ iff $\sigma(\underline{m}) = 1/P_{\underline{m}}$ for all $\underline{m} \in U_X$ iff $\underline{m} \in (U_\sigma)^c$ for all $\underline{m} \in U_X$ iff $U_X \subseteq (U_\sigma)^c$ iff $U_\sigma \cap U_X = \emptyset$. $\square$

**Lemma 4.12.** *For any $\sigma \in A^*$ there is a finite partition $\{U_i \mid i = \overline{1,n}\}$ of $Spec(Co - An(A))$ into nonempty disjoint clopen subsets and there are elements $\{a_i \mid i = \overline{1,n}\}$ of $A$ such that $\sigma = ([a_1] \odot e_{U_1}) \vee \cdots \vee ([a_n] \odot e_{U_n})$.*

*Proof.* Let $\sigma \in A^*$. Since $Spec(Co - An(A))$ is Boolean, we can apply [8, Lemma 3.2] to obtain a finite partition $\{U_i \mid i = \overline{1,n}\}$ of $Spec(Co - An(A))$ into nonempty disjoint clopen subsets and thee elements $\{a_i \mid i = \overline{1,n}\}$ of $A$ such that, for any $\underline{m}$, $\sigma(\underline{m}) = [a_i](\underline{m}) = a_i/P_{\underline{m}}$, where $\underline{m} \in U_i$. It follows that $\sigma = ([a_1] \odot e_{U_1}) \vee \ldots \vee ([a_n] \odot e_{U_n})$. $\square$

**Proposition 4.13.** *If $A$ is a Baer BL-algebra, then $e_U \in \varphi(A)$ for any clopen subset $U$ of $Spec(Co - An(A))$.*

*Proof.* Let $U$ be a clopen subset of $Spec(Co - An(A))$. Hence, there is $H = {}^{\perp}X \in Co - An(A)$ such that $U = N_H = \{\underline{m} \in Spec(Co - An(A)) \mid {}^{\perp}X \in \underline{m}\}$. Since $A$ is a Baer BL-algebra, there is $t \in B(A)$ such that ${}^{\perp}X =< t >$. Applying Proposition 2.9(v), and the fact that $t = t^{--}$, we get that ${}^{\perp}X = {}^{\perp}e^{-}$, so ${}^{\perp\perp}X = {}^{\perp\perp}t^{-}$.

We have that, for any $\underline{m} \in Spec(Co - An(A))$, $A/P_{\underline{m}}$ is a BL-chain, so $A/P_{\underline{m}}$ is directly indecomposable, that is $B(A/P_{\underline{m}}) = \{1/P_{\underline{m}}, 0/P_{\underline{m}}\}$. Since $t^- \in B(A)$, it follows that $[t^-](\underline{m}) = t^-/P_{\underline{m}} \in B(A/P_{\underline{m}})$. Hence,

$$[t^-](\underline{m}) = \begin{cases} 0/P_{\underline{m}} & \text{if } t^- \notin P_{\underline{m}} \\ 1/P_{\underline{m}} & \text{if } t^- \in P_{\underline{m}}. \end{cases}$$

For all $\underline{m} \in Spec(Co - An(A))$, we get that $\underline{m} \in U$ iff ${}^{\perp}X \in \underline{m}$ iff ${}^{\perp\perp}X \notin \underline{m}$ iff ${}^{\perp\perp}t^- \notin \underline{m}$ iff $t^- \in P_{\underline{m}}$, by Proposition 4.2(i). Hence, $e_U(\underline{m}) = 1/P_{\underline{m}}$ iff $[t^-](\underline{m}) = 1/P_{\underline{m}}$ for all for all $\underline{m} \in Spec(Co - An(A))$. Hence, $e_U = [t^-]$, so $e_U = \varphi(t^-) \in \varphi(A)$. □

**Theorem 4.14.** *If $A$ is a Baer BL-algebra, then*

$$\varphi : A \cong \Gamma(Spec(Co - An(A)), F_A).$$

*Proof.* It remains to prove that $\varphi$ is surjective. If $\sigma \in \Gamma(Spec(Co - An(A)), F_A)$, then by Lemma 4.12, there is a finite partition $\{U_i \mid i = \overline{1,n}\}$ of $Spec(Co - An(A))$ into nonempty disjoint clopen subsets and the elements $\{a_i \mid i = \overline{1,n}\}$ of $A$ such that $\sigma = ([a_1] \odot e_{U_1}) \vee \ldots \vee ([a_n] \odot e_{U_n})$. Applying Proposition 4.13, it follows that there are elements $\{t_i \mid i = \overline{1,n}\}$ such that $e_{U_i} = [t_i]$ for any $i = \overline{1,n}$. Hence, $\sigma = [(a_i \odot t_1) \vee \ldots \vee (a_n \odot t_n)] = \varphi((a_i \odot t_1) \vee \ldots \vee (a_n \odot t_n)) \in \varphi(A)$. □

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