Recent results in proof mining

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(joint work with Ulrich Kohlenbach)

The talk is a report on joint work [7, 8] with Ulrich Kohlenbach and presents two applications of proof mining. By proof mining we mean the logical analysis of mathematical proofs with the aim of extracting new numerically relevant information hidden in the proofs (we refer to [5] for a book treatment).

In 1939, Garrett Birkhoff proved the following generalization of von Neumann's Mean Ergodic Theorem.

Theorem 1. [2] *Let X be a uniformly convex Banach space and $T : X \to X$ be a linear operator with $\|Tx\| \leq \|x\|$ for all $x \in X$. Then for any $x \in X$, the Cesaro mean (x_n) is convergent.*

In [1], Avigad, Gerhardy and Towsner address the issue of finding an effective rate of convergence for (x_n) in Hilbert spaces. They show that even for the separable Hilbert space L_2 there are simple computable such operators T and computable points $x \in L_2$ such that there is no computable rate of convergence of (x_n). In such a situation the best one can hope for is an effective bound on the Herbrand normal form of the Cauchy property of (x_n):

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \in \mathbb{N}\, \forall i, j \in [N, N + g(N)]\ (\|x_i - x_j\| < \varepsilon). \tag{1}$$

The mathematical relevance of this reformulation of convergence was recently pointed out by T. Tao ([9, 10]), who also uses the term 'metastability'.

In [4], a general logical metatheorem is proved that guarantees (given a proof of (1)) the extractability of an effective bound $\Phi(\varepsilon, g, b, \eta)$ on '$\exists N$' in (1) that is highly uniform in the sense that it only depends on g, ε, an upper bound $\mathbb{N} \ni b \geq \|x\|$ and a modulus η of uniform convexity for X, but otherwise is independent from x, X and T.

We extract [8, Theorem 2.1] such a bound from the proof of Theorem 1: $\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(1)$, with $\|x\| \leq b \in \mathbb{N}$, $M := \left\lceil \frac{16b}{\varepsilon} \right\rceil$, $K := \left\lceil \frac{b}{\gamma} \right\rceil$, $\gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right)$, $h, \tilde{h} : \mathbb{N} \to \mathbb{N}$, $h(n) := 2(Mn + g(Mn))$, $\tilde{h}(n) := \max_{i \leq n} h(i)$. In the case of Hilbert spaces, $K := \left\lceil \frac{512b^2}{\varepsilon^2} \right\rceil$.

In this way, we provide a finitary version in the sense of T. Tao [9, 10] of the Mean Ergodic Theorem for uniformly convex Banach spaces and so generalize similar results obtained for Hilbert spaces by Avigad, Gerhardy and Towsner [1] and T. Tao [10]. Despite of our result being significantly more general then the Hilbert space case treated in [1], the extraction of our bound is considerably more easy compared to [1] and even numerically better.

The second application is in metric fixed point theory, more specifically in the approximate fixed point theory of asymptotically nonexpansive mappings, introduced in [3].

One typical result is the following theorem which is otained in [6, Corollary 8] as corollary of a quantitative result.

Theorem 2. *Let $(X, \|\cdot\|)$ be a uniformly convex normed space, $C \subseteq X$ a convex subset and $T : C \to C$ an asymptotically nonexpansive mapping with sequence (k_n) in $[0, \infty)$ satisfying $\sum_{i=0}^{\infty} k_i < \infty$. Let (λ_n) be a sequence in $[a, b]$ for $0 < a < b < 1$ and define the Krasnoselski-Mann iteration of T starting from $x \in X$ by*

$$x_0 := x, \ x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T^n(x_n).$$

If T has a fixed point, then $d(x_n, T(x_n)) \stackrel{n\to\infty}{\to} 0$.

While there does not seem to exist a computable rate of convergence (see the discussion in[6]), the general logical metatheorems from [4] guarantee an effective bound on the $\exists N$ in the Herbrand normal form of the convergence of $(\|x_n - T(x_n)\|)$ towards 0:

$$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists N \in \mathbb{N} \, \forall m \in [N, N + g(N)] \, (\|x_m - T(x_m)\| < \varepsilon). \tag{2}$$

Such a bound was extracted in [6, Theorem 22]. In [7] we take the proofs from [6] as our point of departure and generalize the results to uniformly convex hyperbolic spaces. This, in particular, covers the important class of CAT(0)-spaces (in the sense of Gromov) and, a-fortiorily, $\mathbb{R}$-trees in the sense of Tits. For CAT(0)-spaces we get a quadratic bound on the approximate fixed point property of (x_n).

References

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